Categorical Duality between Two Aspects of the Notion of Space
空間概念の二つの側面間の圏論的双対性

by

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Abstract

We attempt to reveal the mathematical relationships between two aspects of the notion of space, namely what is called ontological and epistemological aspects of it in the thesis, on the basis of the language of category theory. On the one hand, there are set-theoretic, point-set concepts of space (e.g., topological space, convexity space, and measurable space), which we consider represents the ontological aspect of the notion of space. On the other hand, there are algebraic, point-free concepts of space (e.g., locale, continuous lattice, and σ-complete Boolean algebra), which we consider represents the epistemological aspect of the notion of space.

Our general picture is that the two aspects of the notion of space are related in a dual way via "logic-of-space" and "space-of-logic" constructions, which in many cases may be seen as leading us to categorical dualities between point-set and point-free spaces. With this picture in mind, our investigation proceeds as follows: We first develop a moderately general theory of categorical dualities, then apply it to each case, and finally analyze individual cases in detail to obtain deeper insights into them that cannot be derived from a neat, general theory. In the process, we exploit categorical concepts such as monad, functor-structured category, and topological axiom. At the final stage, we particularly focus on Scott's continuous lattices in domain theory, which turn out to be dually related to convexity spaces, thus opening up the possibility of the interpretation of domain theory in terms of convex geometry.

Duality indeed has different faces in different fields, which allows us to utilize it in diverse disciplines ranging from mathematics, to logic, to physics, and to computer science, though in the thesis we focus on duality itself. The dichotomy between the ontological and epistemological aspects of the notion of space may be broadly seen as that between varieties and rings in mathematics, that between models and theories in logic, that between states and observables in physics, that between internal structures (resp. denotations) and external behaviors (resp. observable properties) of programs in computer science, that between the notions of point and region in philosophy of geometry, and that between the Cantorian and Brouwerian concepts of real number in foundations of mathematics, which richly amplifies the significance of duality between the two aspects of the notion of space.

We remark that, although we explain some philosophical background for introductory purposes, the thesis itself and all results in it are independent of any philosophical standpoint, and are intended to be of purely mathematical nature without any substantial, philosophical claim and contribution. A part of the thesis has been published as follows: Y. Maruyama, Fundamental results for pointfree convex geometry, *Annals of Pure and Applied Logic* 161 (2010) 1486-1501.

概要

我々は、空間概念の二つの側面間、すなわち空間概念の存在論的側面と認識論的側面と本論文で呼ばれるものの間の数学的連関を、圏論の言語に基づいて明らかにすることを試みる。一方では、集合論に基づく点集合的な空間概念（例えば、位相空間、凸性空間や可測空間など）があるが、我々は、そういった空間概念は空間概念の存在論的側面を表現していると考える。他方では、点概念を仮定しない代数的な空間概念（例えば、ロカール、連続束やシグマ完備ブール代数など）があるが、我々は、そういった空間概念は空間概念の認識論的側面を表現していると考える。

我々の一般的描像は、空間概念のこういった二つの側面は「空間の論理」構成法と「論理の空間」構成法によって双対的に関連づけられる、というものである。これは、多くの事例において、点集合に基づく空間と点概念を仮定しない空間の間の圏論的双対性に我々を導いていると考えられるものである。この描像を念頭に置きながら、我々の探求は次のように進行する。まず最初に圏論的双対性の適度な一般論を展開し、次にそれを個々の事例に適用した後、最後に、きれいな一般論からは得られない、より深い洞察を得るため、個別の事例を詳しく分析する。この過程において我々は、モナド、関手により構造化された圏や位相的公理といった圏論的概念を利用する。最後の段階において我々は、領域理論におけるスコットの連続束に特に焦点を当て、それが実は凸性空間と双対的に関連づけられることを示す。これは、領域理論の凸幾何学的解釈の可能性を切り開くものである。

我々は本論文では双対性そのものに焦点を当てるが、双対性は、幅広い学問領域において様々な顔を持っており、このことが、数学、論理学、物理学から計算機科学にまで至る、多様な分野において双対性を活用することを可能にしている。空間概念の存在論的側面と認識論的側面という二項対立は、広く言えば、数学においては多様体と環の間のそれ、論理学においてはモデルと理論の間のそれ、物理学においては状態と可観測量の間のそれ、計算機科学においてはプログラムの内部構造（あるいは表示的意味）と外的挙動（あるいは観察可能な性質）の間のそれ、幾何学の哲学においては点概念と領域概念の間のそれ、数学基礎論においてはカントール的実数概念とブラウワー的実数概念の間のそれとして捉えられ、こうして空間概念の二つの側面間の双対性というものの意味がより豊かなものとなるのである。

なお、本論文では、主題への導入としてある種の哲学的背景を説明するが、本論文それ自体とその結果は全て、特定の哲学的立場からは独立したものであり、いかなる実質的な哲学的主張・貢献も伴わない、純粋に数学的なものであるように意図されている。また、本論文の一部は、以下のように既に出版されているものである：Y. Maruyama, Fundamental results for pointfree convex geometry, *Annals of Pure and Applied Logic* 161 (2010) 1486-1501.

Contents

1 Introduction and Background

In this section, we first explain what motivates and underlies mathematical developments in the later sections by placing them in a broader context. Then, we give a brief overview of the content of the thesis as well as a comparison with related work. The main results of the thesis are Theorems 2.19, 2.33, 3.45, 3.63, 3.66, and 3.71, the significance of which are explained in Subsection 1.4.

In this section, some points at issue are not fully articulated because of the limitation of space. Footnotes are used to remedy this to some extent and thus are sometimes longer than usual. We have several remarks on the subject matter of the present thesis as follows.

What we shall discuss in this section is the very notion of space and is not primarily real, physical space in any sense. Our aim is not to reveal the actual structure of space, which is the task of physics (of course, this does not mean that we exclude a physical point of view). Rather, we are concerned with our perspectives on the notion of space and clarifying the mutual relationships between them.

We do not claim that the mathematical results of the thesis contribute to a philosophical understanding of the notion of space. Our aim in this section is to present the reader a non-technical introduction to or motivation and ideas behind our mathematical results, which are the main ingredients of the thesis, and we do not intend to claim any philosophical thing throughout the thesis. Philosophical ideas presented in this section are merely motivational or for introductory purposes. We emphasize that the present thesis and all results in it are independent of any philosophical standpoint, and are intended to be solely and exclusively mathematical without any substantial, philosophical claim and contribution.

Our discussion may lack historical precision because of our (more or less) Whiggish view of history, though we try to avoid it as far as the simplicity of the description is maintained.[1]

We finally note that some results and proofs are omitted, especially in Subsection 3.4, because of the limitation of space, and relevant manuscripts are available upon request.

1.1 Ontological and Epistemological Aspects of Space

We (or some of us) usually think with no doubt that space consists of points. A number of major theories in physics and mathematics indeed seem to be based on such an idea of space. For example, the notion of space in Newtonian mechanics is Euclidean space and that in Einstein's relativity theory is Riemannian manifold[2], both of which are set-theoretically defined as collections of points equipped with certain structures to represent the geometry of space (according to the standard formulations of them). The notion of topological space is highly set-theoretical and many concepts of space in mathematics are usually introduced, more or less building on this notion,[3] including

[1] It is interesting that some philosophers are positively Whiggish. For example, Hacking [43] definitely says, "But I as philosopher am decidedly Whiggish. The history that I want is the history of the present. That is Michel Foucault's phrase, implying that we recognize and distinguish historical objects in order to illumine our own predicaments." (p.5).

[2] More properly, we should say "spacetime" here rather than "space", since we cannot separate space and time in the relativity theory. Although we concentrate on the notion of space, most of our arguments apply also to the notion of time or spacetime. Anyway, they are usually represented by collections of points in physics and mathematics.

[3] This does not mean that mathematicians see such a way as being essential. Many (or at least some) mathematicians use set theory while regarding it as inessential for mathematics itself. Nevertheless, they still exploit it, probably because it is useful and gives a rigorous way to describe mathematics. Bourbaki would be a good example of such mathematicians (see, e.g., Bourbaki [14]). We consider that, by talking about spaces based on point sets for

manifolds, Banach spaces, and algebraic varieties (equipped with Zariski topologies)[4].

Several philosophers and mathematicians are against or at least doubt the idea that space ultimately consists of points and, furthermore, some of them even assert that points are merely derived or abstracted from other entities that are more fundamental in some sense. For instance, there is an intriguing remark in Wittgenstein [100] (p.216):

> What makes it apparent that space is not a collection of points, but the realization of a law?

Although we do not know what he really meant, the idea "space is not a collection of points, but the realization of a law" is apparently relevant to a branch of point-free geometry,[5] locale theory (see, e.g., [54, 95]),[6] in which space is not a collection of points, but the body of properties in a certain sense. Since points would be beyond our ability to perceive, it seems plausible to doubt the primary existence of them. Whitehead [97, 98, 99] was more directly involved in developments of point-free geometry. Irvine [51] explains Whitehead's idea as follows:

> Whitehead's basic thought was that we obtain the abstract idea of a spatial point by considering the limit of a real-life series of volumes extending over each other, for example in much the same way that we might consider a nested series of Russian dolls or a nested series of pots and pans. However, it would be a mistake to think of a spatial point as being anything more than an abstraction (paragraph 9, Section 2).

This is almost the same as the following idea of locale theory: Consider a region (or a gunk in the sense of Lewis [63]) as a primitive entity and a point as a secondary one that appears as the limit of shrinking regions (or is a collection of such regions). Phenomenologists such as Brentano [15] and Husserl [49] had relevant perspectives on space continuum. According to Bell [10] (p.206), in a letter to Husserl drafted in 1905, Brentano clearly stated:

> I regard it as absurd to interpret a continuum as a set of points.

Brentano, Husserl, and Whitehead are counted as founders of modern mereology [18, 83], a branch of point-free geometry, as well as Leśniewski [61], who coined the term "mereology." Brouwer's theory of continuum (see, e.g., [47, 91])[7] is also of similar nature, and identifies a real number with a sequence of shrinking intervals of rationals, thinking of intervals as being more fundamental than real numbers.[8] In Subsection 1.2 below, we shall review in more detail mathematical developments

any reason, one is committed to the existence of point sets, according to Quine's criterion of ontological commitment: To be is to be the value of a variable (see [80]).

[4]In some cases, we actually have non-standard formulations of them that carefully avoid the use of the notion of point. For instance, the concept of scheme in algebraic geometry has recently been formulated in point-free terms by Coquand, Lombardi, and Schuster [28] (in order to constructivise geometry).

[5]By point-free geometry, we mean geometry that does not presuppose the notion of point. There are several branches of point-free geometry, but their basic ideas are similar. We use the term "geometry" in the ordinary, informal sense, though it has more specific meaning in mathematics. Especially, geometry in our sense includes topology.

