Lowness for uniform Kurtz randomness

Takayuki Kihara

Japan Advanced Institute of Science and Technology (JAIST)

Joint work with

Kenshi Miyabe (University of Tokyo)

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Background

- "Lowness for Randomness" is a well-studied notion in algorithmic randomness theory.
- The highlighted theorem (Nies) is:

  K-triviality = lowness for ML randomness

- Downey-Griffiths-Laforte (2004) introduced the notion of

  Schnorr triviality

  (the triviality w.r.t. computable measure machines).

- However:

  Schnorr triviality ≠ lowness for Schnorr randomness

- Franklin and Stephan (2010) showed:

  Schnorr triviality = tt-lowness for Schnorr randomness

- Miyabe asked whether a characterization of tt-lowness for Kurtz randomness exists.

Definition

1. An oracle Kurtz test is a partial computable function $T :\subseteq 2^\omega \to \mathcal{A}_-(2^\omega)$ such that:

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i.e., T represents a sequence (T_n^X) of clopen sets uniformly in $X \in 2^\omega$ and $n \in \omega$ such that the measure of $(T_n^X)_{n\in\omega}$ converges to 0 for every X.
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2. A is Kurtz random relative to B if
   $A \notin T(B)$ for every (partial) oracle Kurtz test T.

3. A is Kurtz random tt-relative to B if
   $A \notin T(B)$ for every total oracle Kurtz test T.

4. $A \in 2^\omega$ is (tt-)low for Kurtz randomness if every Kurtz random sequence remains Kurtz random (tt-)relative to A.

Example

1. If $f:\omega\to\omega$ is tt-reducible to A and strictly increasing, then
$$\mathbf{SuperSets}^f=\{B\subseteq\omega:\mathbf{rng}(f)\subseteq B\}$$
is a Kurtz test tt-relative to A.
2. If $\mathcal{V}\subseteq 2^\omega$ is a null Π^0_1 class (i.e., a Kurtz test), then
$$A+\mathcal{V}=\{A+B:B\in\mathcal{V}\}$$
is a Kurtz test tt-relative to A, where the bitwise addition $A+B$ is defined by $(A+B)(n)\equiv A(n)+B(n)$ mod 2.

Problem

Find a characterization of tt-lowness for Kurtz randomness!

- Greenberg-Miller (2009) showed that

  low for Kurtz randomness $=$ non-DNR $+$ hyperimmune-free

- The problem is that there is NO counterpart of hyperimmune-freeness in the tt-degrees:
  - For every $A \in 2^\omega$, every $f \leq_{tt} A$ is majorized by a computable function!
- Hence, it seems that we cannot use the methods
  - in the original paper in Greenberg-Miller,
  - in a simpler proof in the book by Downey-Hirschfeldt.

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Main Theorem

$A \in 2^\omega$ is tt-low for Kurtz randomness if and only if it is h-dimensionally Kurtz null for every computable gauge function h.

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Proof Techniques

We have two methods to show the main theorem:

1. Modification of Greenberg-Miller's svelte tree argument.
2. Modification of Pawlikowski's additivity characterization of strong measure zero in set theory of the real line.

We introduce a Kurtz version of effective Hausdorff dimension.

Definition (K.-Miyabe)

For an order $h : \omega \to \omega$, a set $E \subseteq 2^\omega$ is *Kurtz h-null* ($\mathcal{K}^h$-null) if there is a computable sequence $\{C_n\}_{n\in\omega}$ of finite sets of strings such that

$$E \subseteq [C_n] \text{ and } \sum_{\sigma\in C_n} 2^{-h(|\sigma|)} \leq 2^{-n} \text{ for all } n \in \omega.$$

We also say that $A \in 2^\omega$ is *Kurtz h-null* if $\{A\}$ is Kurtz h-null.

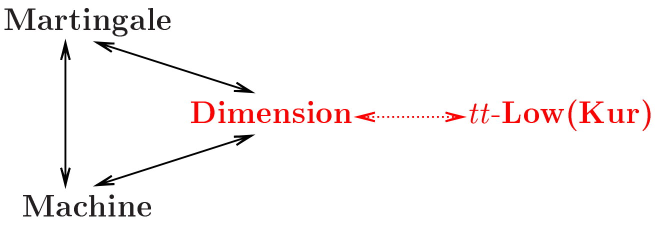
Remark

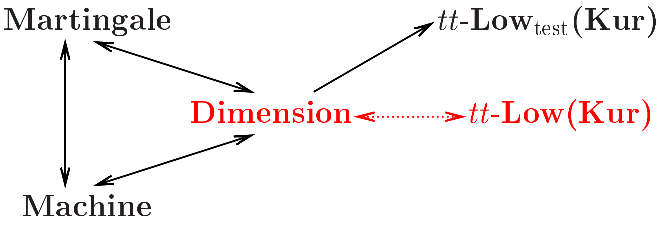
1. Such a function h is called a gauge function or a dimension function in geometric measure theory.
2. If $h : n \mapsto s$ for a fixed real $s \in (0,1]$, then it is a Kurtz version of the s-dimensional Hausdorff measure zero.
3. For $\bar{1} : n \mapsto 1$, a real is Kurtz $\bar{1}$-null iff it is not Kurtz random.

