Categorical Harmony and Degrees of Paradoxity

Yoshihiro Maruyama

Quantum Group
Dept. of Computer Science
Oxford University

CAPE Workshop, Kyoto, Sep 18, 2012

Outline

Outline

Thank you for coming

Someone told me post-modernists would come to the workshop as well as analytic philosophers.

- This talk is basically along the line of philosophy of logic in the analytic tradition, with no real connection to post-mod.
- Still, I want to say something for post-modernists, to express my gratitude to them for coming to the analytic ws.
  - I also thank several post-modernists for reading the categorical articles I contributed to *Mathematics Seminar*.
- I think post-modernists and analytic philosophers share some ideas actually (and could even interact fruitfully).

E.g., both Quine and Lyotard rely on the infinite regress of “proving a proof" in their crucial arguments (Lyotard on legitimation in modernism, and Quine on truth by convention)

Categorical Deconstruction

If I were a post-modernist, I would say:

- Category theory is involved in deconstruction. Deconstruction of what?
- CT deconstructs, for instance, the old dichotomy b/w algebra and geometry (or space), in the sense that one categorical concept can represent both algebraic and geometric structures.
  - E.g., algebras of monads encompass compact T_2 spaces as well as all the varieties in universal algebra.
  - Grothendieck's theory of Galois categories unify Galois theory in algebra and fundamental groups in topology.
  - Categorical duality theory tells us algebra (e.g., rings) is equivalent to geometry (e.g., varieties in geometry).

This might be called Categorical Deconstructionism.

The Dichotomy b/w PTS and MTS

The following are quotes from:
http://www.st-andrews.ac.uk/arche/projects/logic/

- “Logical consequence is the relation between premises and conclusion of a valid piece of reasoning (an argument)."
- “The Foundations of Logical Consequence project concentrates on two principal positive approaches to explicating this notion, model-theoretic and inferentialist."

They contrast model theory and inferentialism; proof-theoretic semantics (PTS), as opposed to model-th. semantics (MTS), is an incarnation of inferentialism. Category theory deconstructs this dichotomy, integrating PTS and MTS into the one concept.

Decorating logical consequence with proofs

PT is CT. Suppose we have the following concepts given:

- Formuli φ; Proofs from φ to ψ; a proof from φ to φ exists;
- Proof-decorated relation $\vdash$:

$$\varphi \vdash_p \psi$$

  where p is a proof from φ to ψ;
- Sequential combination of proofs: from $\varphi \vdash_p \psi$ and $\psi \vdash_q \xi$ derive $\varphi \vdash_{p,q} \xi$; (parallel composition: $\varphi \otimes \varphi' \vdash_{p \otimes p'} \psi \otimes \psi'$;)
- Reduction of proofs that cancels a proof from φ to φ, and has local coherency (i.e., proofs may be reduced locally).

I call this “proof-theoretic consequence", which is exactly the same as the concept of category. I avoid to use full-fledged CT in this talk; it suffices for you to know some order theory.

Categorical Deconstruction Again

PTS is thus part of categorical semantics in the form of syntactic categories, which are important in CT itself, since they give so-called classifying categoires.

- MTS is part of categ. sem. as well, in the form of set-theoretic categories (topos of sets give Tarski Semantics; topos of presheaves give Kripke Semantics). They are certain quotients of syntactic categories.

I recently showed any substructural predicate logic (i.e., axiomatic extensions of full Lambek calculus) has categorical semantics in the sense of Lawvere's hyperdoctrines.

- Categorical semantics is thus consistent with logical pluralism. It is the third way in semantics.

It does not exclude any of PTS and MTS; rather, it allows us to consider both PTS and MTS to be proper semantics.

Inferentialism

More quotes from the same source as in the above.