[6]Locale theory is a formulation of point-free topology. Coquand [27] states that Hilbert's program may be reformulated using pointfree topology.

[7]Van Atten [92] studies the relationships between Brouwer's intuitionism and Husserl's phenomenology.

[8]According to Hayashi [47], Tanabe, a philosopher of Kyoto school, was influenced by Brouwer's theory of continuum when he was working on his logic of species (see, e.g., [88, 89]). Species in Tanabe's sense are racial groups, racially homogenenous societies, or nations according to Hayashi [46]. Tanabe was concerned with the relationships between individuals and species in this sense, which he considered were similar to the relationships between a real number and a continuum of real numbers such as intervals in Brouwer's theory of continuum (for details, see [47, 46]).

of geometry based on the point-free perspective on space.[9]

However, we do not abandon the point-set perspective on space and follow a line of thought that is different from those of the above mentioned philosophers and mathematicians. Rather than abolishing it, we would like to consider that the two perspectives on the notion of space are compatible because they express different aspects of the notion of space. Our idea is that the point-set perspective on space represents the ontological aspect of the notion of space, while the point-free perspective on space represents the epistemological aspect of the notion of space. Przywara [78] summarizes Husserl's phenomenology of space as follows (p.5): "Epistemology of space before ontology of space." According to our idea, the point-free epistemology of space and the point-set ontology of space are on a par and neither of them has priority over each other, the reasons of which are as follows.

On the one hand, pointwise reasoning about space has been successful in physics and mathematics.[10] Most of the major concepts of space in them are indeed of point-set nature in the sense as we have seen above. In contrast to the usefulness of pointwise reasoning, we often have some difficulty in point-free reasoning. For example, convex geometry [42], which is usually based on pointwise reasoning, are scarcely developed on the basis of point-free reasoning, since it strongly depends on the notion of point (recall that combinatorial convex geometers often count the number of points as is typical in some Helly-type theorems such as Radon theorem [31]). These could be a reason why the point-set ontology of space should be maintained, as Lewis [62] argued that one should accept the standard, set-theoretic ontology in mathematics since: "the benefits in theoretical unity and economy are well worth the entities" (p.4). In fact, he adovocated his modal realism or his ontology of possible worlds because it is fruitful and gives rise to many benefits. In a nutshell, pointwise reasoning is more useful than point-free reasoning from an ontological point of view,[11] whence we should maintain the point-set perspective on space. No one has the right to get us banned from the fruitful paradise of point sets.

On the other hand, from an epistemological point of view, the point-free approach to space has some definite advantages over the point-set approach. First of all, we cannot perceive any point in space. From the viewpoints of duality theory [9, 21, 54, 77] and algebraic geometry [41, 45], we notice that a point amounts to a prime ideal (or a model in logical terms), which is an infinitary entity, and we need some indeterministic principle such as (a weaker form of) the axiom of choice in order to show the existence of it and therefore the notion of point is very ideal. In contrast to this ideality of the notion of point, we can actually perceive regions of space in some sense and, from the viewpoints of duality theory and algebraic geometry, a (basic) region can be identified with an algebraic formula, which is a finitary entity. Hence, the notion of region seems to be epistemologically more certain than that of point. If we argue more logically, the consistency strength of Aczel's constructive set theory CZF is strictly weaker than the consistency strength of the standard set theory ZFC (see, e.g., [17]).[12] The point-free approach such as formal topology

[9] We have discussed point-free notions of space in contrast to point-set notions of it. We remark that this contrast would be related also to that between Newtonian and Leibnizian concepts of space. If we turn to a broader context, it appears to be related also to the contrast between classical and representational theories of measurement in the sense of Michell [73] and Gray [40, pp.328-346].

[10] From a logical point of view, this is closely related to the usefulness of semantic methods, since pointwise (resp. point-free) reasoning amounts to semantic (resp. syntactic or algebraic) reasoning.

[11] This never means that in every case the former is superior to the latter.

[12] Note that the consistency strength of intuitionistic set theory IZF equals the consistency strength of ZFC in contrast to the case of CZF.

[81, 82] can be formalized within CZF, whereas the point-set approach cannot be so and can only be formalized in ZFC (see, e.g., [35]).[13] In a nutshell, the point-free approach is less ideal than the point-set one from an epistemological point of view, whence we should also maintain the point-free perspective on space.

Hence, we should maintain both of the two perspectives on space, and this seems possible by supposing that they express different aspects of the notion of space. Now it would also have become evident why we consider that the point-set perspective on space represents the ontological aspect of the notion of space, while the point-free perspective on space represents the epistemological aspect of the notion of space. There are a number of models of space representing each perspective on space in mathematics, which we shall discuss in the next subsection.

1.2 Mathematical Concepts of Space: Point-Set vs. Point-Free

In this subsection, we shall see some point-set and point-free concepts of space in mathematics with their features. We do not claim that any notion of space in mathematics can be classified into either point-set or point-free one (some may be both point-set and point-free; some may be neither point-set nor point-free). We omit explanation about mereology (see [18, 83]) because of limitation of space and particularly focus on locale-theoretic ideas, which are most relevant to developments in the later sections.

Point-set concepts of space include the following. A topological space is a set S equipped with a collection $\mathcal{O}$ of subsets of it, which is called the topology of the space, satisfying the following closure properties: $\mathcal{O}$ is closed under finite intersections and arbitrary unions.[14] An element of $\mathcal{O}$ is called an open set of S. In measure theory, we use the concept of measurable space, which is also called Borel space. A mesurable space is a set S equipped with a collection $\mathcal{B}$ of subsets of it such that $\mathcal{B}$ is closed under countable intersections and countable unions. An element of $\mathcal{B}$ is called a measurable set or a Borel set. In convex geometry, we use the concept of convexity space (see [93]). A convexity space is a set S equipped with a collection $\mathcal{C}$ of subsets of it, which is called the convexity of the space, satisfying the following properties: $\mathcal{C}$ is closed under arbitrary intersections and directed unions.[15] An element of $\mathcal{C}$ is called a convex set. These all are point-set concepts of space.[16] Any concept of space based on these spaces is also of point-set nature. For example, we obtain the concept of manifold by equipping certain topological spaces with differential structures and thus a manifold can be considered as a kind of point-set space.

Now, let us turn to various concepts of point-free space. There are several ways to realize the notion of point-free space in a mathematical form. In mathematics, we often encounter the following phenomenon: A point-set space is recovered from the function algebra on it (a point-set space and the function algebra on it have the same amount of information). Such point-set spaces include manifolds, compact Hausdorff spaces, and affine schemes (see [45, 54, 50]). Moreover, geometric notions can often be translated into algebraic ones via the correspondence between space and algebra (for example, in algebraic geometry, the dimension of an algebraic variety corresponds

[13] As is indicated in this case, point-free thinking is deeply relevant to constructive mathematics.

[14] Index sets of intersections and unions may be empty, so that we do not have to explicitly say that a topology on a set includes the empty set and the whole set.

[15] Note that closedness under directed unions is equivalent to closedness under unions of totally ordered subsets (under the assumption of the axiom of choice).

[16] These all are defined as sets with collections of subsets satisfying some closure properties, and can be unified in a categorical framework using the concepts of functor-structured category and topological axiom as we shall see later.

to the Krull dimension of the coordinate ring of it). These facts give us the idea "Algebra itself is space." This idea has been pursued in several areas of mathematics such as non-commutative geometry (see [22]).

Locale theory is an algebraic theory of topological structures that does not presuppose the notion of point and is primarily based on that of region, since locale theory studies the lattice structure of open sets in an algebraic way, i.e., a space in locale theory is a join-complete lattice with finite meets that distribute over arbitrary joins, which is called a frame or a locale (for locale theory, see [54, 65, 76, 79, 95, 96]).[17] The distributivity condition is always satisfied in the lattice of open sets. Frames are the same as complete Heyting algebras, which are certain algebras of intuitionistic logic. Note that the lattice of open sets of a topological space is naturally seen as a function algebra on the space by considering the indicator functions of open sets. Usually, localic versions of theorems in the ordinary topology do not need non-constructive principles such as the law of excluded middle or the axiom of choice and so locale theory can also be seen as constructive topology (see [3, 23, 25, 26]).

In a similar way, we can consider those algebraic theories of measurable structures and of convex structures that do not presuppose the notion of point and is primarily based on that of region. A point-free analogue of a measurable space is a σ-complete Boolean algebra such that countable meets distribute over countable joins. A point-free analogue of a convexity space has been proposed by the author in [70] for developments of point-free convex geometry. It is called a convexity algebra in [70] and is defined as a meet-complete poset with directed joins that distribute over arbitrary meets.[18] Interestingly, this concept of convexity algebra coincides with the concept of continuous lattice (for details, see Conclusions in [70]), which was introduced by Scott [84] for the denotational semantics of programming languages. Note that the usual definition of continuous lattice is totally different from that of convexity algebra.

If we move from algebra to category theory, we can find several other concepts of topological structure. The most famous one would be a topos (see [55, 65, 95]), which is a certain category axiomatizing the categorical structure of sheaves on a topological space. Topoi and frames (or locales) are closely related: the truth value object of a topos always forms a frame; frame-valued sets form a topos; topoi arising as categories of sheaves on frames are called localic topoi and play an important role in topos theory. It is significant from a logical point of view that a topos represents an intuitionistic higher-order logic. Topoi are categorical, point-free concepts of space, since they have no underlying point-set structure. In contrast, the notion of ionad is a categorical, point-set concept of topological structure, though there are the close relationships between topoi and ionads (see [37]).

In the present thesis, we are interested not only in topological spaces, but also in measurable spaces and convexity spaces. Thus, the concept of ionad does not work for our purposes. However, concrete category theory as in [6] actually leads us to a useful concept encompassing all of such point-set spaces, which shall be studied in detail in Section 2. The corresponding algebraic concepts of point-free space (such as frames, σ-complete Boolean algebra, and continuous lattice) can also be unified under the categorical concept of monad as shall be explained in Section 2.

[17] Frames in this sense are totally different from those in modal logic.

[18] To be precise, there are two versions of convexity algebra. One is defined in terms of directed joins, while the other is defined in terms of joins of chains. We discuss the former in the thesis. At the point-set level, the two definitions coincide, but they seem to be slightly different at the point-free level.

1.3 Duality-Theoretic Perspective on Two Aspects of Space

According to our point of view, duality theory aims to study those relationships between algebraic and geometric structures that are dual in the categorical sense. There are plenty of examples of such duality in logic and mathematics: Stone duality between Boolean algebras and zero-dimensional compact Hausdorff spaces (see [21, 54, 86]); Gelfand-Naimark duality between commutative C^* algebras and compact Hausdorff spaces (see [11, 60, 50]); Grothendieck duality between commutative rings and affine schemes (see [45, 41]), and many more.

