Full Lambek Hyperdoctrine: Categorical Semantics for First-Order Substructural Logics

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Abstract. We pursue the idea that predicate logic is a "fibred algebra" while propositional logic is a single algebra; in the context of intuitionism, this algebraic understanding of predicate logic goes back to Lawvere, in particular his concept of hyperdoctrine. Here, we aim at demonstrating that the notion of monad-relativised hyperdoctrines, which are what we call fibred algebras, yields algebraisations of a wide variety of predicate logics. More specifically, we discuss a typed, first-order version of the non-commutative Full Lambek calculus, which has extensively been studied in the past few decades, functioning as a unifying language for different sorts of logical systems (classical, intuitionistic, linear, fuzzy, relevant, etc.). Through the concept of Full Lambek hyperdoctrines, we establish both generic and set-theoretical completeness results for any extension of the base system; the latter arises from a dual adjunction, and is relevant to the tripos-to-topos construction and quantale-valued sets. Furthermore, we give a hyperdoctrinal account of Girard's and Gödel's translation.

1 Introduction

Categorical logic deconstructs the traditional dichotomy between proof theory and model theory, in the sense that both of them can be represented in certain syntactic and set-theoretical categories (or hyperdoctrines in this paper) respectively. We may thus say that categorical semantics does encompass both proof-theoretic and model-theoretic semantics in terms of philosophy of logic.

Categorical semantics divides into two sub-disciplines: semantics of provability (e.g., semantics via toposes or logoses) and semantics of proofs (e.g., semantics via CCC or monoidal CC). Our focus shall be upon the former wrt. logic and the latter wrt. type theory because we aim at developing categorical semantics for a broad range of logics over type theories, including classical, intuitionistic, linear, and fuzzy logics. Type theories have inherent identities of proofs (or terms), and fully admit semantics of proofs, however, logics in general do not allow semantics of proofs, due to collapsing phenomena on their identities of proofs (for the case of classical logic, refer to the Joyal lemma, e.g., in Lambek-Scott [11]).

Thus, the Curry-Howard paradigm does not make so much sense in this general context of logics over type theories, for the logics of the latter (types) may

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differ from the former original logics (propositions), just as Abramsky-Coecke's type theory of quantum mechanics is distinct from Birkhoff-von Neumann's logic of it. In general, we thus need to treat logic and type theory separately, and the concept of fibred universal algebras does the job, as elucidated below. Aczel's idea of logic-enriched type theory is along a similar line. Fibred algebras to represent logics over monoidal type theories even allow us to reconcile Birkhoff-von Neumann's cartesian logic of quantum propositions and Abramsky-Coecke's monoidal logic (or type theory) of quantum systems; this is future work, however.

Substructural logics over the Full Lambek calculus (FL for short), which encompass a wide variety of logical systems (classical, intuitionistic, linear, fuzzy, relevant, etc.), have extensively been investigated in the past few decades, especially by algebraic logicians in relation to residuated lattices (a major reference is Galatos-Jipsen-Kowalski-Ono [3]). Although some efforts have been made towards the algebraic treatment of logics over quantified FL (see, e.g., Ono [14, 15] and references therein), however, it seems that there has so far been no adequate concept of algebraic models of them. Note that complete residuated lattices can only give complete semantics for those classes of substructural predicate logics for which completions (such as Dedekind-MacNeille's or Crawley's) of Lindenbaum-Tarski algebras work adequately (see, e.g., Ono [14, 15]); for this reason, complete residuated lattices (or quantales) cannot serve the purpose.

In the context as articulated above, we propose fibred algebras as algebraic models of predicate logic, especially substructural logics over quantified FL. Fibred algebras expand Lawvere's concept of hyperdoctrine [12]. According to Pitts' formulation [16], a hyperdoctrine is a functor (presheaf) $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ where **HA** is the category of heyting algebras; there are additional conditions on P (and $\mathbf{C}$) to express quantifiers and other logical concepts (for a fibrational formulation of hyperdoctrine, see Jacobs [8]; the two formulations are equivalent via the Grothendieck construction).[1] We may see a hyperdoctrine as a fibred heyting algebra $(P(C))_{C\in\mathbf{C}}$, a bunch of algebras indexed by $\mathbf{C}$.

Now, a fibred algebra is a universal algebra indexed by a category: categorically, it is a functor (presheaf) $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ (apart from logical conditions to express quantifiers and others) where T is a monad on **Set**, and $\mathbf{Alg}(T)$ is its Eilenberg-Moore algebras; note that monads on **Set** are equivalent to (possibly infinitary) varieties in terms of universal algebra (see, e.g., Adámek et al. [1]). The intuitive meaning of the base category $\mathbf{C}$ is the category of types (aka. sorts) or domains of discourse, and then $P(C)$ is the algebra of predicates on a type C. If a propositional logic L is sound and complete wrt. a variety $\mathbf{Alg}(T)$, then the corresponding fibred algebras $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$ yield sound and complete semantics for the predicate logic that extends L. This may be called the thesis of completeness lifting: the completeness of propositional logic wrt. $\mathbf{Alg}(T)$ lifts to the completeness of predicate logic wrt. $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}(T)$.

The present paper is meant to demonstrate the thesis in the fairly general context of substructural logics over FL, hopefully bridging between algebraic

[1] Toposes amount to higher-order hyperdoctrines via the two functors of taking subobject hyperdoctrines and of the tripos-to-topos construction (see, e.g., Frey [4]).

logic, in which logics over FL have been studied, and categorical logic, in which hyperdoctrines have been pursued. Although the two disciplines are currently separated to the author's eyes, nevertheless, Lawvere's original ideas on categorical logic are of algebraic nature (especially, his functorial semantics directly targets universal algebra), and it would be fruitful to restore lost interactions between them. Along this line of thought, in subsequent work, the author plans to develop "Categorical Universal Logic" qua theory of fibred universal algebras or monad-relativised hyperdoctrines. The paper takes a first step towards it.

Main technical contributions are three-fold as follows: (i) generic completeness wrt. Full Lambek hyperdoctrines, which are hyperdoctrines for logics over FL; (ii) Tarskian completeness wrt. **Set**-based models, which arise from a dual adjunction, with more specialised completeness results; (iii) hyperdoctrinal formulations of Girard's translation and Gödel's translation. The generic and set-theoretical completeness results are established for any axiomatic extension of FL, therefore covering a great majority of standard logical systems (classical, intuitionistic, linear, fuzzy, relevant, and so on). In passing, we briefly discuss the tripos-to-topos construction in the present context, which is originally due to Hyland-Johnstone-Pitts [7]. Our uniform categorical semantics for various logical systems enables us to compare different categorical logics on the one setting, and (iii) indeed embodies such a comparison in terms of logical translation. The paper proceeds in the same order as above, after an introduction to typed FL.