Fine Structure inside “Probability **0**”

Non-SchR	$\leftarrow$	Non-KurR			
$\updownarrow$		$\updownarrow$			
$\mathcal{H}^1$-null	$\leftarrow$	$\mathcal{K}^1$-null	$\leftarrow$	$\mathcal{P}^1$-null	
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathbf{dim}_H \leq \mathbf{1}$	$\leftarrow$	$\mathbf{dim}_K \leq \mathbf{1}$	$\leftarrow$	$\mathbf{dim}_P \leq \mathbf{1}$	
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathbf{dim}_H \leq s$	$\leftarrow$	$\mathbf{dim}_K \leq s$	$\leftarrow$	$\mathbf{dim}_P \leq s$	$(s \in (0,1])$
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathbf{dim}_H = \mathbf{0}$	$\leftarrow$	$\mathbf{dim}_K = \mathbf{0}$	$\leftarrow$	$\mathbf{dim}_P = \mathbf{0}$	
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathcal{H}^h$-null	$\leftarrow$	$\mathcal{K}^h$-null	$\leftarrow$	$\mathcal{P}^h$-null	$(h \to \infty)$
$\uparrow$		$\uparrow$		$\uparrow$	
$(\forall h)\ \mathcal{H}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{K}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{P}^h$-null	
$\updownarrow$		$\updownarrow$		$\updownarrow$	
tt-Low(Sch, Kur)	$\leftarrow$	tt-Low(Kur)	$\leftarrow$	tt-Low(Sch)	

$\textbf{Dimension} \longleftrightarrow tt\text{-}\textbf{Low(Kur)}$


Martingale
Dimension
tt-Low(Kur)
Machine


Martingale
tt-Low_test(Kur)
Dimension
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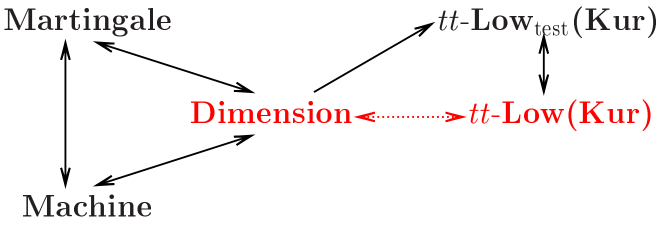
Machine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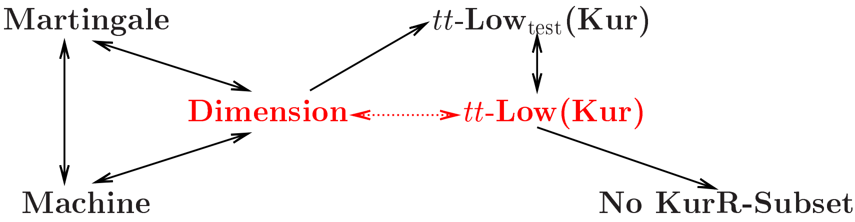
Martingale
tt-Low$_{\mathrm{test}}$(Kur)
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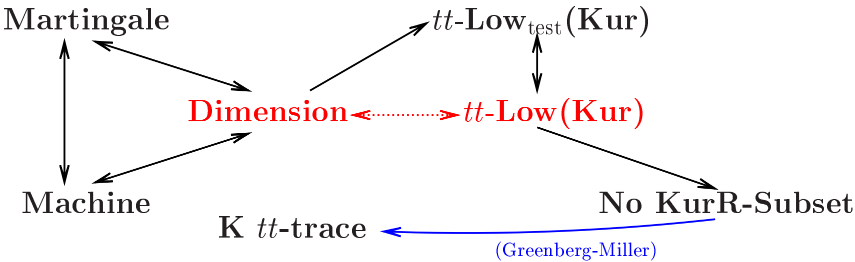
Machine
No KurR-Subset