These dualities all state that certain algebras are dually equivalent to certain point-set-based spatial structures, telling us that those algebras can work as point-free analogues of the corresponding point-set structures. In this sense, duality theory provides foundations of point-free geometry. For instance, non-commutative geometry [22] is indeed founded on Gelfand-Naimark duality and cosndiers non-commutative C^* algebras as "non-commutative spaces."

Duality theory also makes it possible to translate problems about point-free (resp. point-set) spaces into problems about point-set (resp. point-free) spaces. It sometimes happens that a problem can be much more easily solved at the point-set (resp. point-free) level than at the point-free (resp. point-set) level. Here, we can replace "point-free spaces" by "algebras", since they are similar in practice. In this way, duality theory actually have applications in diverse areas of logic and mathematics. Especially, combined with the view that a logic is a point-free space and their models are a point-set space, Stone-type dualities have a number of applications to logical systems and their algebras (see, e.g., [21, 71]).

Taking into account our discussion in Subsection 1.1 above, the ontological and epistemological aspects of space may be thought of as represented by the point-set and point-free perspectives on space respectively. Duality theory thus suggests that the two aspects of space are related in a dual way, since we can find the dual relationships between point-set spaces and point-free spaces in a variety of contexts. One of the best examples of such duality would be Isbell duality (more precisely, adjunction; see [9, 54]) between frames and topological spaces,[19] though the above mentioned dualities are also of similar nature. We consider that what is really significant here is the very way the two perspectives on space are dually related, which we shall discuss in the following.

Given a point-set space, how can we construct the corresponding point-free space? The most typical way would be to consider the algebra of certain predicates or functions on the space. Note that a predicate can be identified with a region in which the predicate holds. In the case of Isbell duality, from a topological space, we derive the algebra of open sets (i.e., regions in topology) in the space. Here, open sets can be seen as (unary) predicates on the space, since an open set determines whether or not a point in the space is included in the open set. We call this construction the "logic-of-space" construction for the reason that the algebra of predicates on a point-set space can be thought of as representing the logic of the space. According to Stone duality, the logic of Cantor space coincides with classical logic. According to Isbell duality, the logic of a topological space in general is a kind of intuitionistic logic (more precisely, a theory over geometric logic; see [95]).

Conversely, given a point-free space, how can we construct the corresponding point-set space? This is where Whitehead's idea explained in Subsection 1.1 comes into the picture. Supposing that a point is a collection of shrinking regions, we can derive the (point-set) space of collections of shrinking regions from a point-free space, where recall that a point-free space looks like an algebra of regions (open sets, measurable sets, convex sets, etc.) in many cases. The idea of shrinking

[19] Recall that frames are point-free, topological structures. Do not confuse "frames" here with those in modal logic.

regions can usually be mathematically formalized as the concept of prime ideal (though the precise definition of prime ideal slightly varies depending on each case). And a prime ideal is a model in logical terms. This is exactly true in a number of logics. Most typically, the two-valued models of classical logic coincide with the prime ideals of the Lindenbaum algebra of classical logic. Thus, the space of collections of shrinking regions amounts to the space of models in logical terms.

This space is still just a point set at the present stage and we have to equip it with a geometric structure like a topology. This is done in the following way. Fix a region a. Consider the set $\langle a \rangle$ of collections P of shrinking regions such that $P \in \langle a \rangle$ iff $a \in P$. All of $\langle a \rangle$ for regions a give the geometric structure of the space of collections of shrinking regions. In the case of Isbell duality, given a frame L, $\{\langle a \rangle \ ; \ a \in L\}$ provides the topology of the space of prime ideals of L. In logical terms, a predicate amounts to a region (consider the region in which the predicate holds) and then $\langle a \rangle$ is the set of models in which a predicate a holds. The work is finally done. This is the way we derive a point-set space from a point-free space.

We call the construction we have just described the "space-of-logic" construction. A point-free space is considered as consisting of regions or predicates, and thus it represents a logic. Then we construct the space of models of the logic (for example, this coincides with Cantor space if the logic concerned is classical logic), which is the point-set space corresponding to the original point-free space. This is why the above construction is called the space-of-logic construction. It would have been evident that logical ideas are lurking throughout the two constructions we have explained. Everything discussed above has a logical meaning.

The two constructions actually give us dual equivalence or dual adjunction between point-set spaces and point-free space in many concrete cases. Here, "dual" means "contravariant." In ordinary terms, it would simply means: The more conditions we require, the less things satisfying them we will have. In logical terms, we may say: The more predicates,[20] the less models. Although the mathematical content of duality is actually more complex, we think that this explanation does not miss the point.

Finally, we would like to emphasize that duality has been found and exploited in a broad range of disciplines. In algebraic geometry, duality between varieties and commutative algebras (Hilbert Nullstellensatz, Grothendieck duality, etc.) appears to be something like the air we breathe, and algebraic geometers freely go back and forth between them (see [45, 41]).[21] In logic, we have duality between models and theories (Stone duality, Jónsson-Tarski duality, etc.), which can be seen as a strengthened version of completeness and has applications in logic and computer science (see [2, 13, 16, 71]). In physics, duality between states and observables (Gelfand-Naimark theorem, Micro-Macro duality, etc.) is important especially for foundations of quantum physics (see [4, 5, 75, 87]). In computer science, we have the following dualities for program semantics: duality between denotations and observable properties of programs, a mathematical representation of which is Isbell duality as Abramsky [1] considers that the space of denotations of programs can be represented by a topological space and that the logic of obseavable properties of programs can be represented by a frame (see [1, 2, 56, 85]); duality between internal structures and external behaviors of computer systems, mathematical representations of which are dualities between coalgebras and modal logics (see [52, 59, 72]). These richly amplify the significance of duality and may be broadly seen as representing the dichotomy between the ontological and epistemological aspects of the notion of space.

[20] We use the term "predicate" in the same sense as "formula."

[21] Algebra and geometry would be the two sides of the same coin in algebraic geometry.

1.4 Overview and Comparison with Related Work

In this subsection, we give an overview of the thesis and a comparison with related work. This subsection is intended mainly for researchers working in a relevant field and so contains some technical explanation about motivation and ideas behind our results.

1.4.1 Overview of the Thesis

In the thesis, we are concerned with dualities between algebraic, point-set spaces (e.g., topological space, convexity space, and measurable space) and set-theoretic, point-free spaces (e.g., frame, continuous lattice, and σ-complete Boolean algebra). Here, recall that a convexity space is a set equipped with a collection of subsets of it that is closed under arbitrary intersections and directed unions (see [93]). Scott's continuous lattices are naturally considered as point-free convexity spaces, since continuous lattices coincide with meet-complete posets with directed joins distributing over arbitrary meets (this follows from [39, Theorem I-2.7]). This observation is important to consider the relationships between continuous lattices and convexity spaces, on which we particularly focus in the concrete dimension of this work.

In Section 2, we first develop a general theory of dualities between algebraic, point-set spaces and set-theoretic, point-free spaces. The basic machinery is concrete category theory as in [6]. We do not try to develop a general theory of every duality, since a too general theory is like saying that a category $\mathbf{C}$ is dually equivalent to $\mathbf{C}^{\mathrm{op}}$, which is the most general duality "theorem", but is also the most trivial one. To avoid this triviality, we stick to duality between point-set spaces and point-free spaces as exemplified by Isbell duality in locale theory.

We employ the concept of (a full subcategory of) the Eilenberg-Moore category of a monad (see [6, 64, 66, 67]) as a general notion of point-free space, since any of frames, continuous lattices, and σ-complete Boolean algebras can be expressed as the Eilenberg-Moore category of a monad. On the other hand, we exploit the concepts of a functor-(co)structured category and topological (co)axioms in it (see [6, 7, 20]) in order to obtain a general notion of point-set space ([6] is a basic reference for these concrete-categorical concepts). In addition, we introduce a new concept of Boolean topological coaxiom, which has certain technical advantages. Any of topological spaces, convexity spaces, and measurable spaces can be expressed as a full subcategory of a (fixed) functor-costructured category that is definable by Boolean topological (co)axioms. We thus consider such a category as a general category of point-set spaces.

In this work, we intend to capture the practice of duality theory for algebraic, point-set spaces and set-theoretic, point-free spaces, and thus stick to concrete ideas in the practice rather than high-level abstractions (e.g., we equip Hom-sets in an algebraic category with generalized topologies similar to Stone and Zariski topologies). And we make explicit the algebraic nature of one category involved and the topological nature of the other category involved (this is discussed in more detail in a comparison with related work below), so that we obtain a moderately general theory of dualities.

We denote by **Alg** and **Spa** a general category of algebraic, point-free spaces and a general category of set-theoretic, point-set spaces respectively. In order to obtain a duality, we assume that there is a "dualizing" object Ω living both in **Alg** and in **Spa**, which is a standard way to develop a general theory of dualities. According to [9], Lawvere said that this idea was due to Isbell. From Ω, we want to construct contravariant Hom functors

$$\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) \text{ and } \mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega).$$

At this stage, however, we have no geometric structure on $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ for $A \in \mathbf{Alg}$ and no algebraic structure on $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ for $S \in \mathbf{Spa}$. In fact, abstracting from Stone and Zariski topology, we can naturally equip $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with a certain geometric structure and consider it as an object in **Spa**, which gives us a functor $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ from **Alg** to **Spa**. On the other hand, although we can provisionally equip $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ with pointwise operations induced by Ω, there is no guarantee that $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is actually closed under those operations. We thus assume "harmony condition", which essentially means that $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is closed under the pointwise operations.[22] The harmony condition gives us a functor $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)$ from **Spa** to **Alg**. Now, under these small assumptions, we can prove that $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ and $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)$ provide a dual adjunction between **Alg** and **Spa**. This is the main result in Section 2.

We emphasize that the harmony condition can be easily checked in concrete cases (for example, in the case of Isbell adjunction between frames and topological spaces, the harmony condition amounts to the fact that any topology is closed under arbitrary unions and finite intersections, which is evident). Hence, the harmony condition is not a big assumption, but a rather small one and easy to verify in concrete cases. This ensures that the adjunction theorem is effective and useful in practice. The adjunction theorem immediately implies: Isbell adjunction between frames and topological spaces; a dual adjunction between continuous lattices and convexity spaces; a dual adjunction between σ-complete Boolean algebras and measurable spaces. The second one was discovered by the author in [70].