2 Typed Full Lambek Calculus

In this section, we define a typed (or many-sorted) version of quantified FL as in Ono [15], which shall be called TFL^q ("T" means "typed"; "q" means "quantified"). In particular, TFL^q follows the typing style of Pitts [16].

Standard categorical logic discusses a typed version of intuitionistic (or coherent or regular) logic, as observed in Pitts [16], Lambek-Scott [11], Jacobs [8], and Johnstone [10]. Typed logic is more natural than single-sorted one from a categorical point of view, and is more expressive in general, since it can encompass various type constructors. If one prefers single-sorted logic to typed logic, the latter can be reduced to the former by allowing for one type (or sort) only.

To put it differently, typed logic is the combination of logic and type theory, and has not only a logic structure but also a type structure, and the latter itself has a rich structure as well as the former. For this reason, syntactic hyperdoctrines constructed from typed systems of logic (which are discussed in relation to completeness in the next section) are amalgamations of syntactic categories obtained from their type theories on the one hand, and Lindenbaum-Tarski algebras obtained from their logic parts on the other; in a nutshell, syntactic hyperdoctrines are type-fibred Lindenbaum-Tarski algebras.

Another merit of typed logic is that the problem of empty domains is resolved because it allows us to have explicit control on type contexts. This was discovered by Joyal, and shall be touched upon later, in more detail.

TFL^q has the following logical connectives:

$$\otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot, \forall, \exists.$$

Note that there are two kinds of implication connectives $\backslash$ and $/$, owing to the non-commutative nature of TFL^q.

In TFL^q, every variable x comes with its type σ. That is, TFL^q has basic types, which are denoted by letters like σ, τ, and $x : \sigma$ is a formal expression meaning that a variable x is of type σ. Then, a (type) context is a finite list of type declarations on variables: $x_1 : \sigma_1, ..., x_n : \sigma_n$. A context is often denoted Γ.

Accordingly, TFL^q has typed predicate symbols (aka. predicates in context) and typed function symbols (aka. function symbols in context): $R(x_1, ..., x_n)\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$ is a formal expression meaning that R is a predicate with n variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ respectively; likewise, $f : \tau\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$ is a formal expression meaning that f is a function symbol with n variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ and with its values in τ. Then, formulae-in-context $\varphi\ [\Gamma]$ and terms-in-context $t : \tau\ [\Gamma]$ are defined in the usual, inductive way. Our terminology is basically following Pitts [16].

In the present paper, we do not consider any specific type constructor. Higher-Order Full Lambek Calculus shall be discussed in a subsequent paper, and have products and function spaces as type constructors. In this paper, however, we shall focus upon plainly typed predicate logic with no complicated type structure; still, products (not as types but as categorical structures) shall be used in categorical semantics in the next section, to the end of interpreting predicate and function symbols (of arity greater than one).

TFL^q thus has both a type structure and a logic structure, dealing with sequents-in-contexts: $\Phi \vdash \varphi\ [\Gamma]$ where Γ is a type context, and Φ is a finite list of formulae: $\varphi_1, ..., \varphi_n$. Although it is common to write $\Gamma \mid \Phi \vdash \varphi$ rather than $\Phi \vdash \varphi\ [\Gamma]$, we employ the latter notation in this paper, following Pitts [16], since TFL^q is an adaptation of Pitts' typed system for intuitionistic logic to the system of the Full Lambek calculus.

The syntax of type contexts Γ in TFL^q is the same as that of typed intuitionistic logic in Pitts [16]; due to space limitations, we do not repeat it here, referring to Pitts [16] for details. Yet we note it is allowed to add a fresh $x : \sigma$ to a context Γ: e.g., $\Phi \vdash \varphi\ [\Gamma, x : \sigma]$ whenever $\Phi \vdash \varphi\ [\Gamma]$. On the other hand, it is not permitted to delete redundant variables; the reason becomes clear in later discussion on empty domains. It is allowed to change the order of contexts (e.g., $[\Gamma, \Gamma']$ into $[\Gamma', \Gamma]$). In the below, we focus upon logical rules of inference, which are most relevant part of TFL^q in the paper, being of central importance for us.

TFL^q has no structural rule other than the following cut rule

$$\frac{\Phi_1 \vdash \varphi\ [\Gamma] \quad \Phi_2, \varphi, \Phi_3 \vdash \psi\ [\Gamma]}{\Phi_2, \Phi_1, \Phi_3 \vdash \psi\ [\Gamma]}\ (cut)$$

where ψ may be empty; this is allowed in the following L (left) rules as well. As usual, we have the rule of identity

$$\frac{}{\varphi \vdash \varphi\ [\Gamma]}\ (id)$$

In the following, we list the rules of inference for the logical connectives of TFL^q. There are two kinds of conjunction in TFL^q: multiplicative or monoidal $\otimes$ and additive or cartesian $\wedge$:

$$\frac{\Phi, \varphi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \otimes \psi, \Psi \vdash \chi\ [\Gamma]}\ (\otimes L) \qquad \frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi \vdash \psi\ [\Gamma]}{\Phi, \Psi \vdash \varphi \otimes \psi\ [\Gamma]}\ (\otimes R)$$

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \wedge \psi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L_1) \qquad \frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \psi \wedge \varphi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L_2)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi \wedge \psi\ [\Gamma]}\ (\wedge R)$$

There is only one disjunction in TFL^q, which is additive, since TFL^q is intuitionistic in the sense that only one formula is allowed to appear on the right-hand side of sequents. Nevertheles, we can treat classical logic as an axiomatic extension of TFL^q, by adding to TFL^q exchange, weakening, contraction, and the excluded middle; note that structural rules can be expressed as axioms.

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma] \quad \Phi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \vee \psi, \Psi \vdash \chi\ [\Gamma]}\ (\vee L)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma]}{\Phi \vdash \varphi \vee \psi\ [\Gamma]}\ (\vee R_1) \qquad \frac{\Phi \vdash \varphi\ [\Gamma]}{\Phi \vdash \psi \vee \varphi\ [\Gamma]}\ (\vee R_2)$$

Due to non-commutativity, there are two kinds of implication in TFL^q, $\backslash$ and $/$, which are a right adjoint of $\varphi \otimes (\text{-})$ and a right adjoint of $(\text{-}) \otimes \psi$ respectively.