- “According to inferentialism, the conclusion is a logical consequence of the premises if it may be derived from them by step-wise application of primitive inference-rules, conceived (according to some inferentialists) as implicitly defining the logical expressions they contain, whose acceptance (some hold) is constitutive of understanding those expressions."
- “The classic objection to inferentialism is posed by Arthur Prior’s demonstration that not every characterisation of inferential role determines an admissible logical operation."

What is the classic objection from Prior?

The Problem of Prior's Tonk

Arthur Prior came up with a weird connective “tonk":

- It is defined by the following two rules of inference:

$$\frac{\Phi \vdash \varphi}{\Phi \vdash \varphi \text{tonk} \psi}\ \text{(tonk-intro)} \qquad \frac{\Phi \vdash \varphi \text{tonk} \psi}{\Phi \vdash \psi}\ \text{(tonk-elim)}$$

- Combining the two, we can show $\varphi \vdash \psi$ for any φ, ψ. This means tonk causes the trivialisation of the deductive relation $\vdash$. Is tonk a proper logical constant?
  - We always assume $\vdash$ admits identity (i.e., $\varphi \vdash \varphi$) and cut (i.e., $\varphi \vdash \psi$ and $\psi \vdash \xi$ together imply $\varphi \vdash \xi$).
- If not, how can we conceptually discriminate b/w proper logical constants and other paradoxical connectives?

What is wrong with tonk? This is called the problem of tonk, a sort of demarcation problem in philosophy of logic.

Disclaimer

I largely agree upon Wittgenstein's (abused) idea of meaning as use, as most inferentialists do.

- This does not mean, however, that I agree upon the inferentialistic claim that there are formal, static rules governing the use of a word or expression.
  - This would conform to Wittgenstein's original idea; he was against scientism, and superficial formalisation.
  - Even if there were such rules, there would be yet another, Kripkensteinian problem of following those rules.
- I think the meaning of a word is of more dynamic, variable, and reflexive nature as some post-modernists say.
  - Cf. dynamic turn in logic (dynamic logic, game, GoI, ...).

In this talk, I leave aside such intriguing issues on meaning, focusing solely on the logical demarcation problem.

Disclaimer (cont.)

I distinguish b/w connectives, those connectives that have meanings, and those connectives that are logical constants.

- I shall show tonk is not a logical constant according to categorical harmony.
- I do not show tonk has no meaning.
- It would be possible for tonk to have a meaning if meaning is use, since it is crystal clear how to use tonk.
- Tonk could be a connective who has a meaning.
- I only show tonk is not a logical constant.

In the main part, I only discuss logical constants; no discussion on meaning at all. Meaning is too difficult for me.

Outline

Overview

Lawvere understood logical constants as adjoint functors.

- I propose to see Adjointness as Harmony.
- Proof-theoretically, an adjunction amounts to the validity of a double line rule of certain form.

Categorical, Lawverian harmony tells us there are three different *degrees of paradoxity* of connectives (the last two below are pseudo-paradoxes caused by equivocation):

	right adjoint to	left adjoint to
Genuine Paradox	itself	itself
Disconjunction	diagonal	diagonal
Tonk	true diagonal	false diagonal

Acknowledgements: The picture above was obtained through discussion with Peter Schröder-Heister.

Adjunction in a Logical Context

Fix a logic L with a deductive relation $\vdash$.
Consider unary operations $t, s : L \to L$.

Definition (unary adjointness)

t is left adjoint to s ($t \dashv s$) iff the following double line rule holds:

$$\frac{t(\varphi) \vdash \psi}{\varphi \vdash s(\psi)}$$

Example: Let $t(\varphi) := \varphi \wedge \xi$, and $s(\xi) := \xi \to \psi$. Then,

$$\frac{\varphi \wedge \xi \vdash \psi}{\varphi \vdash \xi \to \psi}$$

Thus, $\wedge$ is left adjoint to $\to$, and $\to$ is right adjoint to $\wedge$.

Adjunction in a Logical Context (cont.)