What we have constructed is a dual adjunction and is not a dual equivalence in general. There are automatic ways to derive a dual equivalence from a dual adjunction, by which we can obtain a dual equivalence from our dual adjunction. However, there seems to be no substantial way to achieve it categorically in a moderately general context. Some ways are very general and so trivial. Others seem to be too specific and depend on particular situations. Concerning this matter, [77], which appears to be a standard reference for a categorical general theory of dualities, makes an interesting remark (for the definitions of Fix(**Alg**) and Fix(**Spa**), see Definition 2.27):

> The main task for establishing a duality in a concrete situation is now to identify Fix(**Alg**) and Fix(**Spa**). This can be a very hard problem, and this is where categorical guidance comes to an end (p.102).

Note that we have a dual equivalence between Fix(**Alg**) and Fix(**Spa**), which is the maximal one that can be derived from the dual adjunction between **Alg** and **Spa**. Note also that in the above quotation we use our own notations instead of the original ones in [77]. We may say that we can get to a certain point from a generalist point of view, but, to go further, we have to abondon it and analyze individual cases from a particularist point of view. The aim of the next section is to attack the (possibly hard) problem mentioned above in several cases.

In Section 3, we discuss two types of dualities, namely sober-type and T_1-type dualities, between point-set spaces and point-free spaces in concrete cases. We emphasize that there are two notions of point (i.e., prime and maximal ones) for point-free spaces, which we show induce the two types of dualities in many cases. In this section, we specially focus on Scott [84]'s concept of continuous lattice in domain theory and establish sober-type and T_1-type dualities between convexity spaces and continuous lattices, which we shall explain in detail and give full proofs of them. These are the main results in Section 3. These dualities are considered as fundamental for pointfree convex

[22] The harmony condition intuitively means that the "continuous" maps ($\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$) are closed under the "algebraic" operations defined pointwise.

geometry as Isbell duality is for locale theory. Although we have also obtained T_1-type dualities for topology and measurable spaces, we can only describe them without any proof because of limitation of space.

Polytopes are important objects of study in convex geometry (see [24, 42, 93]), and play a crucial role also in our research. It is important that the polytopes in any convexity space form a basis of the convexity. Since there is no similar fact in topology, the concept of polytope seems to indicate a striking difference between topology and convex geometry. In this work, the prime spectrum (Spec in Definition 3.21) of any continuous lattice turns out to be the space of polytopes in a convexity space. Conversely, the space of polytopes in any convexity space is the prime spectrum of a continuous lattice. Thus, polytopes are prime spectrums. Related to this, the space of polytopes in a convexity space is the soberification of the original space.

Regarding the notion of point in point-free convex geometry, we conclude that a Scott-open meet-complete filter (the prime notion of point) actually represents not a point, but a polytope, and an maximal meet-complete filter (the maximal notion of point) properly represents a point in many ordinary cases. In a nutshell, the latter is an actual point, while the former is a "generic" point or an "ideal" point, which makes it possible to generate a polytope as the convex hull of a unique point, as in algebraic geometry a prime ideal is considered as a generic point, which makes it possible to generate an irreducible algebraic variety as the closure of a unique point (see [45]). In this sense, the notion of polytope in convex geometry corresponds to that of irreducible algebraic variety in algebraic geometry.

1.4.2 Comparison with Related Work

Concerning Section 2, [77] would be a basic reference for a categorical general theory of dualities and is closely related to our general theory. We thus compare our general theory with that in [77]. Especially, Theorem 1.7 in [77], which is a dual adjunction theorem, is most relevant to our dual adjunction theorem (Theorem 2.19). The setting of [77] is more general than our one. Indeed, [77] starts with two concrete categories **C** and **D**, and then assume a "schizophrenic" object Ω in both of them with the two conditions on initial lifting. The two initial-lifting conditions seem to be most important assumptions in [77] and make it possible to equip Hom-sets into Ω with suitable structures to induce Hom-functors providing a dual adjunction.

What is inadequate about this result in [77] from our point of view is that the two categories **C** and **D** are symmetric (the assumed initial-lifting conditions are symmetric). In the framework of [77], there is no difference between (the conditions about) the two categories involved. Why is this inadequate? That is because the two categories are not symmetric in practice. In practice, one is an algebraic category (e.g., the category of frames) and the other is a topological category (e.g., the category of topological spaces). Categories **C** and **D** are indeed of very different natures in practice. Thus, we want to endow **C** with an algebraic flavor and **D** with a topological flavor so as to simplify their initial-lifting conditions and lead to a result that is more effective and useful in practice. In the present thesis, we attempt to achieve this by using categorical algebra (a monad and its algebras) and categorical topology (a functor-structured category and topological axioms).

Regarding Section 3, since Isbell duality is sober-type duality for frames, one might think that Isbell duality and our sober-type duality for continuous lattices are more or less similar. However, it does not necessarily hold that dualities of the same type play the same role in each area. Indeed, sober-type dualities for frames and for continuous lattices are of rather different natures. Most ordinary topological spaces (e.g., $\mathbb{R}$ as a topological space) fall into sober-type duality for frames (i.e., according to the sober-type duality, we can recover points from such spaces), while most ordinary convexity spaces (e.g., $\mathbb{R}$ as a convexity space) do not fall into sober-type duality for continuous lattices (i.e., according to the sober-type duality, we cannot recover points from such spaces), which shows a striking difference between the two sober-type dualities.

On the other hand, it turns out that most ordinary convexity spaces fall into T_1-type duality for continuous lattices rather than sober-type one. In this respect, T_1-type duality is better than sober-type one. However, there seems to be no broader dual adjunction behind T_1-type duality, while there is a broader dual adjunction behind sober-type duality. In this respect, sober-type duality seems to be superior to T_1-type duality. We thus may consider that each duality has its own advantage.

Researchers in locale theory have considered the only one notion of point (i.e., completely prime filter or the prime notion of point) and so the only one duality (i.e., Isbell duality), which is sober-type duality. Building on another notion of point, we can show that there is also T_1-type duality in locale theory, which has a certain advantage over Isbell duality. T_1-type duality in locale theory makes it possible to recover points from non-sober T_1 topological spaces such as real and complex algebraic varieties. Note that, in the usual way based on Isbell duality, we cannot recover points from such spaces.

2 General Theory of Concrete Dualities

In this section, we develop a moderately general theory of dualities between algebraic, point-free spaces and set-theoretic, point-set spaces. We first review categorical algebra and categorical topology, and then turn to a general theory based on the concepts of monad, functor-structured category, and topological axiom. Among other things, we introduce a new concept of Boolean topological axiom for the general theory.

Preliminaries

We review basic definitions, and fix terminology and notation for later developments. For the basics of convexity spaces, we refer to [93]. For the basics of frames and continuous lattices, we refer to [39, 54]. For the basics of universal algebra, we refer to [19]. For the basics of category theory, we refer to [6, 64].

We define an infinitary operation on a set S as a function $f : S^A \to S$ such that A is a (possibly infinite) set. An infinitary operation can actually be finitary. We mean by an infinitary algebra a set equipped with a set of infinitary operations on it. We define a partial operation on a set S as a function $f : X^A \to S$ such that X is a (possibly proper) subset of S and that A is a finite set. Note that a partial operation may be total. We mean by a partial algebra a set equipped with a set of partial operations on it. We define a partial infinitary operation on a set S as a function $f : X^A \to S$ such that X is a subset of S and that A is a set. We mean by a partial infinitary algebra a set equipped with a set of partial infinitary operations on it. A frame and a σ-complete Boolean algebra are infinitary algebras. We consider a continuous lattice as a partial infinitary algebra, since a homomorphisms of continuous lattices only preserve joins of directed sets and does not necessarily preserve all joins. A field is a partial algebra.

For a category $\mathbf{C}$ and a faithful functor $U : \mathbf{C} \to \mathbf{Sets}$, a tuple $(\mathbf{C}, U)$ is called a concrete category, where $\mathbf{Sets}$ denotes the category of sets and functions. U is called the underlying functor of the concrete category. Note that U may map different objects of $\mathbf{C}$ to the same object of **Sets**. We often omit the functor U of a concrete category $(\mathbf{C}, U)$ when the choice of U is obvious, and then we simply talk about $\mathbf{C}$ as a concrete category. We can also define the notion of a concrete category over a general category as in [6]. For a category $\mathbf{C}$ and a faithful functor $U : \mathbf{C} \to \mathbf{D}$, $(\mathbf{C}, U)$ is called a concrete category over $\mathbf{D}$. A concrete category (over **Sets**) in this thesis is called a construct in [6].

Top denotes the category of topological spaces and continuous functions. **Conv** denotes the category of convexity spaces and convexity preserving maps (we review this in detail in Section 3). **Meas** denotes the category of measurable spaces and measurable functions. **Frm** denotes the category of frames and their homomorphisms. **ContLat** denotes the category of continuous lattices and their homomorphisms. $\mathbf{BA}_\sigma$ denotes the category of σ-complete Boolean algebras with σ-distributivity and their homomorphisms, where σ-distributivity means that countable joins distribute over countable meets. Let $\mathbf{C}$ be a category. 1_C denotes the identity arrow on $C \in \mathbf{C}$. $1_{\mathbf{C}}$ denotes the identity functor on $\mathbf{C}$.

$\mathcal{Q} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ denotes a contravariant power-set functor.

2.1 Categorical Notion of Algebra

We first introduce the notion of an algebra for an endofunctor. An endofunctor is a functor from a category to the same category.

Definition 2.1. For a category $\mathbf{C}$ and an endofunctor $T : \mathbf{C} \to \mathbf{C}$, an algebra for T or a T-algebra is defined as a tuple $(C, h : T(C) \to C)$ where C is an object and h is an arrow in $\mathbf{C}$. A morphism from a T-algebra $(C, h : T(C) \to C)$ to a T-algebra $(C', h' : T(C') \to C')$ is defined as an arrow $f : C \to C'$ in $\mathbf{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
T(C) & \xrightarrow[h]{} & C \\
\big\downarrow{\scriptstyle T(f)} & & \big\downarrow{\scriptstyle f} \\
T(C') & \xrightarrow[h']{} & C'
\end{array}
.
$$

For simplicity, we sometimes write C instead of $(C, h : T(C) \to C)$.

In mathematics, we often discuss a class of algebras that can be defined by equations or closed under formations of subalgebras, direct products, and homomorphic images, where recall Birkhoff theorem stating that closedness under these constructions is equivalent to definability by equations (see [19]). Such a class of algebras is called a variety in universal algebra. A categorical analogue of a variety is the Eilenberg-Moore category of a monad, which we review in the following.