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi\ [\Gamma]}{\Psi_1, \Phi, \varphi\backslash\psi, \Psi_2 \vdash \chi\ [\Gamma]}\ (\backslash L) \qquad \frac{\varphi, \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi\backslash\psi\ [\Gamma]}\ (\backslash R)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi\ [\Gamma]}{\Psi_1, \psi/\varphi, \Phi, \Psi_2 \vdash \chi\ [\Gamma]}\ (/L) \qquad \frac{\Phi, \varphi \vdash \psi\ [\Gamma]}{\Phi \vdash \psi/\varphi\ [\Gamma]}\ (/R)$$

There are two kinds of truth and falsity constants, monoidal and cartesian ones.

$$\frac{\Psi_1, \Psi_2 \vdash \varphi\ [\Gamma]}{\Psi_1, 1, \Psi_2 \vdash \varphi\ [\Gamma]}\ (1L) \qquad \frac{}{\vdash 1\ [\Gamma]}\ (1R)$$

$$\frac{}{0 \vdash\ [\Gamma]}\ (0L) \qquad \frac{\Phi \vdash\ [\Gamma]}{\Phi \vdash 0\ [\Gamma]}\ (0R)$$

$$\frac{}{\Phi \vdash \top\ [\Gamma]}\ (\top R) \qquad \frac{}{\Phi_1, \bot, \Phi_2 \vdash \varphi\ [\Gamma]}\ (\bot L)$$

Finally, we have the following rules for quantifiers $\forall$ and $\exists$, in which type contexts explicitly change; notice that type contexts do not change in the rest of the rules presented above.

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}{\Phi_1, \forall x\varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}\ (\forall L) \qquad \frac{\Phi \vdash \varphi\ [x : \sigma, \Gamma]}{\Phi \vdash \forall x\varphi\ [\Gamma]}\ (\forall R)$$

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi \ [x : \sigma, \Gamma]}{\Phi_1, \exists x\varphi, \Phi_2 \vdash \psi \ [\Gamma]} \ (\exists L) \qquad \frac{\Phi \vdash \varphi \ [x : \sigma, \Gamma]}{\Phi \vdash \exists x\varphi \ [x : \sigma, \Gamma]} \ (\exists R)$$

As usual, there are eigenvariable conditions on the rules above: x does not appear as a free variable in the bottom sequent of Rule $\forall R$; likewise, x does not appear as a free variable in the bottom sequent of Rule $\exists L$. The other two rules do not have eigenvariable conditions, and this is why contexts do not change in them.

The deducibility of sequents-in-context in TFL^q is defined in the usual way. In this paper, we denote by FL the propositional (and hence no contextual) part of TFL^q. Note that what is called FL in the literature often lacks $\bot$ and $\top$.

As is well known, the following propositional (resp. predicate) logics can be represented as axiomatic (to be precise, axiom-schematic) extensions of FL (resp. TFL^q): classical logic, intuitionistic logic, linear logic (without exponentials), relevance logics, fuzzy logics such as Gödel-Dummett logic (see, e.g., Galatos et al. [3]). Given a set of axioms (to be precise, axiom schemata), say X, we denote by FL_X (resp. TFL^q_X) the corresponding extension of FL (resp. TFL^q) via X.

Lemma 1 *The following sequents-in-context are deducible in* TFL^q*:*

- $\varphi \otimes (\exists x\psi) \vdash \exists x(\varphi \otimes \psi) \ [\Gamma]$ *and* $\exists x(\varphi \otimes \psi) \vdash \varphi \otimes (\exists x\psi) \ [\Gamma]$.
- $(\exists x\psi) \otimes \varphi \vdash \exists x(\psi \otimes \varphi) \ [\Gamma]$ *and* $\exists x(\psi \otimes \varphi) \vdash (\exists x\psi) \otimes \varphi \ [\Gamma]$.

where it is supposed that φ *does not contain* x *as a free variable, and* Γ *contains type declarations on those free variables that appear in* φ *and* $\exists x\psi$*.*

A striking feature of typed predicate logic is that domains of discourse in semantics can be empty; they are assumed to be non-empty in the usual Tarski semantics of predicate logic. This means that a type σ can be interpreted as an initial object in a category. We therefore need no ad hoc condition on domains of discourse if we work with typed predicate logic. This resolution of the problem of empty domains is due to Joyal as noted in Marquis and Reyes [13].

A proof-theoretic manifestation of this feature is that the following sequent-in-context is not necessarily deducible in TFL^q: $\forall x\varphi \vdash \exists x\varphi \ [\ ]$ where the context is empty. Nonetheless, the following is deducible in TFL^q: $\forall x\varphi \vdash \exists x\varphi \ [x : \sigma, \Gamma]$ where Γ is an appropriate context including type declarations on free variables in φ. This means that we can prove the sequent above when a type σ is inhabited. Here, it is crucial that it is not allowed to delete redundant free variables (e.g., $[x : \sigma, \Gamma]$ cannot be reduced into $[\Gamma]$ even if x does not appear as a free variable in formulae involved); however, it is allowed to add fresh free variables to a context.

3 Full Lambek Hyperdoctrine

It is well known that FL algebras (defined below) provide sound and complete semantics for propositional logic FL (see, e.g., Galatos et al. [3]). In this section we show that fibred FL algebras, or FL hyperdoctrines (defined below), yield sound and complete semantics for typed (or many-sorted) predicate logic TFL^q.

We again emphasise the simple, algebro-logical idea that single algebras (symbolically, A with no indexing) correspond to propositional logic, and fibred algebras (symbolically, $(A_C)_{C \in \mathbf{C}}$ indexed by a category $\mathbf{C}$) correspond to predicate logic. As universal algebra gives foundations for algebraic propositional logic, so fibred universal algebra lays foundations for algebraic predicate logic.

Definition 2 ([3]) $(A, \otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot)$ *is called an* FL *algebra iff*

- $(A, \otimes, 1)$ *is a monoid;* 0 *is a (distinguished) element of* A*;*
- $(A, \wedge, \vee, \top, \bot)$ *is a bounded lattice, which induces a partial order* $\leq$ *on* A*;*
- *for any* $a \in A$*,* $a\backslash(\text{-}) : A \to A$ *is a right adjoint of* $a \otimes (\text{-}) : A \to A$*: i.e.,* $a \otimes b \leq c$ *iff* $b \leq a \backslash c$ *for any* $a, b, c \in A$*;*
- *for any* $b \in A$*,* $(\text{-})/b : A \to A$ *is a right adjoint of* $(\text{-}) \otimes b : A \to A$*: i.e.,* $a \otimes b \leq c$ *iff* $a \leq c/b$ *for any* $a, b, c \in A$*.*

A homomorphism of FL *algebras is defined as a map preserving all the operations* $(\otimes, \wedge, \vee, \backslash, /, 1, 0, \top, \bot)$*.* **FL** *denotes the category of* FL *algebras and their homomorphisms.*

Although 0 is just a neutral element of A with no axiom, the rules for 0 are automatically valid by the definition of interpretations defined below.