Martingale
tt-Low$_{\text{test}}$(Kur)
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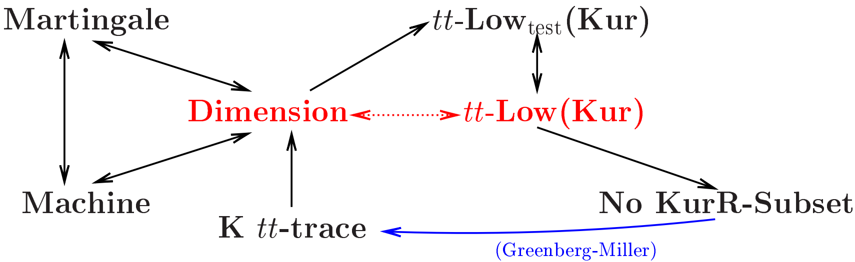
No KurR-Subset
K tt-trace
(Greenberg-Miller)



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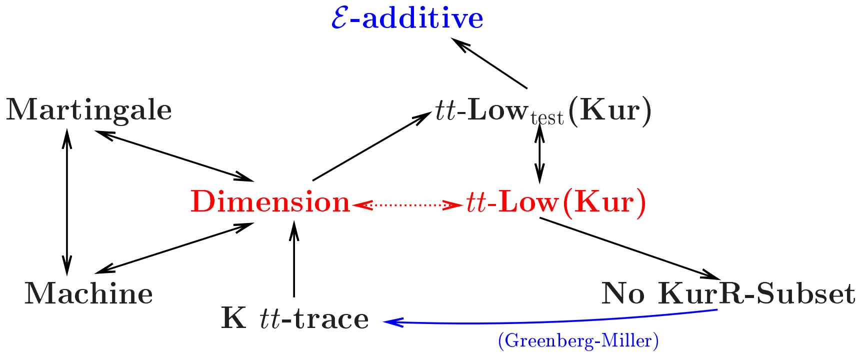
$\mathcal{E}$-additive
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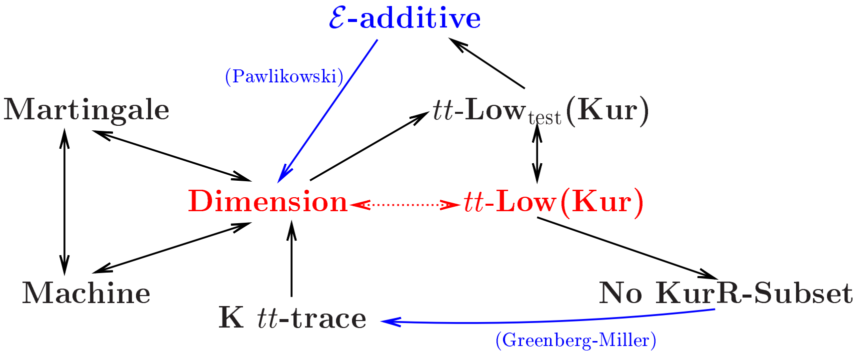
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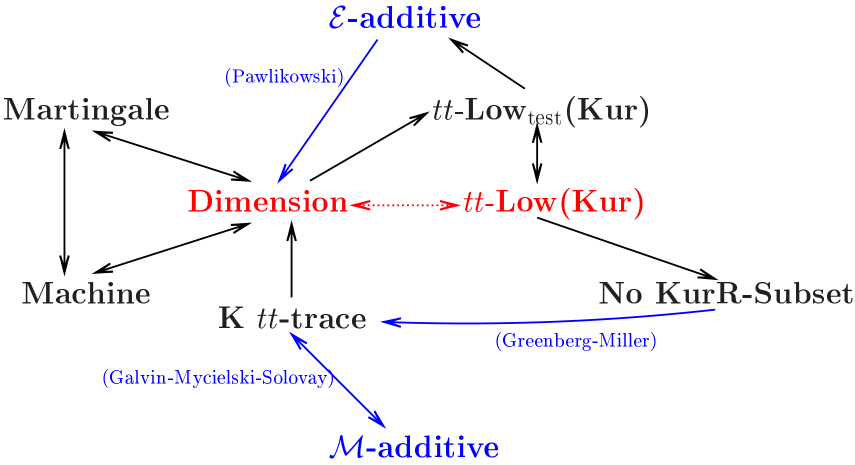
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(Galvin-Mycielski-Solovay)
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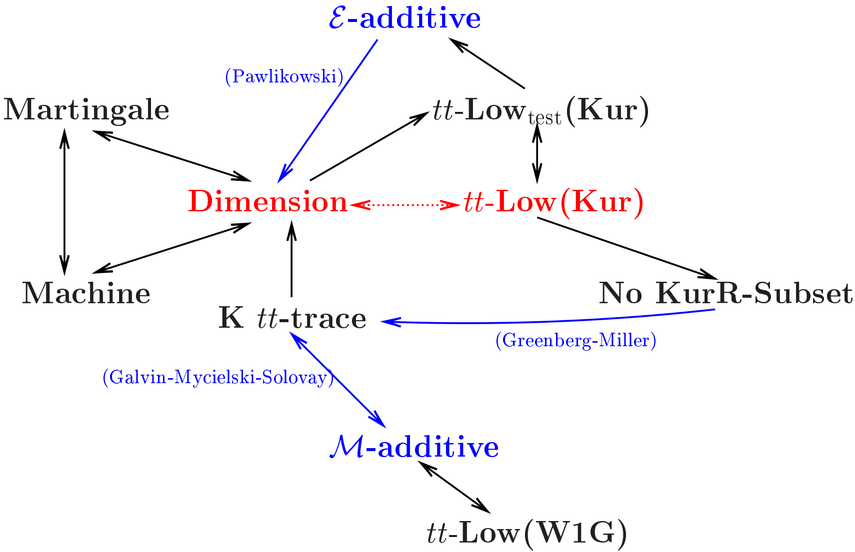