Definition 2.2. A monad on a category $\mathbf{C}$ is a triple (T, η, μ) such that T is an endofunctor T on $\mathbf{C}$ and that $\eta : 1_{\mathbf{C}} \to T$ and $\mu : T \circ T \to T$ are natural transformations satisfying the following equations between natural transformations:

$$
\begin{array}{rcl}
\mu \circ T\mu & = & \mu \circ \mu T \\
\mu \circ T\eta & = & \mu \circ \eta T = 1_T
\end{array}
$$

where 1_T denotes the identity natural transformation from T to T. An (Eilenberg-Moore) algebra of a monad (T, η, μ) on $\mathbf{C}$ is defined as a tuple $(C, h : T(C) \to C)$ with C and h in $\mathbf{C}$ such that

$$\begin{aligned} h \circ \eta_C &= 1_C \\ h \circ T(h) &= h \circ \mu_C. \end{aligned}$$

A morphism of algebras of a monad is defined as a morphism of algebras for the underlying endofunctor of the monad. The Eilenberg-Moore category of a monad is the category of algebras of a monad and their morphisms.

Broadly speaking, an algebra $(C, h : T(C) \to C)$ of a monad T on **Sets** is a set C equipped with operations induced by h, which is called a structure map. In general, an algebra $(C, h : T(C) \to C)$ of a monad T on a category $\mathbf{C}$ may intuitively be seen as an object C in $\mathbf{C}$ endowed with operations induced by h. For instance, if $\mathbf{C}$ is **Top**, then an algebra of a monad $T : \mathbf{Top} \to \mathbf{Top}$ is a topological algebra, i.e., a topological space with operations on it. In the present thesis, we are mainly interested in algebraic structures with no extra structure (such as topology) and thus:

- we shall mainly focus on a monad on **Sets** rather than an arbitrary category.

The categorical notion of algebra encompasses the universal algebraic notion of algebra as in the following proposition. (for the definition of infinitary algebra, see Subsubsection 2).

Proposition 2.3. *For any (possibly infinitary) algebra A, there is a monad T on* **Sets** *with A an algebra of T.*

Proof. It is straightforward to see that A is an algebra for a (possibly infinitary) polynomial functor P on **Sets**, since A has a set of operations whose arities are sets. We can verify by a standard argument that the category of algebras for P has free objects (i.e., P is a varietor in the sense of [6, Definition 20.53]). Then it follows from [6, Theorem 20.56] that the category of algebras for P is monadic, whence A is an algebra of the monad associated with the adjunction between **Sets** and the category of algebras for P. □

Even some partial infinitary algebras (such as continuous lattices) can be seen as algebras of monads on **Sets** (see Proposition 2.7 below).

- This is the reason why we employ the categorical notion of algebra rather than the universal algebraic notion of algebra.

Incorporating partial and infinitary algebras into universal algebra indeed seems to require some unnecessary complications.

However, it is also possible to formulate the main result (Theorem 2.19) of this section in terms of universal algebra. Hence, the issue is essentially a matter of beauty.

We consider that it is significant to be able to formulate the result both in terms of category theory and in terms of universal algebra, since it implies that the result is to some degree independent of the choice of a framework and may thus be thought of as robust in a certain sense.

Proposition 2.4. *Let T be a monad on* **Sets**. *The forgetful functor from the Eilenberg-Moore category of T to* **Sets** *is representable.*

Proof. For the free algebra F_0 generated by a single element, the forgetful functor is naturally isomorphic to $\mathrm{Hom}(F_0, \text{-})$. □

Concerning the relationships between monads and varieties, the following fact is well known (for its proof, see [6, Theorem 24.7]).

Proposition 2.5. *For any (finitary) variety $\mathcal{V}$, the category of algebras in $\mathcal{V}$ and their homomorphisms is equivalent to the Eilenberg-Moore category of a finitary monad on* **Sets**.

Conversely, for any finitary monad on **Sets**, *there is a (finitary) variety $\mathcal{V}$ such that the category of algebras in $\mathcal{V}$ and their homomorphisms is equivalent to the Eilenberg-Moore category of the monad.*

There are similar facts also in the cases of infinitary variety and of quasivariety (see [6, Remark 24.11]). We remark that thinking of a monad on **Sets** amounts to thinking of a (possibly infinitary) variety in terms of universal algebras.

If (the category corresponding to) a class of algebras can be expressed as (the Eilenberg-Moore category of) algebras of a monad, then it follows from the general theory of monads and varieties that the class of algebras has many nice properties.

However, there are some classes of algebras of interest that cannot be expressed as Eilenberg-Moore algebras of a monad. For example, the category of complete Boolean algebras and the category of complete lattices are not equivalent to the Eilenberg-Moore category of any monad (see [54]).

On the other hand, most of classes of algebras can be expressed as full subcategories of algebras of monads on **Sets**. For this reason, in the present thesis,

- we shall focus on a full subcategory of the Eilenberg-Moore category of a monad in order to develop a general duality theory.

We have the following fact concerning the relationships between a full subcategory of the algebras for an endofunctor and a full subcategory of the Eilenberg-Moore algebras of a monad:

Proposition 2.6. *If a category is a full subcategory of the Eilenberg-Moore category of a monad, then it is a full subcategory of the category of algebras for an endofunctor.*

Let T be an endofunctor on a category **C** *such that the category of algebras for T has free objects. If a category is a full subcategory of the category of algebras for T, then it is a full subcategory of the Eilenberg-Moore category of a monad on* **C**.

Proof. The first claim is evident. By [6, Theorem 20.56], we have the free monad T' generated by T. And the Eilenberg-Moore category of T' is equivalent to the category of the algebras for T. The second claim follows immediately from this fact. □

For future developments, let us see how to consider the category **Frm** of frames, the category **ContLat** of continuous lattices, and the category $\mathbf{BA}_\sigma$ of σ-complete Boolean algebras as Eilenberg-Moore categories of monads on **Sets**.

Proposition 2.7. *(i)* **Frm** *is the Eilenberg-Moore category of a monad on* **Sets**. *(ii)* **ContLat** *is the Eilenberg-Moore category of a monad on* **Sets**. *(iii)* $\mathbf{BA}_\sigma$ *is the Eilenberg-Moore category of a monad on* **Sets**.

Proof. (i) For a set X, let $F(X)$ be the set of downward closed families of finite subsets of X. Then, $F(X)$ is the free frame generated by X. The free construction yields a left adjoint functor to the forgetful functor from **Frm** to **Sets**, and this adjunction can be shown to be monadic.

(ii) For a set X, let $F(X)$ be the set of filters of the power-set of X. Then, $F(X)$ is the free continuous lattice generated by X. The free construction yields a left adjoint functor to the forgetful functor from **ContLat** to **Sets**, and this adjunction can be shown to be monadic.

(iii) It is straightforward to construct free σ-complete Boolean algebras, since the term algebra over a set of variables always exists as a set in the case of σ-complete Boolean algebras. The free construction yields a left adjoint functor to the forgetful functor from $\mathbf{BA}_\sigma$ to **Sets**, and this adjunction can be shown to be monadic. □

2.2 Categorical Notion of Space

In this subsection, we consider a general notion of space that encompasses topological spaces, convexity spaces, and measurable spaces. Our duality theory shall be developed based on this notion of (generalized) space. For the basics of functor-structured category, topological category, and topological (co)axiom, we refer to [6].

We first define the notion of functor-structured category.

Definition 2.8 ([6])**.** Let $(\mathbf{C}, U : \mathbf{C} \to \mathbf{Sets})$ be a concrete category. Then, a category $\mathbf{Spa}(U)$ is defined as follows.

1. An object of $\mathbf{Spa}(U)$ is a tuple $(C, \mathcal{O})$ where $C \in \mathbf{C}$ and $\mathcal{O} \subset U(C)$.
2. An arrow of $\mathbf{Spa}(U)$ from $(C, \mathcal{O})$ to $(C', \mathcal{O}')$ is an arrow $f : C \to C'$ of $\mathbf{C}$ such that $U(f)[\mathcal{O}] \subset \mathcal{O}'$.

A category of the form $\mathbf{Spa}(U)$ is called a functor-structured category. A category of the form $(\mathbf{Spa}(U))^{\mathrm{op}}$ is called a functor-costructured category.

We consider $\mathbf{Spa}(U)$ as a concrete category equipped with a faithful functor $U \circ F : \mathbf{Spa}(U) \to \mathbf{Sets}$ where $F : \mathbf{Spa}(F) \to \mathbf{C}$ is the forgetful functor that maps $(C, \mathcal{O})$ to C.

Then we can show the following (for the definition of topological category, see [6]; note that there are different notions of "topological" category, but we follow the terminology of [6]).

Proposition 2.9 ([6])**.** *Let $(\mathbf{C}, U : \mathbf{C} \to \mathbf{Sets})$ be a concrete category. A functor-structured category $\mathbf{Spa}(U)$ is topological.*

The concept of topological (co)axiom is defined as follows.

Definition 2.10 ([6])**.** Let $(\mathbf{C}, U : \mathbf{C} \to \mathbf{Sets})$ be a concrete category. A topological axiom in $(\mathbf{C}, U)$ is defined as an arrow $p : C \to C'$ of $\mathbf{C}$ such that

1. $U(C) = U(C')$;
2. $U(p) : U(C) \to U(C)$ is the identity on $U(C)$.

An object C of $\mathbf{C}$ satisfies a topological axiom $p : D \to D'$ in $(\mathbf{C}, U)$ iff, for any arrow $f : C \to D$ of $\mathbf{C}$, there is an arrow $f' : C \to D'$ of $\mathbf{C}$ such that $U(f) = U(f')$.

A topological coaxiom is defined as a topological axiom. An object C of $\mathbf{C}$ satisfies a topological coaxiom $p : D' \to D$ in $(\mathbf{C}, U)$ iff, for any arrow $f : C \to D$ of $\mathbf{C}$, there is an arrow $f : C \to D'$ of $\mathbf{C}$ such that $U(f) = U(f')$.

Note that topological axioms and coaxioms are the same, but the corresponding notions of satisfaction are defined in a dual way. For various examples of topological (co)axiom, we refer to [6].

Definition 2.11 ([6])**.** Let X be a class of topological (co)axioms in a concrete category $\mathbf{C}$. A full subcategory $\mathbf{D}$ of $\mathbf{C}$ is definable by X in $\mathbf{C}$ iff the objects of $\mathbf{D}$ coincide with those objects of $\mathbf{C}$ that satisfy any topological (co)axiom in X.

As in the following proposition, we can show a topological analogue of Birkhoff theorem in universal algebra (for details, see Theorem 22.3 and Corollary 22.4 in [6]; note that a construct in [6] is called a concrete category in the thesis). Recall that Birkhoff theorem states the equivalence between definability by equational axioms and closedness under some constructions.

Proposition 2.12 ([6])**.** *Let* $\mathbf{C}$ *be a concrete category. The following are equivalent:*

1. $\mathbf{C}$ *is fibre-small and topological;*
2. $\mathbf{C}$ *is concretely isomorphic to a subcategory of a functor-structured category that is definable by a class of topological axioms in the functor-structured category.*
3. $\mathbf{C}$ *can be concretely embedded in a functor-structured category as a full subcategory that is closed under the formation of products, initial subobjects, and indiscrete objects.*

Now we introduce a new concept of Boolean topological (co)axiom, which shall play a crucial role in establishing our adjunction theorem.

