On Paradoxes in Proof-Theoretic Semantics

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2nd PTS Conference, Tübingen, March 8, 2013

Outline

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Overview

The aim of the talk:

- Compare “logicality as adjointness" with other ideas.
- Analyse two “paradoxes" in PTS from such a categorial perspective.
  - This leads us to connections with quantum physics.

The paradox of Prior’s tonk:

- Tonk is a connective defined by $\vee$-intro. and $\wedge$-elim., making logic inconsistent.
- What is (not) wrong with tonk? Any model of tonk? What effect tonk has on identity of proofs?

The paradox of classical proofs:

- Identity of proofs trivialises in classical logic.
- What is (not) wrong with classical proofs? What precisely causes the phenomena? LEM? Structural rules?

Analogy b/w Logic and Quantum Physics

The logic of quantum physics is substructural:

- No-Cloning Thm: quantum states cannot be copied.
  - Contraction $\varphi \vdash \varphi \otimes \varphi$ in logic $=$ diagonal $\delta : H \rightarrow H \otimes H$ in cats $=$ copying of states in a Hilbert space H in physics.
- No-Deleting Thm: quantum states cannot be deleted.
  - Weakening $\varphi \otimes \varphi \vdash \varphi =$ projection $\pi : H \otimes H \rightarrow H =$ deleting of states in H.
- This is not merely an analogy, but can be a theorem.

$\otimes$ in substructural logic is $\wedge$ iff contraction and weakening hold.
$\otimes$ in a monoidal cat. is $\times$ iff it has diagonals and projections.
$\otimes$ in **Hilb** is not $\times$ ($\otimes$ in **Frob**(**Hilb**) is $\times$); quantum non-local correlations are not reduced to classical local correlations.

Decorating logical consequence with proofs

PT is CT. Suppose we have the following concepts given:

- Formulas φ; Proofs from φ to ψ; a proof from φ to φ exists;
- Proof-decorated relation $\vdash$:

$$\varphi \vdash_p \psi$$

  where p is a proof from φ to ψ;
- Sequential combination of proofs: from $\varphi \vdash_p \psi$ and $\psi \vdash_q \xi$ derive $\varphi \vdash_{p,q} \xi$; (parallel composition: $\varphi \otimes \varphi' \vdash_{p \otimes p'} \psi \otimes \psi'$;)
- Reduction of proofs that cancels a proof from φ to φ, and has local coherency (i.e., proofs may be reduced locally).

I call $\vdash_p$ "proof-theoretic consequence", which is the same as the notion of (monoidal) categories. CT enables to derive from $\vdash_p$ logical constants with inference rules via adjunctions.

Adjointness as Logicality

Lawvere understood logical constants as adjoint functors.

- We could see adjointness as a sort of harmony.
- Proof-theoretically, an adjunction amounts to the validity of a double line rule of certain form.

Standard logical constants can be characterised by double line rules, i.e., adjunctions.

- So let us regard adjointess as a criterion for logicality.
- A logical constant must be defined (to be precise, characterised) by (the double line rule of) an adjunction.

Lots of Subtleties Behind

Certain subtleties are lurking behind “adjointess as logicality":

- For example, it turns out that being defined by ONE adjunction is crucial, since tonk can be defined by TWO adjunctions.

Even more subtleties: need to be extremely careful in order to avoid “revenge" paradoxes; another kind of subtleties on treatment of multiplicatives, involving $\otimes$ as primitive vocabulary.

Belnap's Harmony

Belnap's harmony consists of conservativity and uniqueness.

- Any logical constant introduced according to categorial harmony is unique, since an adjoint of a functor is unique.
  - Uniqueness is not something assumed in the first place; rather, it is just a consequence of categorial harmony.
- Conservativity naturally fails in categorial harmony, since right (resp. left) adjoints preserve limits (resp. colimits) by Freyd's adjoint functor thm. Avron also is against it.
  - E.g., consider logic with $\wedge$ and $\vee$ specified by adjunctions. This logic does not validate distributivity b/w $\wedge$ and $\vee$. But adding $\rightarrow$ as the right adjoint of $\wedge$ makes distributivity valid.

Conservativity could be contested from a Quinean, holistic point of view. Meaning of a logical constant depends on the whole system. Adding a new constant may well change the meaning of older ones.

Violating Uniqueness

Both the adjointness harmony and Belnap's hamorny might miss the point, since:

- E.g., Girard's exponential ! does not have uniqueness.
- Thus, it cannot count as a logical constant according to any of the two.

But, the role of ! is to have certain control on structural rules or resources. As such, ! may be said to be a technical device or a "computational" constant, rather than a proper logical constant.

Sambin et al.'s Harmony

Sambin et al. introduce logical constants by what they call reflection principle and definitional equalities, like:

- $\varphi \vee \psi \vdash \xi$ iff $\varphi \vdash \xi$ and $\psi \vdash \xi$. $\varphi, \psi \vdash \xi$ iff $\varphi \otimes \psi \vdash \xi$.
- $\Gamma \vdash \varphi \to \psi$ iff $\Gamma \vdash (\varphi \vdash \psi)$.