Definition (binary adjointness)

$t : L \times L \to L$ is left adjoint to $s : L \to L \times L$ (or $t \dashv u$) iff the following double line rule holds:

$$\frac{t(\varphi, \psi) \vdash \xi}{(\varphi, \psi) \vdash s(\xi)}$$

Example: Let $t := \vee$, and $s(\xi) := \Delta(\xi) = (\xi, \xi)$. Then,

$$\frac{\varphi \vee \psi \vdash \xi}{\varphi \vdash \xi \quad \psi \vdash \xi}$$

$\vee$ is left adjoint to diagonal Δ. $\wedge$ is right adjoint to diagonal Δ:

$$\frac{\xi \vdash \varphi \quad \xi \vdash \psi}{\xi \vdash \varphi \wedge \psi}$$

Adjointness as Logicality

Standard logical constants can be characterised by the corresponding double line rule, i.e., adjunction.

- It thus seems natural to see adjointess as logicality.
- Cat. Harmony: a logical constant must be defined by (the double line rule of) an adjunction wrt a existing operation.

Certain subtleties are lurking behind the definition above:

- It turns out that being defined by ONE adjunction is crucial, since tonk can be defined by TWO adjunctions.
- A logical constant must be defined as an adjoint of an existing operation, since paradox can be defined as an adjoint of itself.

This is different from Došen's logicality by double line rules; actually better than that wrt Bonnay-Simmenauer's blonk.

Relativisaton to Primitive Vocabulary

We can relativise the concept of categorical harmony to choice of primitive vocabulary in an obvious way.

- If we include a monoidal conjunction $\otimes$ in primitive vocabulary, multiplicative connectives count as logical constants as well as additive ones.

But in that case we have to be careful of possibility of inconsistencies caused by primitive vocabulary.

Belnap's Harmony

Belnap's harmony consists of conservativity and uniqueness.

- Any logical constant introduced according to categorical harmony is unique, since an adjoint of a functor is unique.
  - Uniqueness is not something assumed in the first place; rather, it is just a consequence of categorical harmony.
- Conservativity naturally fails in categorical harmony, since right (resp. left) adjoints preserve limits (resp. colimits) by Freyd's adjoint functor thm. Avron also is against it.
  - E.g., consider logic with $\wedge$ and $\vee$ specified by adjunctions. This logic does not validate distributivity b/w $\wedge$ and $\vee$. But adding $\rightarrow$ as the right adjoint of $\wedge$ makes distributivity valid.

Conservativity may be contested from a Quinean, holistic point of view. Meaning of a logical constant depends on the whole system. Adding a new constant may well change the meaning of old ones.

Against Uniqueness

Both concepts of harmony might miss the point, since:

- Girard's exponential ! does not have uniqueness.
- Thus, it cannot count as a logical constant according to any of the two.

But, the role of ! is to have control on structural rules or resources. As such, ! may be said to be a technical device or a "computational" constant, rather than a proper logical constant.

Sambin et al.'s Harmony

Sambin et al. introduce logical constants by what they call reflection principle and definitional equalities, like:

- $\varphi \vee \psi \vdash \xi$ iff $\varphi \vdash \xi$ and $\psi \vdash \xi$. $\varphi, \psi \vdash \xi$ iff $\varphi \otimes \psi \vdash \xi$.
- $\Gamma \vdash \varphi \to \psi$ iff $\Gamma \vdash (\varphi \vdash \psi)$.

Definitional equalities are similar to adjointness conditions. There are crucial differences, however. It might be “non-logic".

- Definitional equalities do not always imply adjoitness, due to their “visibility" condition (restriction of context formulas). E.g., $\to$ is not an adjoint of $\wedge$ nor $\otimes$ in their Basic Logic.

Deviation from adjointness is inevitable for Sambin et al.; they want to include quantum logic with some $\to$, but quantum $\to$ if any cannot be an adjoint of $\wedge$, because of non-distributivity.

Is linear logic logic?