$\mathcal{E}$-additive
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$\mathcal{M}$-additive
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The following are equivalent for a real $A \in 2^\omega$:

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4. A is Kurtz tt-traceable.
5. $\forall$comp. order h $\exists$comp. order g $\exists$comp. martingale d s.t. $$\forall n \exists k \in [g(n), g(n+1))\ d(A \upharpoonright k) \geq 2^n 2^{n-h(k)}.$$
6. $\forall$comp. order h $\exists$comp. order g $\exists$c.m.m. M s.t. $$\forall n \exists k \in [g(n), g(n+1))\ K_M(A \upharpoonright k) \geq h(k) - n.$$

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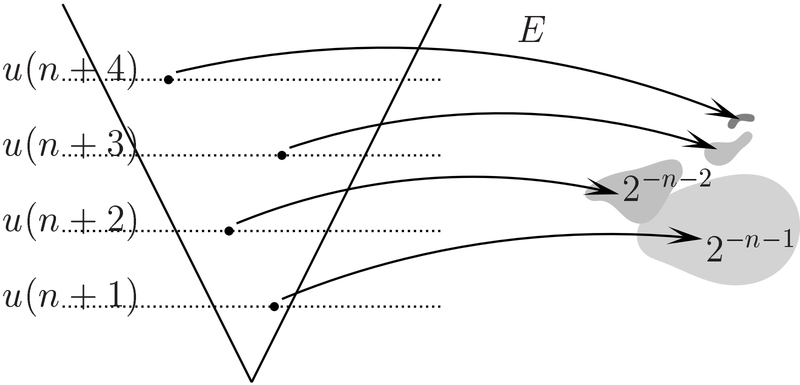
$$\forall n \exists k \in [g(n), g(n+1))\ K_M(A \upharpoonright k) \geq h(k) - n.$$
7. $A + B$ is Kurtz random iff B is Kurtz random.
8. $A + B$ is weakly 1-generic iff B is weakly 1-generic.

Lemma

Assume that $\mathcal{V} \subseteq 2^\omega$ is Kurtz h-null for every computable order h, and $E : 2^\omega \to \mathcal{A}_-(2^\omega)$ is a total Kurtz test.
Then, $E[\mathcal{V}] = \bigcup_{A \in \mathcal{V}} E(A)$ is covered by a Kurtz test.

- u: $E_n^{Z \upharpoonright u(n)}$ is determined for every $Z \in 2^\omega$.
- h: $2^{-h(u(n))} \geq \frac{1}{n+1}$.
- Since $\mathcal{V}$ is Kurtz h-null, $\exists \mathcal{V}_n$: seq. of clopen s.t. $\mathcal{V} \subseteq \bigcap_n \mathcal{V}_n$ and $\sum_{\sigma \in \mathcal{V}_n} 2^{-h(|\sigma|)} < \frac{1}{n+1}$.
- Every $\mathcal{V}_n$ has at most k strings length $\leq u(n+k)$, since
$$\sum_{\sigma \in \mathcal{V}_n} 2^{-h(|\sigma|)} < \frac{1}{n+1} \leq \frac{k+1}{n+k+1} \leq (k+1)2^{-h(u(n+k))}.$$
- $\mathcal{V}_n$ can be think of as a seq. $(\sigma_k)_k$ of strings with $|\sigma_k| = u(n+k)$.