Definition 2.13. For a concrete category $(\mathbf{C}, U : \mathbf{C} \to \mathbf{Sets})$, let us consider $\mathbf{Spa}(U)$. A Boolean topological axiom in $\mathbf{Spa}(U)$ is defined as a topological axiom $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ in $\mathbf{Spa}(U)$ such that

- Any element of $\mathcal{O}' \setminus \mathcal{O}$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}$.

A Boolean topological coaxiom in $(\mathbf{Spa}(U))^{\mathrm{op}}$ is defined as a topological coaxiom $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ in $(\mathbf{Spa}(U))^{\mathrm{op}}$ such that

- Any element of $\mathcal{O} \setminus \mathcal{O}'$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}'$.

Let $\mathcal{Q} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ denote a contravariant power-set functor. Any of the category **Top** of topological spaces, the category **Conv** of convexity spaces, and the category **Meas** of measurable spaces is a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of Boolean topological coaxioms as in the following proposition.

Proposition 2.14. **Top** *is definable by the following class of Boolean topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*:*

$$\begin{aligned} 1_S : (S, \{\emptyset, S\}) &\to (S, \emptyset) \\ 1_S : (S, \{X, Y, X \cap Y\}) &\to (S, \{X, Y\}) \\ 1_S : (S, \mathcal{O} \cup \{\bigcup \mathcal{O}\}) &\to (S, \mathcal{O}) \end{aligned}$$

for all sets S, all subsets X, Y of S and all subsets $\mathcal{O}$ of the power-set of S.

Conv *is definable by the following class of Boolean topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*:*

$$\begin{array}{rcl} 1_S : (S, \{\emptyset, S\}) & \to & (S, \emptyset) \\ 1_S : (S, \mathcal{C} \cup \{\bigcap \mathcal{C}\}) & \to & (S, \mathcal{C}) \\ 1_S : (S, \mathcal{C}' \cup \{\bigcup \mathcal{C}'\}) & \to & (S, \mathcal{C}') \end{array}$$

for all sets S, all subsets $\mathcal{C}$ of the power-set of S, and all those subsets $\mathcal{C}'$ of the power-set of S that are directed with respect to the set inclusion.

Meas *is definable by the following class of Boolean topological coaxioms in* $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$*:*

$$\begin{array}{rcl} 1_S : (S, \{\emptyset, S\}) & \to & (S, \emptyset) \\ 1_S : (S, \{X, X^c\}) & \to & (S, \{X\}) \\ 1_S : (S, \mathcal{B} \cup \{\bigcup \mathcal{B}\}) & \to & (S, \mathcal{B}) \end{array}$$

for all sets S, all subsets X of S, and all those subsets $\mathcal{B}$ of the power-set of S that are of the cardinality $\leq \omega$.

Proof. These are straightforward to prove. Note that the intersection of an empty set is the whole set and that the union of an empty set is an empty set. □

Thus, in order to develop a general duality theory,

- we shall focus on a full subcategory **Spa** of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of Boolean topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$.

We call $(S, \mathcal{O}) \in (\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ a generalized space and $\mathcal{O}$ a generalized topology.

Proposition 2.15. *The forgetful functor from* **Spa** *to* **Sets** *is representable.*

Proof. Define $(S_0, \mathcal{O}_0) \in \mathbf{Spa}$ by $S_0 = \{*\}$ and $\mathcal{O}_0 = \{\emptyset, \{*\}\}$. The forgetful functor from **Spa** to **Sets** is naturally isomorphic to $\mathrm{Hom}_{\mathbf{Spa}}((S_0, \mathcal{O}_0), \text{-})$. □

Given a subset $\mathcal{P}$ of the power-set of a set S, we can generate a topology on S from $\mathcal{P}$, which is the weakest topology containing $\mathcal{P}$. We can do the same thing also in the case of generalized topology defined above.

Proposition 2.16. *For a set S, let $\mathcal{P}$ be a subset of the power-set of S. Then, there is a weakest generalized topology on S containing $\mathcal{P}$ in* **Spa***, i.e., there is $(S, \mathcal{O}) \in \mathbf{Spa}$ such that, if $\mathcal{P} \subset \mathcal{O}'$ for $(S, \mathcal{O}') \in \mathbf{Spa}$, then $\mathcal{O} \subset \mathcal{O}'$.*

Proof. Define

$$\mathcal{O} = \bigcap \{\mathcal{X} \ ; \ \mathcal{P} \subset \mathcal{X} \text{ and } (S, \mathcal{X}) \in \mathbf{Spa}\}.$$

It is sufficient to show that $\mathcal{O}$ is a generalized topology on S in **Spa**, i.e., $(S, \mathcal{O})$ satisfies the class of topological coaxioms that define **Spa**. Assume that $p : (X, \mathcal{B}') \to (X, \mathcal{B})$ is one of such coaxioms and that $f : (S, \mathcal{O}) \to (X, \mathcal{B})$ is an arrow in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$. For $B \in \mathcal{B}' \setminus \mathcal{B}$, we have

$$f^{-1}(B) \in \mathcal{X} \text{ for any } \mathcal{X} \text{ with } \mathcal{P} \subset \mathcal{X} \text{ and } (S, \mathcal{X}) \in \mathbf{Spa},$$

which implies that $f^{-1}(B) \in \mathcal{O}$. This completes the proof. □

We say that $\mathcal{O}$ above is generated in **Spa** by $\mathcal{P}$.

2.3 Dual Adjunction via Harmony Condition

Throughout this subsection, let

- **Alg** denote a full subcategory of the Eilenberg-Moore category of a monad T on **Sets**;
- **Spa** denote a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of Boolean topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$.

Our aim in this subsection is to show a dual adjunction between **Alg** and **Spa** under the following assumptions:

- there is an object Ω living both in **Alg** and in **Spa**, i.e., there is $\Omega \in \mathbf{Sets}$ both with a structure map $h_\Omega : T(\Omega) \to \Omega$ such that $(\Omega, h_\Omega) \in \mathbf{Alg}$ and with a generalized topology $\mathcal{O}_\Omega \subset \mathcal{Q}(\Omega)$ such that $(\Omega, \mathcal{O}_\Omega) \in \mathbf{Spa}$;
- $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ satisfies the harmony condition in Definition 2.17 below.

As a duality theorist expects, Ω shall work as a so-called dualizing object (we do not use the term "schizophrenic", since it has a different technical meaning in some context). We often write just Ω instead of (Ω, h_Ω) or $(\Omega, \mathcal{O}_\Omega)$ when there is no confusion.

The harmony condition intuitively means that the algebraic structure of **Alg** and the geometric structure of **Spa** are in harmony via Ω. The precise definition is given below.

Definition 2.17. $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ is said to satisfy the harmony condition iff, for each $S \in \mathbf{Spa}$,

$$(\ \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega), \ h_S : T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \ )$$

is an object in **Alg** such that, for any $s \in S$ (let p_s be the corresponding projection from $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ to Ω), the following diagram commutes:

$$\begin{array}{ccc} T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) & \xrightarrow{h_S} & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \\ \downarrow{\scriptstyle T(p_s)} & & \downarrow{\scriptstyle p_s} \\ T(\Omega) & \xrightarrow{h_\Omega} & \Omega \end{array} .$$

Remark 2.18. The commutative diagram above means that the induced operations of $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ are defined pointwise. The harmony condition then consists of the following two parts:

(i) $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is closed under the pointwise operations;

(ii) $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ with the operations is in **Alg**.

Here, (ii) is not so important, which is because we can drop the condition (ii) with certain modifications of **Alg**:

- If we let **Alg** be the Eilenberg-Moore category of a monad on **Sets**, then **Alg** has products, and any product in **Alg** is preserved by the forgetful functor from **Alg** to **Sets** (see [54, 66]). This implies that Ω^S equipped with the pointwise operations is in **Alg**. Since the Eilenberg-Moore algebras of a monad on **Sets** have subalgebras in the sense of universal algebras (see [54, 66]), it follows from (i) above and $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \subset \Omega^S$ that $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is in **Alg**.

- If we let **Alg** be an implicational full subcategory of the category of algebras for an endofunctor on **Sets**, then **Alg** has products preserved by the forgetful functor and also has subalgebras, whence, by the same argument as above, $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is in **Alg** (for the definition of an implicational subcategory of algebras and its properties, see [6]).

If we see $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ as a collection of generalized continuous functions on S, then (i) above means that the *continuous* functions are closed under the pointwise *algebraic* operations defined pointwise, which is the most important part of the harmony condition, and after which the *harmony* condition is named.

We assume the harmony condition in the following part of this subsection. The algebraic structure of $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is provided by h_S above.

The geometric structure of $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ for $A \in \mathbf{Alg}$ can be provided in the following way. By Proposition 2.16, we can equip $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with the generalized topology generated in **Spa** by

$$\{\langle a \rangle_O \ ; \ a \in A \text{ and } O \in \mathcal{O}_\Omega\}$$

where

$$\langle a \rangle_O := \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \ ; \ v(a) \in O\}.$$

The induced contravariant Hom-functors

$$\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg} \to \mathbf{Spa} \text{ and } \mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega) : \mathbf{Spa} \to \mathbf{Alg}$$

can be shown to be well defined and form a dual adjunction between categories **Alg** and **Spa**. More precisely:

Theorem 2.19. $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ *is left adjoint to* $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)^{\mathrm{op}}$.

The proof of the dual adjunction theorem is given in Subsubsection 2.3.1 below.

This theorem encompasses a dual adjunction between frames and topological spaces, a dual adjunction between continuous lattices and convexity spaces, and a dual adjunction between σ-complete Boolean algebras and measurable spaces. The first dual adjunction is well known. In Subsection 3.1, we explain in detail how to get the second dual adjunction by the above general theorem. The third one can be obtained in a similar way.

2.3.1 Proof of Dual Adjunction Theorem

We first show that the two Hom-functors are well defined.

Lemma 2.20. *A contravariant functor*

$$\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg} \to \mathbf{Spa}$$

is well defined.