Multiplicative connectives in Girard's sense are logical constants according to Sambin et al.

- They cannot be defined by adjointness, since there can be different monoidal structures on one category.
- Thus, they are not logical constants according to categorical harmony.

Is this is an advantage or disadvantage of categorical harmony.

- Linear logicians claim linear logic is not a new, non-classic. logic, but a decomposition of classical (or int.) logic.
- As such, !, $\otimes$, and the like may be just convenient tools for decomposition. If so, they are arguably not logical const.

It is sometimes said linear logic did not fit into Lawvere's mind.

Dummett, Prawitz, etc.

Dummett has made distinction b/w local harmony (e.g., inversion) and total harmony (e.g., conservativity).

- Inversion is built in categorical harmony in the sense that adjunction rules can always be reversed.
- If local or global harmony depends on how to present a proof system, it is not acceptable from a categorical, structural point of view.
  - Lawvere also emphasised semantics should be independent of syntactic presentations.
- Categorical harmony is structural, global harmony; in particular, it is robust to choice of a proof system.

I personally think the concept of logic should be independent of the concept of proof system, as manifolds are independent of coordinate systems. Do LJ and NJ represent the same logic?

Remarks

Relativisation helps:

- If we want to accept multiplicatives and exponentials, we can use categorical harmony relativised to primitive vocabulary including them.

Classical negation:

- Classical negation is a combination of two adjoints. It is dually adjoint to itself. But it is not self-adjoint.

Is logic with $\wedge$ only possible?

- Totally possible. Δ always exists, i.e., we can always put sequents in parallel (thanks to Takuro Onishi).

Outline

$\Delta_\bot \dashv$ tonk $\dashv \Delta_\top$

A naive way to express categorical harmony: Logical constants must be adjoint functors.

- Is tonk an adjoint functor, or a logical constant from the viewpoint of categorical harmony?
  - Tonk is a connective with $\varphi \vdash \varphi$ tonk ψ and φ tonk $\psi \vdash \psi$.

Observation

Tonk is an adjoint functor in a system with tonk. Indeed, tonk has both right and left adjoints, namely $\Delta_\top$ and $\Delta_\bot$ below.

$\Delta_\top : L \to L \times L$ is defined by

$$\Delta_\top(\varphi) := (\top, \top)$$

and $\Delta_\bot : L \to L \times L$ by $\Delta_\bot(\varphi) := (\bot, \bot)$.

Categorical Harmony accepts Tonk?

Is tonk then a logical constant according to cat. harmony?

- That's right according to the naive formulation.
- But the formulation I first gave was different: a logical constant must be *defined* by an adjunction wrt ...
  - Tonk is not a logical constant according to this formulation of categorical harmony, since the following holds.

Theorem

Tonk cannot be defined in a system without tonk by a single adjunction; there is no functor F s.t. an adjoint of F is tonk.

Still, tonk can be defined by two adjunctions: $\Delta_{\perp} \dashv \mathrm{tonk} \dashv \Delta_{\top}$. This characterises tonk.

Tonk as a Bi-Adjoint or Equivocation

Tonk cannot be defined as a single adjoint, but can be defined as a “bi-adjoint", i.e., being adjoints of two operations at once.

- What’s wrong with tonk?
- The bi-adjoitness of tonk is equivocation.
- The right and left adjoints of $\Delta_{\perp}$ and $\Delta_{\top}$ represent different logical constants (“binary" truth const. and falsity const.).
- Tonk confuses those constants as the same one.
  - We can therefore resolve the paradoxity of tonk by giving different names to the right and left adjoints.

The categorical analysis thus tells us the problem of tonk is the problem of equivocation.

$\Delta \dashv$ disconjunction $\dashv \Delta$

$\wedge$ is right adjoint to diagonal $\Delta : L \to L \times L$; $\vee$ is left adjoint to Δ.