FL is an algebraic category (i.e., a category monadic over **Set**), or a variety in terms of universal algebra, since the two adjointness conditions can be rephrased by equations (see, e.g., Galatos et al. [3]). An axiomatic extension FL_X of FL corresponds to an algebraic full subcategory (or sub-variety) of **FL**, denoted $\mathbf{FL}_X$ (algebraicity follows from definability by axioms); this is the well-known, logic-variety correspondence for logics over FL (see Galatos et al. [3]).

Definition 3 *An* FL *(Full Lambek) hyperdoctrine is an* **FL***-valued presheaf* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *such that* $\mathbf{C}$ *is a category with finite products, and the following conditions on quantifiers hold:*

- *For any projection* $\pi : X \times Y \to Y$ *in* $\mathbf{C}$*,* $P(\pi) : P(Y) \to P(X \times Y)$ *has a right adjoint, denoted*

$$\forall_\pi : P(X \times Y) \to P(Y).$$

And the corresponding Beck-Chevalley condition holds, i.e., the following diagram commutes for any arrow $f : Z \to Y$ *in* $\mathbf{C}$ *(*$\pi' : X \times Z \to Z$ *below denotes a projection):*

$$\begin{array}{ccc} P(X \times Y) & \xrightarrow{\forall_\pi} & P(Y) \\ {\scriptstyle P(X \times f)} \downarrow & & \downarrow {\scriptstyle P(f)} \\ P(X \times Z) & \xrightarrow[\forall_{\pi'}]{} & P(Z) \end{array}$$

– *For any projection* $\pi : X \times Y \to Y$ *in* $\mathbf{C}$*,* $P(\pi) : P(Y) \to P(X \times Y)$ *has a left adjoint, denoted*
$$\exists_\pi : P(X \times Y) \to P(Y).$$
The corresponding Beck-Chevalley condition holds:
$$\begin{array}{ccc} P(X \times Y) & \xrightarrow{\exists_\pi} & P(Y) \\ {\scriptstyle P(X \times f)} \big\downarrow & & \big\downarrow {\scriptstyle P(f)} \\ P(X \times Z) & \xrightarrow[\exists_{\pi'}]{} & P(Z) \end{array}$$
Furthermore, the Frobenius Reciprocity conditions hold: for any projection $\pi : X \times Y \to Y$ *in* $\mathbf{C}$*, any* $a \in P(Y)$*, and any* $b \in P(X \times Y)$*,*
$$a \otimes (\exists_\pi b) = \exists_\pi (P(\pi)(a) \otimes b)$$
$$(\exists_\pi b) \otimes a = \exists_\pi (b \otimes P(\pi)(a)).$$

For an axiomatic extension FL_X *of* FL*, an* FL_X *hyperdoctrine is defined by restricting the value category* $\mathbf{FL}$ *into* $\mathbf{FL}_X$*. An* FL *(resp.* FL_X*) hyperdoctrine is also called a fibred* FL *(resp.* FL_X*) algebra.*

The category $\mathbf{C}$ of an FL hyperdoctrine $P : \mathbf{C} \to \mathbf{FL}$ is called its base category or type category, and P is also called its predicate functor; intuitively, $P(C)$ is the algebra of predicates on a type, or domain of discourse, C.

Note that, in the definition above, we need two Frobenius Reciprocity conditions due to the non-commutativity of FL algebras.

An FL hyperdoctrine may be seen as an indexed category, and so as a fibration via the Grothendieck construction. Although we discuss in terms of indexed categories in this paper, we can do the job in terms of fibrations as well. In the view of fibrations, each $P(C)$ is called a fibre of an FL hyperdoctrine P.

The FL (resp. FL_X) hyperdoctrine semantics for TFL^q (resp. TFL^q_X) is defined as follows.

Definition 4 *Fix an* FL *hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$*. An interpretation* $[\![\text{-}]\!]$ *of* TFL^q *in the* FL *hyperdoctrine* P *consists of the following:*

– *assignment of an object* $[\![\sigma]\!]$ *in* $\mathbf{C}$ *to each basic type* σ *in* TFL^q*;*
– *assignment of an arrow* $[\![f : \tau\ [\Gamma]]\!] : [\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!] \to [\![\sigma]\!]$ *in* $\mathbf{C}$ *to each typed function symbol* $f : \tau\ [\Gamma]$ *in* TFL^q *where* Γ *is supposed to be* $x_1 : \sigma_1, ..., x_n : \sigma_n$ *(note that* $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$ *makes sense because* $\mathbf{C}$ *has finite products);*
– *assignment of an element* $[\![R\ [\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*, which is an* FL *algebra, to each typed predicate symbol* $R\ [\Gamma]$ *in* TFL^q*; if the context* Γ *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*, then* $[\![\Gamma]\!]$ *denotes* $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$*.*

Then, terms are inductively interpreted in the following way:

– $[\![x : \sigma\ [\Gamma_1, x : \sigma, \Gamma_2]]\!]$ *is defined as the following projection in* $\mathbf{C}$*:*
$$\pi : [\![\Gamma_1]\!] \times [\![\sigma]\!] \times [\![\Gamma_2]\!] \to [\![\sigma]\!].$$

– $[\![f(t_1,...,t_n):\tau\ [\Gamma]]\!]$ *is defined as:*

$$[\![f]\!]\circ\langle[\![t_1:\sigma_1\ [\Gamma]]\!],...,[\![t_n:\sigma_n\ [\Gamma]]\!]\rangle$$

where it is supposed that $f:\tau\ [x_1:\sigma_1,...,x_n:\sigma_n]$*, and* $t_1:\sigma_1\ [\Gamma],...,t_n:\sigma_n\ [\Gamma]$*. Note that* $\langle[\![t_1:\sigma_1\ [\Gamma]]\!],...,[\![t_n:\sigma_n\ [\Gamma]]\!]\rangle$ *above is the product (or pairing) of arrows in* $\mathbf{C}$*.*

Formuli are then interpreted inductively in the following manner:

– $[\![R(t_1,...,t_n)\ [\Gamma]]\!]$ *is defined as*

$$P(\langle[\![t_1:\sigma_1[\Gamma]]\!],...,[\![t_n:\sigma_n[\Gamma]]\!]\rangle)([\![R\ [x:\sigma_1,...,x_n:\sigma_n]]\!])$$

where R *is a predicate symbol in context* $x_1:\sigma_1,...,x_n:\sigma_n$*.*

– $[\![\varphi\otimes\psi\ [\Gamma]]\!]$ *is defined as* $[\![\varphi\ [\Gamma]]\!]\otimes[\![\psi\ [\Gamma]]\!]$*. The other binary connectives* $\wedge,\vee,\backslash,/$ *are interpreted in the same way.* $[\![1\ [\Gamma]]\!]$ *is defined as the monoidal unit of* $P([\![\Gamma]\!])$*. The other constants* $0,\top,\bot$ *are interpreted in the same way.*