- $\mu(E_n^{Z \restriction u(n)}) \leq 2^{-n}$.
- $\mathcal{V}_n$ is a seq. $(\sigma_k)_k$ of strings with $|\sigma_k| = u(n+k)$.
- Then $E[\mathcal{V}_n] \subseteq \bigcup_k E_k^{\sigma_k}$ has measure 2^{-n+1}

$(\forall \text{comp. } h)\ V$ is $\mathcal{K}^h$-null $\iff$ $(\forall \text{Kurtz test } E)\ V + E$ is Kurtz null.

Lemma (K.-Miyabe [3])

Assume that V is $\mathcal{K}^h$-null for every computable h. Then, $\bigcup_{y \in V} E(y)$ is Kurtz null for every uniform Kurtz test $E : 2^\omega \to \mathcal{A}_-$.

(∀comp. h) V is $\mathcal{K}^h$-null $\iff$ (∀Kurtz test E) $V + E$ is Kurtz null.

Lemma

Assume that $V + E$ is Kurtz null for every Kurtz null set E.
Then, V is $\mathcal{K}^h$-null for every computable h.

1. h: given. $g(n) = g(n-1) + h(n) + 2^{h(n)}$.
2. $E_k \subseteq 2^{g(k)}$: strings of the form $\tau^\frown\sigma_i{}^\frown\rho$ s.t.

$$|\tau| = g(k-1),\ |\sigma_i| = h(k),\ |\rho| = 2^{h(k)},\text{ and } \rho(i) = 0,$$

where $\{\sigma_i : i < 2^{h(k)}\}$ is an enumeration of $2^{h(k)}$.

3. By assumption, $V + E$ is covered by a Kurtz test $D = \bigcap_n D_n$.
4. $D_{e(k)}$: $\mu(D_{e(k)}|\tau) < 1/8$ for any $\tau \in 2^{g(k-1)}$.

(∀comp. h) V is $\mathcal{K}^h$-null $\Leftarrow$ (∀Kurtz test E) $V + E$ is Kurtz null.

1. $d(k) = e(k)$ if $E_{d(k-1)} \subseteq 2^{\leq g(k-1)}$; $d(k) = d(k-1)$ o.w.
2. Given $\tau \in 2^{g(k-1)}$, $\sigma \in 2^{h(k)+2^{h(k)}}$ gets k-closer to D/τ if
$$(1 - 2^{-k-1})\,\mu(D_{d(k)}|\tau\sigma) > \mu(D_{d(k)}|\tau).$$
$D_\tau[k]$: all σ which get k-closer to D/τ.
3. (Remark) $\mu(D_\tau[k]) \leq 1 - 2^{-k-1}$.
4. $V_\tau[k] = \{\sigma \in 2^{h(k)} : (\exists \sigma' \succeq \sigma)\ \sigma' + E \in D_\tau[k]\}$.
5. $V[k] = \{\tau\sigma : \sigma \in D_\tau[k]\}$.

Claim

$\#V[k] \leq (k+1) \cdot 2^{h(k)}$.

1. Note that $V_\tau[k] + E_k \subseteq D_\tau[k]$.
2. By probability independence, $\mu(V_\tau[k] + E_k) = 1 - 2^{-|V_\tau[k]|}$.
3. However, $\mu(D_\tau[k]) \leq 1 - 2^{-k-1}$.
4. Hence, $|V_\tau[k]| \leq k + 1$.

Claim

Let $\{k(l)\}_{l\in\omega}$ be the list of all k s.t. $d(k) \neq d(k-1)$.
Then, $V \subseteq \bigcap_l \bigcup_{j=k(l-1)}^{k(l)-1} V[j]$.

1. Otherwise, there is $x \in V$ s.t. $x \notin \bigcup_{j=k(l-1)}^{k(l)-1} V[j]$ for some l.
2. $a := k(l-1)$, $b = k(l) - 1$
3. By def., $\mu(D_{d(a)}|x + \tau) < 1/8$ for any $\tau \in 2^{g(a-1)}$.
4. $\mu(D_{d(a)}|x + \tau) = 1$ for $\forall\tau \in E \upharpoonright g(b)$, since $V + E \subseteq D_{d(a)}$.
5. This is impossible, since $x \notin \bigcup_{j=a}^{b} V[j]$ implies we can find sequence $\tau_a, \tau_{a+1}, \dots \in E$ s.t. $\mu(D_{d(a)}|x + \tau_a) \geq \prod_{j=a}^{b}(1 - 2^{-j-1}) \cdot \mu(D_{d(a)}|x + \tau_b) > 1/8$.