- Define disconjunction to be the right and left adjoints of Δ at once. It makes the system trivial as tonk does.
- Disconjunction confuses "and" and "or" as the same const.
  - Thus, the paradoxity of disconjunction, as well as tonk, is caused by equivocation.

What's the (intensional) difference b/w tonk and disconjunct.?

- Tonk is a right adj. of some F, and is a left adj. of some G. Disconjunction is a "uniformly" bi-adjoint functor: it is the right and left adjoint of the same one $F = G$.

Thus, disconjunction is more paradoxical than tonk:
bi-adjointness $<$ uniform bi-adjointness (w.r.t. impossibility).

Paradox $\dashv$ Paradox $\dashv$ Paradox

Let us think of a paradoxical nullary connective R:

$$\frac{\vdash \neg R}{\vdash R}$$

Reformulate this as follows:

$$\frac{R \vdash}{\vdash R}$$

Think of R as a unary constant connective $\tilde{R} : L \to L$ with $\tilde{R}(\varphi) := R$. Then, the rule above means R is right and left adjoint to R. Paradox is thus a self-adjoint functor.

Three Degrees of Paradoxity

We have finally led to three degrees of paradoxity:
bi-adjointness $<$ uniform bi-adjointness $<$ self-adjoitness.

	right adjoint to	left adjoint to
Genuine Paradox	itself	itself
Disconjunction	diagonal Δ	diagonal Δ
Tonk	true diagonal $\Delta_\top$	false diagonal $\Delta_\bot$

The last two are caused by equivocation.

- Genuine Paradox is not so, since self-adjoitness can be given by a single adjunction: if a functor is right (resp. left) adjoint to itself, it is left (resp. right) adjoint to itself.

Genuine Paradox cannot be solved by giving right and left adj. different names; we can do this for tonk and disconjunction.

Lawvere's Hyperdoctrine

Lawvere's hyperdoctrines are functors of the form:

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$$

satisfying logical conditions: e.g., $P(C)$ is a bCCC for $C \in \mathbf{C}$; a right adjoint of $P(\pi : X \times Y \to Y)$ exists; recall

$$\frac{\varphi(y) \vdash \psi(x, y)}{\varphi(y) \vdash \forall x \psi(x, y)}$$

This is the "quantifiers as adjoints" idea.
Note1: This includes topoi for HOL and Heyt. cat. for FOL.
Note2: Lawvere emphasised duality b/w the formal and the conceptual, or PT and MT, or essentially Stone Duality.

PTS and MTS become the one in Categ. Semantics

Tarski Semantics is given by the powerset hyperdoctrine

$$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}.$$

Kripke Semantics is given by presheaf hyperdoctrines:

$$\mathrm{Sub}(\text{-}) : \mathbf{PreSheaf}(P)^{\mathrm{op}} \to \mathbf{Cat}.$$

PTS is given by the syntactic hyperdoctrine $G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$:

- **C** is the cat. of types or sorts. If there is only one basic type, it is the cat. generated by that type w.r.t. type const.
- G maps a type to the category of formulas on it in which arrows are proofs. Identity of proofs is equality of arrows.

This can be adapted for logical pluralism; e.g., substructural logics can be encompassed as well as intuitionistic logic.

Conclusions

The concept of categorical harmony was introduced based on Lawvere's idea of logical constants as adjoint functors.

- The precise formulation of cat. harmony was given.
- Cat. harmony led us to the idea of degrees of paradoxity: tonk $<$ disconjunction $<$ genuine paradox, according to: bi-adj. $<$ uniform bi-adj. $<$ self-adj.
  - What's wrong with tonk is equivocation. The right and left adjoints represent different constants, which tonk confuses; gen. paradox or self-adj. cannot be solved by re-naming.
- Relationships with Belnap's, Prawitz' and Sambin et al.'s hamorny were clarified.

Categ. sem., as the third way in semantics, integrates PTS and MTS into the one concept (bilateralism into unilateralism?).