– $[\![\forall x\varphi\ [\Gamma]]\!]$ *is defined as*

$$\forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])$$

where $\pi:[\![\sigma]\!]\times[\![\Gamma]\!]\to[\![\Gamma]\!]$ *is a projection in* $\mathbf{C}$*, and* φ *is a formula in context* $[x:\sigma,\Gamma]$*. Similarly,* $[\![\exists x\varphi\ [\Gamma]]\!]$ *is defined as*

$$\exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]).$$

Finally, satisfaction of sequents is defined:

– $\varphi_1,...,\varphi_n\vdash\psi\ [\Gamma]$ *is satisfied in an interpretation* $[\![\text{-}]\!]$ *in an* FL *hyperdoctrine* P *iff the following holds in* $P([\![\Gamma]\!])$*:*

$$[\![\varphi_1\ [\Gamma]]\!]\otimes...\otimes[\![\varphi_n\ [\Gamma]]\!]\leq[\![\psi\ [\Gamma]]\!].$$

In case the right-hand side of a sequent is empty, $\varphi_1,...,\varphi_n\vdash[\Gamma]$ *is satisfied in* $[\![\text{-}]\!]$ *iff* $[\![\varphi_1\ [\Gamma]]\!]\otimes...\otimes[\![\varphi_n\ [\Gamma]]\!]\leq 0$ *in* $P([\![\Gamma]\!])$*. In case the left-hand side of a sequent is empty,* $\vdash\varphi\ [\Gamma]$ *is satisfied in* $[\![\text{-}]\!]$ *iff* $1\leq[\![\varphi[\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*.*

An interpretation of TFL^q_X *in an* FL_X *hyperdoctrine is defined by replacing* $\mathbf{FL}$ *and* TFL^q *above with* $\mathbf{FL}_X$ *and* TFL^q_X *respectively.*

In the following, we show that the FL (resp. FL_X) hyperdoctrine semantics is sound and complete for TFL^q (resp. TFL^q_X). Let $[\![\Phi\ [\Gamma]]\!]$ denote $[\![\varphi_1\ [\Gamma]]\!]\otimes...\otimes[\![\varphi_n\ [\Gamma]]\!]$ if Φ is $\varphi_1,...,\varphi_n$.

Intuitively, an arrow f in $\mathbf{C}$ is a term, and $P(f)$ is a substitution operation (this is exactly true in syntactic hyperdoctrines defined later); then, the Beck-Chevalley conditions and the functoriality of P tell us that substitution commutes with all the logical operations (namely, both propositional connectives and quantifiers). From such a logical point of view, the meaning of the Beck-Chevalley conditions is crystal clear; they just say that substitution after quantification is the same as quantification after substitution.

Proposition 5 *If $\Phi \vdash \psi\ [\Gamma]$ is deducible in* TFL^q *(resp.* TFL^q_X*), then it is satisfied in any interpretation in any* FL *(resp.* FL_X*) hyperdoctrine.*

Proof. Fix an FL or FL_X hyperdoctrine P and an interpretation $[\![\text{-}]\!]$ in P. Initial sequents in context are satisfied because $a \leq a$ in any fibre $P(C)$. The cut rule preserves satisfaction, since tensoring preserves $\leq$ and $\leq$ has transitivity. It is easy to verify that all the rules for the logical connectives preserve satisfaction.

Let us consider universal quantifier $\forall$. To show the case of Rule $\forall R$, assume that $[\![\Phi\ [x:\sigma,\Gamma]]\!] \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$ in $P([\![\sigma]\!] \times [\![\Gamma]\!])$. It then follows that $[\![\Phi\ [x:\sigma,\Gamma]]\!] = P(\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!])([\![\Phi\ [\Gamma]]\!])$ where π is a projection in $\mathbf{C}$, and note that Φ does not include x among its free variables by the eigenvariable condition. We thus have $P(\pi)([\![\Phi\ [\Gamma]]\!]) \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$. Since $\forall_\pi : P([\![\sigma]\!] \times [\![\Gamma]\!]) \to P([\![\Gamma]\!])$ is a right adjoint of $P(\pi)$, it follows that $[\![\Phi\ [\Gamma]]\!] \leq \forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]) = [\![\forall x\varphi\ [\Gamma]]\!]$. We next show the case of $\forall L$. Assume that $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq [\![\psi\ [x:\sigma,\Gamma]]\!]$. The adjunction condition for universal quantifier gives us $P(\pi)(\forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])) \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$ where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. Yet we also have $P(\pi)(\forall_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])) = P(\pi)([\![\forall x\varphi\ [\Gamma]]\!]) = [\![\forall x\varphi\ [x:\sigma,\Gamma]]\!]$. Since tensoring respects $\leq$, these together imply that $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\forall x\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq [\![\psi\ [x:\sigma,\Gamma]]\!]$.

It remains to show the case of existential quantifier $\exists$. In order to prove that Rule $\exists L$ preserves satisfaction, assume that $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq [\![\psi\ [x:\sigma,\Gamma]]\!]$. This is equivalent to the following: $[\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!] \leq P(\pi)([\![\psi\ [\Gamma]]\!])$ where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. Since $\exists_\pi : P([\![\sigma]\!] \times [\![\Gamma]\!]) \to P([\![\Gamma]\!])$ is left adjoint to $P(\pi)$, it follows that $\exists_\pi([\![\Phi_1\ [x:\sigma,\Gamma]]\!] \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes [\![\Phi_2\ [x:\sigma,\Gamma]]\!]) \leq [\![\psi\ [\Gamma]]\!]$. This is equivalent to the following: $\exists_\pi(\ P(\pi)([\![\Phi_1\ [\Gamma]]\!]) \otimes [\![\varphi\ [x:\sigma,\Gamma]]\!] \otimes P(\pi)([\![\Phi_2\ [\Gamma]]\!])\ ) \leq [\![\psi\ [\Gamma]]\!]$. Repeated applications of the two Frobenius Reciprocity conditions give us $[\![\Phi_1\ [\Gamma]]\!] \otimes \exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]) \otimes [\![\Phi_2\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!]$. Then we finally have the following: $[\![\Phi_1\ [\Gamma]]\!] \otimes [\![\exists x\varphi\ [\Gamma]]\!] \otimes [\![\Phi_2\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!]$. To show the case of $\exists R$, assume that $[\![\Phi\ [x:\sigma,\Gamma]]\!] \leq [\![\varphi\ [x:\sigma,\Gamma]]\!]$. The adjunction condition for existential quantifier tells us that $[\![\varphi\ [x:\sigma,\Gamma]]\!] \leq P(\pi)(\exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!]))$ where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. We thus have the following: $[\![\Phi\ [x:\sigma,\Gamma]]\!] \leq P(\pi)(\exists_\pi([\![\varphi\ [x:\sigma,\Gamma]]\!])) = [\![\exists x\varphi\ [x:\sigma,\Gamma]]\!]$. This completes the proof.