Definitional equalities are similar to adjointness conditions. There are crucial differences, however.

- Definitional equalities do not always imply adjoitness, due to their "visibility" condition (restriction of context formulas). E.g., $\to$ is not an adjoint of conjunction in their Basic Logic.

Non-adjointness is inevitable in quantum logic with some $\to$. Quantum $\to$ if any cannot be an adjoint of $\wedge$, because of non-distributivity.

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The Problem of Prior's Tonk

Arthur Prior came up with a weird connective "tonk":

- It is defined by the following two rules of inference:

$$\frac{\Phi \vdash \varphi}{\Phi \vdash \varphi \text{tonk} \psi} \text{ (tonk-intro)} \qquad \frac{\Phi \vdash \varphi \text{tonk} \psi}{\Phi \vdash \psi} \text{ (tonk-elim)}$$

- Combining the two, we can show $\varphi \vdash \psi$ for any φ, ψ. Tonk thus causes the trivialisation of the deductive relation $\vdash$.
  - We always assume $\vdash$ admits identity (i.e., $\varphi \vdash \varphi$) and cut (i.e., $\varphi \vdash \psi$ and $\psi \vdash \xi$ together imply $\varphi \vdash \xi$).

This raised a sort of demarcation problem in philosophy of logic.

Categorical Harmony Rejects Tonk

The following thm. tells us the categorial criterion allows us to exclude tonk from logical constants.

Theorem

Tonk cannot be defined in a system without tonk by a single adjunction: i.e., there is no functor F s.t. an adjoint of F is tonk.

Still, tonk can be define by two adjunctions:

$$\Delta_{\bot} \dashv \text{tonk} \dashv \Delta_{\top}$$

where $\Delta_{\top} : L \to L \times L$ is defined by $\Delta_{\top}(\varphi) := (\top, \top)$ and $\Delta_{\bot} : L \to L \times L$ by $\Delta_{\bot}(\varphi) := (\bot, \bot)$.

Tonk as a Double Adjoint or Equivocation

Tonk cannot be defined as an adjoint, but can be defined as a “double" adjoint, i.e., being adjoints of two operations at once.

- What’s wrong with tonk?
- The double adjointness of tonk is equivocation.
- The right and left adjoints of $\Delta_{\perp}$ and $\Delta_{\top}$ represent different logical constants (“binary" truth const. and falsity const.).
- Tonk confuses those constants as the same one, thereby causing inconsistency.
  - We can therefore resolve the paradox of tonk by discriminating b/w the right and left adjoints properly.

The categorial analysis thus tells us the problem of tonk is a problem of equivocation.

$\Delta \dashv$ disconjunction $\dashv \Delta$

$\wedge$ is right adjoint to diagonal $\Delta : L \to L \times L$; $\vee$ is left adjoint to Δ.

- Define disconjunction to be the right and left adjoints of Δ at once. It makes the system trivial as tonk does.
- Disconjunction confuses “and" and “or" as the same const.

What's the (intensional) difference b/w tonk and disconjunct.?

- Tonk is a right adj. of some F, and is a left adj. of some G. Disconjunction is a “uniformly" double adjoint: it is the right and left adjoint of the same one $F = G$.

Thus, disconjunction is more paradoxical than tonk, since uniform double adjointness is stronger than double adjointness.

Paradox $\dashv$ Paradox $\dashv$ Paradox

Let us think of a paradoxical nullary connective R:

$$\frac{\vdash \neg R}{\vdash R}$$

Reformulate this as follows:

$$\frac{R \vdash}{\vdash R}$$

Think of R as a unary constant connective $\tilde{R} : L \to L$ with $\tilde{R}(\varphi) := R$. Then, the rule above means R is right and left adjoint to R. Paradox is thus a self-adjoint functor.

Three Degrees of Paradoxity

We have finally led to three degrees of paradoxity:

	right adjoint to	left adjoint to
Genuine Paradox	itself	itself
Disconjunction	diagonal Δ	diagonal Δ
Tonk	true diagonal $\Delta_\top$	false diagonal $\Delta_\bot$

- The last two are caused by equivocation, and can be made innocuous by discriminating right and left adjonits properly.
- Genuine Paradox is not so, since self-adjoitness can be given by a single adjunction: if a functor is right (resp. left) adjoint to itself, it is left (resp. right) adjoint to itself.

Note: properties like double or self adjointness do not make paradoxes; they make distinction between already paradoxical constants.

Different levels of inconsistency

Inconsistency on (plain) consequence $\vdash$:

- $\varphi \vdash \psi$ for any φ and ψ.
  - Nothing to do with identity of proofs.

Inconsistency on proof-theoretic consequence $\vdash_p$:

- $\varphi \vdash_p \psi$ and $\psi \vdash_q \varphi$ such that the sequential combinations of proofs p and q equal identities.
  - p and q are mutually inverse; φ and ψ are isomorphic; objects collapse.
  - Logical equivalence $\leftrightarrow$ on formulas is different from proof-theoretic isomorphism witnessed by identity of proofs.