Proof. The object part is well defined by Proposition 2.16. We show that the arrow part is well defined. Let $f : A \to A'$ be an arrow in **Alg**. We prove that

$$\mathrm{Hom}_{\mathbf{Alg}}(f, \Omega) : \mathrm{Hom}_{\mathbf{Alg}}(A', \Omega) \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$$

is an arrow in **Spa**. For $a \in A$ and $O \in \mathcal{O}_\Omega$, we have:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{Alg}}(f,\Omega)^{-1}(\langle a \rangle_O) & = & \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A',\Omega) \ ; \ \mathrm{Hom}_{\mathbf{Alg}}(f,\Omega)(v) \in \langle a \rangle_O\} \\ & = & \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A',\Omega) \ ; \ v \circ f(a) \in O\} \\ & = & \langle f(a) \rangle_O. \end{array}$$

Since **Spa** is definable by a class of Boolean topological coaxioms and since Boolean set operations are preserved by the inverse image function f^{-1}, this implies that $\mathrm{Hom}_{\mathbf{Alg}}(f,\Omega)$ is an arrow in **Spa**. □

Lemma 2.21. *A contravariant functor*

$$\mathrm{Hom}_{\mathbf{Spa}}(\text{-},\Omega) : \mathbf{Spa} \to \mathbf{Alg}$$

is well defined.

Proof. The object part is well defined by the harmony condition (or can be verified as in (i) or (ii) in Remark 2.18 if we employ either of the other two definitions of **Alg**).

We show that the arrow part is well defined. Let $f : S \to S'$ be an arrow in **Spa**. We prove that

$$\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega) : \mathrm{Hom}_{\mathbf{Spa}}(S',\Omega) \to \mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$$

is an arrow in **Alg**, i.e., the following diagram commutes:

$$\begin{array}{ccc} T(\mathrm{Hom}_{\mathbf{Spa}}(S',\Omega)) & \xrightarrow{h_{S'}} & \mathrm{Hom}_{\mathbf{Spa}}(S',\Omega) \\ {\scriptstyle T(\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Spa}}(f,\Omega)} \\ T(\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S,\Omega) \end{array}$$

By the harmony condition applied to S (or the commutativity of the lower square in the figure below), this is equivalent to the commutativity of the outermost square in the following diagram for any $s \in S$:

$$\begin{array}{ccc} T(\mathrm{Hom}_{\mathbf{Spa}}(S',\Omega)) & \xrightarrow{h_{S'}} & \mathrm{Hom}_{\mathbf{Spa}}(S',\Omega) \\ {\scriptstyle T(\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Spa}}(f,\Omega)} \\ T(\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S,\Omega) \\ {\scriptstyle T(p_s)}\downarrow & & \downarrow{\scriptstyle p_s} \\ T(\Omega) & \xrightarrow[h_\Omega]{} & \Omega \end{array}$$

where recall that p_s denotes the corresponding projection. By the harmony condition applied to S', we have: for any $s' \in S'$,

$$h_\Omega \circ T(p_{s'}) = p_{s'} \circ h_{S'}.$$

By taking $s' = f(s)$ in this equation, we have

$$h_\Omega \circ T(p_{f(s)}) = p_{f(s)} \circ h_{S'}.$$

It is straightforward to verify that

$$p_{f(s)} = p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega).$$

Thus we obtain

$$h_\Omega \circ T(p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega)) = p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega) \circ h_{S'}.$$

Since T is a functor, this yields the commutativity of the outermost square above. Hence, the arrow part is well defined. □

Now we define two natural transformations in order to show the dual adjunction.

Definition 2.22. Natural transformations

$$\Phi : 1_{\mathbf{Alg}} \to \mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega), \Omega)$$

and

$$\Psi : 1_{\mathbf{Spa}} \to \mathrm{Hom}_{\mathbf{Alg}}(\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega), \Omega)$$

are defined as follows. For $A \in \mathbf{Alg}$, define Φ_A by

$$\Phi_A(a)(v) = v(a)$$

where $a \in A$ and $v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$. For $S \in \mathbf{Spa}$, define Ψ_S by

$$\Psi_S(x)(f) = f(x)$$

where $x \in S$ and $f \in \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$.

We have to show that Φ and Ψ are well defined.

Lemma 2.23. *For $A \in \mathbf{Alg}$ and $a \in A$, $\Phi_A(a)$ is an arrow in* **Spa**.

Proof. For $O \in \mathcal{O}_\Omega$, we have

$$\Phi_A(a)^{-1}(O) = \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \ ; \ \Phi_A(a)(v) \in O\} = \langle a \rangle_O.$$

Thus, $\Phi_A(a)$ is an arrow in **Spa**. □

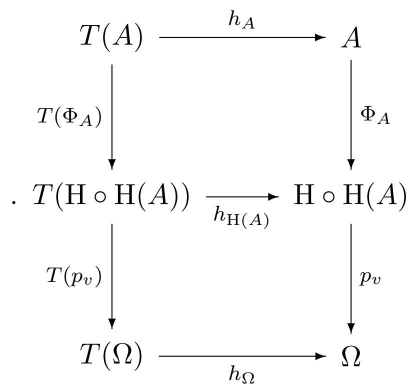
Lemma 2.24. *For $S \in \mathbf{Spa}$ and $x \in S$, $\Psi_S(x)$ is an arrow in* **Alg**.

Proof. This lemma follows immediately from the harmony condition applied to $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ together with the fact that $p_x = \Psi_S(x)$. □

We also have to show that Φ_A is an arrow in **Alg** and that Ψ_S is an arrow in **Spa**.

Lemma 2.25. *For $A \in \mathbf{Alg}$, Φ_A is an arrow in* **Alg**.

Proof. Let $h_A : T(A) \to A$ denote the structure map of A. For the simplicity of description, let $\mathrm{H}(A)$ denote $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ and $\mathrm{H} \circ \mathrm{H}(A)$ denote $\mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega), \Omega)$. In order to show the commutativity of the upper square in the diagram below, it is sufficient to prove that the outermost square is commutative for any $v \in \mathrm{H}(A)$, since the lower square is commutative because of the harmony condition applied to $\mathrm{H} \circ \mathrm{H}(A)$.



It is straightforward to verify that

$$p_v \circ \Phi_A = v.$$

Then, it suffices to show that

$$v \circ h_A = h_\Omega \circ T(v).$$

This is nothing but the fact that $v \in \mathrm{H}(A)$. $\square$

Lemma 2.26. *For $S \in \mathbf{Spa}$, Ψ_S is an arrow in $\mathbf{Spa}$.*

Proof. For $f \in \mathrm{Hom}_{\mathbf{Alg}}(\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega), \Omega)$ and $O \in \mathcal{O}_\Omega$, we have

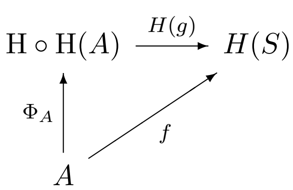
$$\Psi_S^{-1}(\langle f \rangle_O) = \{x \in S \ ; \ \Psi_S(x) \in \langle f \rangle_O\} = f^{-1}(O).$$

Since $\mathbf{Spa}$ is definable by a class of Boolean topological coaxioms and since Boolean set operations are preserved by the inverse image function Ψ_S^{-1}, this implies that Ψ_S is an arrow in $\mathbf{Spa}$. $\square$

Now it is straightforward to verify that Φ and Ψ are actually natural transformations. We finally obtain the dual adjunction theorem.

Theorem 2.19 $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ is left adjoint to $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)^{\mathrm{op}}$ with Φ the unit and Ψ^{op} the counit of the adjunction.

Proof. Let $A \in \mathbf{Alg}$ and $S \in \mathbf{Spa}$. It is enough to show that, for any $f : A \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ in $\mathbf{Alg}$, there is a unique $g : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ in $\mathbf{Spa}$ such that the following diagram commutes:



where $\mathrm{H}(S)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$, $\mathrm{H}(g)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(g, \Omega)$, $\mathrm{H}(A)$ denotes $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ and $\mathrm{H} \circ \mathrm{H}(A)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega), \Omega)$. We first show that such g exists. Define $g : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ by

$$g(x)(a) = \Psi_S(x)(f(a))$$

where $x \in S$ and $a \in A$. Then we have

$$\begin{aligned}(\mathrm{Hom}_{\mathbf{Spa}}(g, \Omega) \circ \Phi_A(a))(x) &= (\Phi_A(a) \circ g)(x) = g(x)(a) \\ &= \Psi_S(x)(f(a)) = f(a)(x).\end{aligned}$$

Thus, the above diagram commutes for this g. It remains to show that g is an arrow in **Spa**. For $a \in A$ and $O \in \mathcal{O}_\Omega$, we have

$$\begin{aligned}g^{-1}(\langle a \rangle_O) &= \{x \in S \ ; \ g(x) \in \langle a \rangle_O\} \\ &= \{x \in S \ ; \ g(x)(a) \in O\} \\ &= \{x \in S \ ; \ f(a)(x) \in O\} \\ &= f(a)^{-1}(O).\end{aligned}$$

Since $f(a) \in \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ and since **Spa** is definable by a class of Boolean topological coaxioms, this implies that g is an arrow in **Spa**.

Finally, in order to show the uniqueness of such g, we assume that $g' : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ in **Spa** makes the above diagram commute. Then we have

$$f(a)(x) = (\mathrm{Hom}_{\mathbf{Spa}}(g', \Omega) \circ \Phi_A(a))(x) = (\Phi_A(a) \circ g')(x) = g'(x)(a).$$

Since we also have $f(a)(x) = g(x)(a)$, it follows that $g = g'$. This completes the proof. □

We shall discuss how to derive a dual equivalence from a dual adjunction in Subsubsection 2.4.1 below.

2.4 Derivation of Dual Equivalence and Extended Adjunction for Additional Structures

In this subsection, we consider how to derive a dual equivalence from a dual adjunction and how to deal with additional algebraic structures by extending our framework.

2.4.1 Deriving Equivalence from Adjunction

Assume that $F : \mathbf{C} \to \mathbf{D}$ is left adjoint to $G : \mathbf{D} \to \mathbf{C}$ with Φ and Ψ the unit and the counit of the adjunction respectively. How can we derive a categorical equivalence from the adjunction? We define a full subcategory $\mathrm{Fix}(\mathbf{C})$ of $\mathbf{C}$ and a full subcategory $\mathrm{Fix}(\mathbf{D})$ of $\mathbf{D}$ as follows.

Definition 2.27. $\mathrm{Fix}(\mathbf{C})$ is a full subcategory of $\mathbf{C}$ such that $C \in \mathrm{Fix}(\mathbf{C})$ iff Φ_C is an isomorphism in $\mathbf{C}$. $\mathrm{Fix}(\mathbf{D})$ is a full subcategory of $\mathbf{D}$ such that $D \in \mathrm{Fix}(\mathbf{D})$ iff Ψ_D is an isomorphism in $\mathbf{D}$.

Then we have the following proposition, which is easy to prove.

Proposition 2.28. $\mathrm{Fix}(\mathbf{C})$ *and* $\mathrm{Fix}(\mathbf{D})$ *are categorically equivalent. Moreover, this equivalence is the maximal one that can be derived from the adjunction between* $\mathbf{C}$ *and* $\mathbf{D}$*.*

If we require a condition about the original adjunction, we have another way to describe Fix($\mathbf{C}$) and Fix($\mathbf{D}$). We first introduce the following notations.