Syntactic hyperdoctrines are then defined as follows towards the goal of proving completeness. They are the categorificaton of Lindenbaum-Tarski algebras.

Definition 6 *The syntactic hyperdoctrine of* TFL^q *is defined as follows; that of* TFL^q_X *is defined by replacing* $\mathbf{FL}$ *and* TFL^q *below with* $\mathbf{FL}_X$ *and* TFL^q_X*.*

We first define the base category $\mathbf{C}$*. An object in* $\mathbf{C}$ *is a context* Γ *up to* α*-equivalence (i.e., the naming of variables does not matter). An arrow in* $\mathbf{C}$ *from an object* Γ *to another* Γ' *is a list of terms* $[t_1, ..., t_n]$ *(up to equivalence) such that* $t_1 : \sigma_1\ [\Gamma], ..., t_n : \sigma_n\ [\Gamma]$ *where* Γ' *is supposed to be* $x_1 : \sigma_1, ..., x_n : \sigma_n$*.*

The syntactic hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *is then defined in the following way. For an object* Γ *in* $\mathbf{C}$*, let* $\mathrm{Form}_\Gamma = \{\varphi \mid \varphi \text{ is a formula in context } \Gamma\}$*. Define an equivalence relation* $\sim$ *on* Form_Γ *as follows: for* $\varphi, \psi \in \mathrm{Form}_\Gamma$*,* $\varphi \sim \psi$

iff both $\varphi \vdash \psi$ $[\Gamma]$ *and* $\psi \vdash \varphi$ $[\Gamma]$ *are deducible in* TFL^q. *We then define*

$$P(\Gamma) = \mathrm{Form}_\Gamma / \sim$$

with an FL *algebra structure induced by the logical connectives.*

The arrow part of P *is defined as follows. Let* $[t_1, ..., t_n] : \Gamma \to \Gamma'$ *be an arrow in* $\mathbf{C}$ *where* Γ' *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$. *Then we define* $P([t_1, ..., t_n]) : P(\Gamma') \to P(\Gamma)$ *by*

$$P([t_1, ..., t_n])(\varphi) = \varphi[t_1/x_1, ..., t_n/x_n]$$

where it is supposed that $t_1 : \sigma_1$ $[\Gamma], ..., t_n : \sigma_n$ $[\Gamma]$, *and that* φ *is a formula in context* $x_1 : \sigma_1, ..., x_n : \sigma_n$.

Intuitively, $P(\Gamma)$ above is a Lindenbaum-Tarski algebra sliced with respect to each Γ. It is straightforward to verify that the operations of $P(\Gamma)$ above are well defined, and $P(\Gamma)$ forms an FL algebra. We still have to check that P defined above is a hyperdoctrine; this is done in the following lemma.

Lemma 7 *The syntactic hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$ *(resp.* $\mathbf{FL}_X$*) is an* FL *(resp.* FL_X*) hyperdoctrine. In particular, it has quantifier structures satisfying the Beck-Chevalley and Frobenius Reciprocity conditions.*

Proof. Since substitution commutes with all the logical connectives, $P([t_1, ..., t_n])$ defined above is always a homomorphism of FL algebras. Thus, P is a contravariant functor.

We have to verify that the base category $\mathbf{C}$ has finite products, or equivalently, binary products. For objects Γ, Γ' in $\mathbf{C}$, we define their product $\Gamma \times \Gamma'$ as follows. Suppose that Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, and Γ' is $y_1 : \tau_1, ..., y_m : \tau_m$. Then, $\Gamma \times \Gamma'$ is defined as $x_1 : \sigma_1, ..., x_n : \sigma_n, y_1 : \tau_1, ..., y_m : \tau_m$. An associated projection $\pi : \Gamma \times \Gamma' \to \Gamma'$ is defined as $[y_1, ..., y_m] : \Gamma \times \Gamma' \to \Gamma'$ where the context of each y_i is taken to be $x_1 : \sigma_1, ..., x_n : \sigma_n, y_1 : \tau_1, ..., y_m : \tau_m$ (rather than $y_1 : \tau_1, ..., y_m : \tau_m$). The other projection is defined in a similar way. It is easily verified that these indeed form a categorical product in $\mathbf{C}$.

In order to show that P has quantifier structures, let $\pi : \Gamma \times \Gamma' \to \Gamma'$ denote the projection in $\mathbf{C}$ defined above, and then consider $P(\pi)$, which we have to show has right and left adjoints. The right and left adjoints of $P(\pi)$ can be constructed as follows. Recall Γ is $x : \sigma_1, ..., x_n : \sigma_n$. Let $\varphi \in P(\Gamma \times \Gamma')$; here we are identifying φ with the equivalence class to which φ belongs, since every argument below respects the equivalence. Then define $\forall_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by $\forall_\pi(\varphi) = \forall x_1 ... \forall x_n \varphi$ where the formula on the right-hand side actually denotes the corresponding equivalence class. Similarly, we define $\exists_\pi : P(\Gamma \times \Gamma') \to P(\Gamma')$ by $\exists_\pi(\varphi) = \exists x_1 ... \exists x_n \varphi$. Let us show that $\forall_\pi$ is the right adjoint of $P(\pi)$. We first assume $P(\pi)(\psi) \leq \varphi$ in $P(\Gamma \times \Gamma')$ for $\psi \in P(\Gamma')$ and $\varphi \in P(\Gamma \times \Gamma')$. Then it follows from the definition of P and π that $P(\pi)(\psi\ [\Gamma]) = \psi\ [\Gamma, \Gamma']$ where we are making explicit the two different contexts of ψ; the role of $P(\pi)$ just lies in changing contexts. Since the $\leq$ of $P(\Gamma \times \Gamma')$ is induced by its lattice structure, we have $\varphi \wedge \psi = \psi$. It follows from the definition of $P(\Gamma \times \Gamma')$ that

$\varphi \wedge \psi \vdash \psi$ $[\Gamma, \Gamma']$ and $\psi \vdash \varphi \wedge \psi$ $[\Gamma, \Gamma']$ are deducible in TFL^q (resp. TFL^q_X), whence $\psi \vdash \varphi$ $[\Gamma, \Gamma']$ is deducible as well. By repeated applications of rule $\forall R$, it follows that $\psi \vdash \forall x_1 ... \forall x_n \varphi$ $[\Gamma']$ is deducible. This implies that both $\psi \vdash \psi \wedge \forall x_1 ... \forall x_n \varphi$ $[\Gamma']$ and $\psi \wedge \forall x_1 ... \forall x_n \varphi \vdash \psi$ $[\Gamma']$ are deducible, whence $\psi \leq \forall x_1 ... \forall x_n \varphi$ in $P(\Gamma')$.