Inconsistency on identity of proofs:

- Any two proofs of the same type are equal.
  - Categories collapse into preorders. Arrows collapse.

How inconsistent tonk is?

In terms of consequence relation $\vdash$:

- Tonk is surely inconsistent.

In terms of proof-theoretic consequence $\vdash_p$:

- Tonk dose not cause inconsistency.
  - Tonk (disconj., g.p.) can consistently be modeled in cats with biproducts $\oplus$ incl. **Hilb** and **Rel**.
  - This inconsis. only arises by requiring logical equivalences $\varphi \vdash_p \psi$ and $\psi \vdash_q \varphi$ to give isomorphisms. Tonk with such a property is inconsistent.

In terms of identity of proofs:

- Tonk is not inconsistent. In conjunction with the above, tonk perfectly makes sense categorially or type-theoretically.
  - Joyal's lemma typically exhibits this sort of inconsistency.

Inconsistency in terms of PT consequence $\vdash_p$

An example of inconsistency in terms of proof-theoretic consequence $\vdash_p$:

- Toposes with the fixpoint property trivialise: if any endomorphsim $f : C \to C$ in a topos has a fixpoint, all objects in the topos are isomorphic.
- Logically rephrasing, intuitionistic HOL with the fixpoint property trivialises: the structure of types collapses.

Moral: toposes do not make so much sense in domain theory.

Outline

What is Joyal Lem conceptually?

Joyal Lem tells us:

- incompatibility b/w computational content and classic. log.;
- more specifically, collapsing of semantics of proofs for classical logic.
  - The identity of proofs becomes trivial.
    Semantics becomes proof-irrelevant.

We aim at articulating the nature of Joyal-type paradoxes, rather than making them vanish. Some have attempted to overcome them, though.

What is Joyal Lem technically?

Let **C** be a cartesian closed category, which represents intuitionistic logic (without $\vee$).

- **C** is said to have a dualising object iff there is $\bot$ in **C** such that the canonical arrow $d : C \to \bot^{\bot^C}$ is iso. for $\forall C$ in **C**.
    - d is obtained by currying $ev : \bot^C \times C \to \bot$.
    - Having $\bot$ is having DNE (double negation elim.).

Lemma (Joyal Lem)

If **C** *has a dualising object, then* **C** *is a Bool. alg. up to equiv.: namely, the structure of arrows ("proofs") collapses.*

Note: classical logic = CCC with a dualising object. Any two arrows in a CCC with a dualising object are actually equal. Interpretation of proofs in classical logic becomes trivial.

Substructural Joyal Lemma

We discuss Joyal Lem in a monoidal setting, i.e., in the context of substructural logic.

- Some claim both diagonals (contraction) and projections (weakening) are essential when proving Joyal Lem.
- In our analysis, however, a diagonal is only needed for one specific object $\bot$, and not for any other object.

Basically, $*$-autonomus cat. $=$ classical linear logic.

Lemma (Substructural Joyal Lem)

Any $*$-autonomous cat. $(\mathbf{C}, \otimes, I, \bot)$ with I being terminal and $\bot \otimes \bot \simeq \bot \times \bot$ collapses into an ordered monoid up to equiv.

"I being terminal" (or $1 = \top$) means (full) weakening.
$\bot \otimes \bot \simeq \bot \times \bot$ means $\otimes$ and $\times$ (or $*$ and $\wedge$) coincide for $\bot$ only.

Quantum No-Deleting Thm

Subst. Joyal Lem. yields a unification of Joyal Lem. in logic and No-Deleting Thm. in quantum physics.

- †-compact cats. give a categorial framework for quantum mechanics and computation; “compact" implies “$*$-auto".
  - Abramsky-Coecke’s Categorical Quantum Mechanics.
- The assumption $\bot \otimes \bot \simeq \bot \times \bot$ is not needed in †-compact categories. This special case in †-compact categories was independently known as No-Deleting Thm in CQM.
- “I being terminal" means the (uniform) existence of projections $p : H \times H \to H$, deleting of quantum states.
- Subst. Joyal Lem. tells us the presence of deleting in quantum cats. leads to inconsistency.

Conclusions

What is and is not wrong with tonk:

- Tonk is not an adjoint, but a double adjoint (equivocation).
    - Disconjunction is uniformly so. Paradox is self-adjoint.
- Tonk is inconsistent in terms of $\vdash$, but not at all in terms of proof-theoretic consequence $\vdash_p$ or identity of proofs.
    - Having $\varphi \vdash \psi$ and $\psi \vdash \varphi$ for any φ, ψ is not contradictory in categories or proof-theoretic consequences.

Classical logic trivialises the identity of proofs (Joyal Lem).

- Substructural Joyal Lem tells us contraction is only needed for $\bot$; weakening is fully needed.
- It is a unification of Joyal Lem and No-Deleting Thm.

“From structured to substructural logic" $=$ “from cats. to monoidal cats" $=$ “from classical to quantum physics".