We show the converse. Assume that $\psi \leq \forall x_1 ... \forall x_n \varphi$ in $P(\Gamma')$. By arguing as in the above, $\psi \vdash \forall x_1 ... \forall x_n \varphi$ $[\Gamma']$ is deducible. By enriching the context, $\psi \vdash \forall x_1 ... \forall x_n \varphi$ $[\Gamma, \Gamma']$ is deducible. Since $\forall x_1 ... \forall x_n \varphi \vdash \varphi$ $[\Gamma, \Gamma']$ is deducible by rule $\forall L$, the cut rule tells us that $\psi \vdash \varphi$ $[\Gamma, \Gamma']$ is deducible; note that the contexts of two sequents-in-context must be the same when applying the cut rule to them. It finally follows that $P(\pi)(\psi) \leq \varphi$ in $P(\Gamma \times \Gamma')$. Thus, $\forall_\pi$ is the right adjoint of $P(\pi)$. Similarly, $\exists_\pi$ can be shown to be the left adjoint of $P(\pi)$.

The Beck-Chevalley condition for $\forall$ can be verified as follows. Let $\varphi \in P(\Gamma \times \Gamma')$, $\pi : \Gamma \times \Gamma' \to \Gamma'$ a projection in $\mathbf{C}$, and $\pi' : \Gamma \times \Gamma'' \to \Gamma''$ another projection in $\mathbf{C}$ for objects $\Gamma, \Gamma', \Gamma''$ in $\mathbf{C}$. Then, we have $P([t_1, ..., t_n]) \circ \forall_\pi(\varphi) = (\forall x_1 ... \forall x_n \varphi)[t_1/y_1, ..., t_n/y_m]$ where it is supposed that Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, Γ' is $y_1 : \tau_1, ..., y_m : \tau_m$, and $t_1 : \tau_1$ $[\Gamma''], ..., t_m : \tau_m$ $[\Gamma'']$. We also have the following $\forall_{\pi'} \circ P([t_1, ..., t_n])(\varphi) = \forall x_1 ... \forall x_n (\varphi[t_1/y_1, ..., t_n/y_m])$. The Beck-Chevalley condition for $\forall$ thus follows. The Beck-Chevalley condition for $\exists$ can be verified in a similar way. The two Frobenius Reciprocity conditions for $\exists$ follow immediately from Lemma 1.

The syntactic hyperdoctrine is a free or classifying hyperdoctrine in a suitable sense. It is the combination of the classifying category $\mathbf{C}$ above and the free algebras $P(\Gamma)$ above, which has the universal property inherited from both of them, though we do not have space to work out the details in this paper.

Now, there is the obvious, canonical interpretation of TFL^q (resp. TFL^q_X) in the syntactic hyperdoctrine of TFL^q (resp. TFL^q_X); it is straightforward to see:

Lemma 8 *If $\Phi \vdash \psi$ $[\Gamma]$ is satisfied in the canonical interpretation in the syntactic hyperdoctrine of* TFL^q *(resp.* TFL^q_X*), it is deducible in* TFL^q *(resp.* TFL^q_X*).*

The lemmata above give us the completeness result: If $\Phi \vdash \psi$ $[\Gamma]$ is satisfied in any interpretation in any FL (resp. FL_X) hyperdoctrine, then it is deducible in TFL^q (resp. TFL^q_X). Combining soundness and completeness, we obtain:

Theorem 9 *$\Phi \vdash \psi$ $[\Gamma]$ is deducible in* TFL^q *(resp.* TFL^q_X*) iff it is satisfied in any interpretation in any* FL *(resp.* FL_X*) hyperdoctrine.*

4 Duality-Induced Set-Theoretical Hyperdoctrines

In this section, we discuss hyperdoctrines induced from dual adjunctions between **Set** and **FL**, which are, so to say, many-valued powerset hyperdoctrines, and give many-valued Tarski semantics with soundness and completeness, generalising the powerset hyperdoctrine $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathbf{2})$, which is equivalent to Tarski semantics. We mostly omit proofs in this section due to space limitations.

Theorem 10 *Let $\Omega \in \mathbf{FL}$. The following dual adjunction holds between* **Set** *and* **FL***, induced by Ω as a dualising object:*

$$\mathrm{Hom}_{\mathbf{FL}}(\text{-}, \Omega)^{\mathrm{op}} \dashv \mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}.$$

Proposition 11 *Let $\Omega \in \mathbf{FL}$ with Ω complete. Then,* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{FL}$ *(resp. $\mathbf{FL}_X$) is an* FL *(resp.* FL_X*) hyperdoctrine.*

Proof. Let $\pi : X \times Y \to Y$ be a projection in **Set**. We define $\forall_\pi$ and $\exists_\pi$ as follows: given $v \in \mathrm{Hom}(X \times Y, \Omega)$ and $y \in Y$, let $\forall_\pi(v)(y) := \bigwedge\{v(x, y) \mid x \in X\}$ and $\exists_\pi(v)(y) := \bigvee\{v(x, y) \mid x \in X\}$. These yield the required quantifier structures with the Beck-Chevalley and Frobenius Reciprocity conditions; details omitted.

Now, we aim at obtaining completeness with respect to models of form $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$. The above proof tells us that $\forall$ and $\exists$ in $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ are actually meets and joins in Ω. This implies that if Ω is not complete, in general, $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ cannot interpret quantifiers. At the same time, however, assuming completeness prevents us from obtaining completeness for any axiomatic extension TFL^q_X of TFL^q; this is why we do not assume it. Such incompleteness phenomena have already been observed (see, e.g., Ono [14]). A standard remedy to this problem is to restrict attention to "safe" interpretations while considering general Ω. In our context, a safe interpretation $[\![\text{-}]\!]$ in $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ is such that $[\![\text{-}]\!]$ uses those joins and meets only that exist in Ω, i.e., quantifiers are always interpreted via existing joins and meets only. We then have completeness with respect to the special class of set-theoretical models $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$.

Theorem 12 *$\Phi \vdash \psi$ $[\Gamma]$ is deducible in* TFL^q_X *iff it is satisfied in any safe interpretation in* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ *for any $\Omega \in \mathbf{FL}$.*

In the special case of TFL^q*, it suffices to consider complete Ω's only: $\Phi \vdash \psi$ $[\Gamma]$ is deducible in* TFL^q *iff it is satisfied in any interpretation in any* FL *hyperdoctrine* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ *with $\Omega \in \mathbf{FL}$ complete.*

Focusing on a more specific context, we can further reduce the class of models $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ into a smaller one. In the strongest case of classical logic, it suffices to consider $\{0, 1\}$ only in the place of Ω; this is exactly the Tarski completeness.

For an intermediate case, consider MTL (monoidal t-norm logic; see Hájek et al. [5]), which is FL expanded with exchange, weakening, and the pre-linearity axiom, $(\varphi \to \psi) \vee (\psi \to \varphi)$. The algebras of MTL are denoted by **MTL**. We denote by MTL^q the quantified version with the additional axiom of $\forall$-$\vee$ distributivity, i.e., $\forall x(\varphi \vee \psi) \leftrightarrow \forall x\varphi \vee \psi$ where x does not occur in ψ as a free variable, and by MTL^q_X an axiomatic extension of MTL^q.

Theorem 13 *$\Phi \vdash \psi$ $[\Gamma]$ is deducible in* MTL^q_X *iff it is satisfied in any interpretation in* $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ *for any linearly ordered $\Omega \in \mathbf{MTL}_X$.*

We briefly discuss the tripos-topos construction in the present context of FL hyperdoctrines; it is originally due to Hyland-Johnstone-Pitts [7]. To this end, we work in the internal logic of FL hyperdoctrines $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}$: i.e., we have types X and function symbols f corresponding to objects X and arrows f in $\mathbf{C}$ respectively, and also those predicate symbols R on a type $C \in \mathbf{C}$ that correspond to elements $R \in P(C)$.

Definition 14 *Let P be an* FL *hyperdoctrine. We define a category $\mathbf{T}[P]$ as follows. An object of $\mathbf{T}[P]$ is a partial equivalence relation, i.e., a pair (X, E_X) such that X is an object in the base category $\mathbf{C}$, and E_X is an element of $P(X \times X)$ and is symmetric and transitive in the internal logic of P: $E_X(x, y) \vdash E_X(y, x)$ $[x, y : X]$ and $E_X(x, y), E_X(y, z) \vdash E_X(x, z)$ $[x, y, z : X]$.*

An arrow from (X, E_X) to (Y, E_Y) is $F \in P(X \times Y)$ such that (i) extensionality: $E_X(x_1, x_2), E_Y(y_1, y_2), F(x_1, y_1) \vdash F(x_2, y_2)$ $[x_1, x_2 : X, y_1, y_2 : Y]$; (ii) strictness: $F(x, y) \vdash E_X(x, x) \wedge E_Y(y, y)$ $[x : X, y : Y]$; (iii) single-valuedness: $F(x, y_1), F(x, y_2) \vdash E_Y(y_1, y_2)$ $[x : X, y_1, y_2 : Y]$; (iv) totality: $E_X(x, x) \vdash \exists y\ F(x, y)$ $[x : X]$. Such an F is called a functional relation.

For a complete FL algebra Ω, which is a quantale with additional operations, $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$ may be called the category of Ω-valued sets. Quantale sets in the sense of Höhle et al. [6] are objects in $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$, but not vice versa: our Ω-valued sets are slightly more general than their quantale sets.

Note that if Ω is a locale, $\mathbf{T}[\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)]$ is the Higgs topos of Ω-valued sets, which is in turn equivalent to the category of sheaves on Ω.

5 Girard's and Gödel's translation hyperdoctrinally

In this section, we discuss Girard's and Gödel's translation theorems on the hyperdoctrinal setting. The former embeds intuitionistic logic into linear logic via exponential !; the latter embeds classical logic into intuitionistic logic via double negation $\neg\neg$. Since logic is dual to algebraic semantics, we construct intuitionistic (resp. classical) hyperdoctrines from linear (resp. intuitionistic) hyperdoctrines. We omit proofs in this section as well for space limitations.

We first consider Gödel's translation. We think of $\neg\neg$ as a functor $\mathrm{Fix}_{\neg\neg}$ from $\mathbf{HA}$, the category of heyting algebras, to $\mathbf{BA}$, the category of boolean algebras: i.e., define $\mathrm{Fix}_{\neg\neg}(A) = \{a \in A \mid \neg\neg a = a\}$; the arrow part is defined by restriction. Here, $\mathrm{Fix}_{\neg\neg}(A)$ forms a boolean algebra.

Let us define IL hyperdoctrines as FL hyperdoctrines with values in $\mathbf{HA}$. Likewise, CL hyperdoctrines are defined as FL hyperdoctrines with values in $\mathbf{BA}$. Note that both kinds of hyperdoctrines are TFL_X^q hyperdoctrines with suitable choices of axioms X. Finally, Gödel's translation theorem can be understood in terms of hyperdoctrines as follows.

Theorem 15 *Let $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ be an* IL *hyperdoctrine. Then, the following composed functor $\mathrm{Fix}_{\neg\neg} \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$ forms a* CL *hyperdoctrine.*

This is a first-order and hyperdoctrinal version of the construction of boolean toposes from given toposes via double negation topologies on them.

We can treat Girard's translation along a similar line. An exponential ! on an FL algebra A is defined as a unary operation satisfying: (i) $a \leq b$ implies $!a \leq !b$; (ii) $!!a = !a \leq a$; (iii) $!\top = 1$; (iv) $!a \otimes !b = !(a \wedge b)$ (see Coumans et al. [2]). We denote by $\mathbf{FL}_c^!$ the category of commutative FL algebras with ! and maps preserving both ! and FL algebra operations; they give the algebraic counterpart of intuitionistic linear logic with !, denoted ILL.

We regard exponential ! as a functor $\mathrm{Fix}_!$ from $\mathbf{FL}_c^!$ to $\mathbf{HA}$: define $\mathrm{Fix}_!(A) = \{a \in A \mid !a = a\}$; the arrow part is defined by restriction. $\mathrm{Fix}_!(A)$ is the set of those elements of A that admit structural rules, and forms a heyting algebra. ILL hyperdoctrines are defined as FL hyperdoctrines with values in $\mathbf{FL}_c^!$.

Theorem 16 *Let* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_c^!$ *be an* ILL *hyperdoctrine. Then, the following composed functor* $\mathrm{Fix}_! \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ *forms an* IL *hyperdoctrine.*

The theorem above is slightly more general than Girard's translation theorem, in the sense that the latter corresponds to the case of syntactic hyperdoctrines in the former. Although in this paper we do not explicitly discuss substructural logics enriched with modalities and their hyperdoctrinal semantics, nevertheless, our method perfectly works for them as well, yielding the corresponding soundness and completeness results in terms of hyperdoctrines with values in FL algebras with modalities; Girard's ! is just a special case.

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