Definition 2.29. Img($\mathbf{C}$) is a full subcategory of $\mathbf{C}$ such that $C \in \text{Img}(\mathbf{C})$ iff $C \simeq G(D)$ for some $D \in \mathbf{D}$. Img($\mathbf{D}$) is a full subcategory of $\mathbf{D}$ such that $D \in \text{Img}(\mathbf{D})$ iff $D \simeq F(C)$ for some $C \in \mathbf{C}$.

Then we have the following proposition, which is also easily verified.

Proposition 2.30. *Assume that* $F(C) \in \text{Fix}(\mathbf{D})$ *for any* $C \in \mathbf{C}$ *and that* $G(D) \in \text{Fix}(\mathbf{C})$ *for any* $D \in \mathbf{D}$*. It then holds that* $\text{Img}(\mathbf{C}) = \text{Fix}(\mathbf{C})$ *and* $\text{Img}(\mathbf{D}) = \text{Fix}(\mathbf{D})$*. Hence,* Img($\mathbf{C}$) *and* Img($\mathbf{D}$) *are categorically equivalent.*

Note that the above assumption is satisfied in the case of Isbell duality between spatial frames and sober spaces.

2.4.2 Extended Adjunction: How to Deal with Additional Structures

Spa is a category of "pure" (point-set) spaces. We mean by "pure" that there is no additional structure on them and **Spa** does not encompass spaces with algebraic operations. In some dualities, **Spa** is actually equipped with algebraic structures. Thus, we discuss how to deal with additional structures by extending our framework,

In order to discuss (point-set) spaces with algebraic structures, we replace the contravariant power-set functor on **Sets** (i.e., $\mathcal{Q} : \mathbf{Sets}^{\text{op}} \to \mathbf{Sets}$) with the contravariant power-set functor on the Eilenberg-Moore category of a monad on **Sets**, which is defined below.

Definition 2.31. Let $(\mathbf{C}, U)$ be a concrete category. Then, the contravariant power-set functor $\mathcal{Q}_{\mathbf{C}} : \mathbf{C}^{\text{op}} \to \mathbf{Sets}$ on $\mathbf{C}$ is defined as follows. Given an object $C \in \mathbf{C}$, define

$$\mathcal{Q}_{\mathbf{C}}(C) = \mathcal{Q}(U(C)).$$

Given an arrow $f : C \to D$ in $\mathbf{C}$, define

$$\mathcal{Q}_{\mathbf{C}}(f) = Q(U(f)).$$

In a nutshell, $\mathcal{Q}_{\mathbf{C}}$ is defined as $\mathcal{Q} \circ U$.

Since $\mathcal{Q}_{\mathbf{Sets}}$ coincides with $\mathcal{Q}$, this definition seems to be right.

Now, we can define **Spa** with additional algebraic structures in the following way.

- Let $\mathbf{D}$ be a full subcategory of the Eilenberg-Moore category of a monad F on **Sets**. Define $\mathbf{Spa}_{\mathbf{D}}$ as a full subcategory of $\mathbf{Spa}(\mathcal{Q}_{\mathbf{D}})^{\text{op}}$ that is definable by Boolean topological coaxioms in $\mathbf{Spa}(\mathcal{Q}_{\mathbf{D}})^{\text{op}}$.
- Let **Alg** be a full subcategory of the Eilenberg-Moore category of a monad T on **Sets**.

Then, $\mathbf{Spa}_{\mathbf{D}}$ can be naturally considered as a concrete category (the underlying functor is obvious). Intuitively, an obejct in $\mathbf{Spa}_{\mathbf{D}}$ may be seen as a space with an algebraic structure.

In order to obtain a dual adjunction between **Alg** and $\mathbf{Spa}_{\mathbf{D}}$, we assume the following:

- We have Ω in **Sets** satisfying the following two conditions:

(1) there is a structure map $h_\Omega : T(\Omega) \to \Omega$ such that $(\Omega.h_\Omega) \in \mathbf{Alg}$.

(2) there are a generalized topology $\mathcal{O}_\Omega \subset \mathcal{Q}(\Omega)$ and a structure map $g_\Omega : F(\Omega) \to \Omega$ such that $((\Omega, g_\Omega), \mathcal{O}_\Omega) \in \mathbf{Spa_D}$.

- $(\mathbf{Alg}, \mathbf{Spa_D}, \Omega)$ satisfies the (extended) harmony condition in Definition 2.32 below.

By Ω, we often mean $((\Omega, g_\Omega), \mathcal{O}_\Omega)$ or $(\Omega.h_\Omega)$ when there is no confusion.

For $A \in \mathbf{Alg}$, the generalized topology of $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ can be provided as in the above subsection and is denoted by $\mathcal{O}_A$.

Definition 2.32. $(\mathbf{Alg}, \mathbf{Spa_D}, \Omega)$ is said to satisfy the (extended) harmony condition iff the following conditions hold:

(1) for each $S \in \mathbf{Spa_D}$,

$$(\ \mathrm{Hom}_{\mathbf{Spa_D}}(S, \Omega),\ h_S : T(\mathrm{Hom}_{\mathbf{Spa_D}}(S, \Omega)) \to \mathrm{Hom}_{\mathbf{Spa_D}}(S, \Omega)\ )$$

is an object in $\mathbf{Alg}$ such that, for any $s \in S$, the following diagram commutes:

$$\begin{array}{ccc} T(\mathrm{Hom}_{\mathbf{Spa_D}}(S, \Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa_D}}(S, \Omega) \\ \downarrow{\scriptstyle T(p_s)} & & \downarrow{\scriptstyle p_s} \\ T(\Omega) & \xrightarrow[h_\Omega]{} & \Omega \end{array} .$$

(2) for each $A \in \mathbf{Alg}$,

$$(\ (\ \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega),\ h_A : F(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)) \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)\ ),\ \mathcal{O}_A\ )$$

is an object in $\mathbf{Spa_D}$ such that, for any $a \in A$, the following diagram commutes:

$$\begin{array}{ccc} F(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)) & \xrightarrow[h_A]{} & \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \\ \downarrow{\scriptstyle F(p_a)} & & \downarrow{\scriptstyle p_a} \\ F(\Omega) & \xrightarrow[g_\Omega]{} & \Omega \end{array} .$$

The induced contravariant Hom-functors

$$\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg} \to \mathbf{Spa_D} \text{ and } \mathrm{Hom}_{\mathbf{Spa_D}}(\text{-}, \Omega) : \mathbf{Spa_D} \to \mathbf{Alg}$$

can be shown to be well defined and form a dual adjunction between categories $\mathbf{Alg}$ and $\mathbf{Spa_D}$. More precisely:

Theorem 2.33. $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ *is left adjoint to* $\mathrm{Hom}_{\mathbf{Spa_D}}(\text{-}, \Omega)^{\mathrm{op}}$.

We omit the proof of this theorem, since it is a straightforward extension of the proof of Theorem 2.19.

3 Two Types of Dualities between Point-Set and Point-Free Spaces

In this section, we study two types of dualities between point-set spaces (topological spaces, convexity spaces, measurable spaces) and point-free spaces (frames, continuous lattices, σ-complete Boolean algebras). Because of limitation of space, we particularly focus on dualities between continuous lattices and convexity spaces.

The general theory we have developed leads us to a dual adjunction between continuous lattices and convexity spaces. However, to go further and obtain a full duality, we have to analyze this case in more detail from a particularist point of view by paying our attention to what is specific to convex structures. Among ohter things, the concept of polytope is peculiar to this case and shall play a crucial role in the following investigation.

We also discuss dualities related to topology and measure theory, and briefly state several theorems without giving their proofs. Manuscripts giving their proods are available upon request.

3.1 Convexity Spaces and Scott's Continuous Lattices

In this subsection we review the basics of convexity spaces and continuous lattices with related basic concepts and propositions.

3.1.1 Convexity Spaces

We first review the notion of convexity space. For more detailed exposition, we refer to [24, 93]. Convexity spaces are sometimes called aligned spaces as in [24]. Let 2^X denote the powerset of X.

Definition 3.1 ([24, 93])**.** For a set S and a subset $\mathcal{C}$ of 2^S, $(S,\mathcal{C})$ is a convexity space iff $(S,\mathcal{C})$ satisfies the following conditions:

1. $\mathcal{C}$ is closed under arbitrary intersections;
2. if $\{X_i \in \mathcal{C} \; ; \; i \in I\}$ is directed, then $\bigcup\{X_i \; ; \; i \in I\} \in \mathcal{C}$.

We call $\mathcal{C}$ the convexity of S and an element of $\mathcal{C}$ a convex set in S. The complement of a convex set in S is called a concave set in S.

Note that $\emptyset, S \in \mathcal{C}$ by letting the index sets be empty in the above conditions.

A convexity space $(S,\mathcal{C})$ is often denoted by its underlying set S.

Remark 3.2. In the above definition of convexity space, we can replace the condition that $\mathcal{C}$ is closed under unions of directed subsets with the condition that $\mathcal{C}$ is closed under unions of totally ordered subsets. The two conditions are equivalent under the assumption of the axiom of choice (see [93]).

Let us denote by **2** the two-element distributive lattice $\{0,1\}$ equipped with the Sierpiński convexity $\{\emptyset,\{1\},\{0,1\}\}$.

Example 3.3. Consider a vector space V over the real number field $\mathbb{R}$. We can equip V with a natural convexity determined by the condition that $X \subset V$ is convex iff for any $x, y \in X$ and any $t \in [0,1]$, $tx + (1-t)y \in X$. In particular, the n-dimensional Euclidean space $\mathbb{R}^n$ for an integer $n \geq 1$ is naturally equipped with a convexity in this way.

Consider the n-sphere S^n for an integer $n \geq 1$. We can equip S^n with a natural convexity determined by the condition that $X \subset S^n$ is convex iff the following hold: (i) for any $x, y \in X$, the antipodal point of x is not y; (ii) for any $x, y \in X$, the shortest path (i.e., geodesic) between x and y on S^n is a subset of X.

Let P be a poset. Then, we can equip P with a convexity in the following way: $X \subset P$ is convex iff, if $x \leq z \leq y$ for $x, y \in X$ and $z \in P$, then $z \in X$. For example, closed intervals $[a, b] = \{x \in P \ ; \ a \leq x \leq b\}$ are convex in this sese.

Given an algebra (in the sense of universal algebra), the set of congruences on it forms a convexity on the algebra. For example, the set of ideals of a lattice forms a convexity on the lattice.

We can also equip the n-dimensional real projective space with a convexity (see [24, Example 6.2.7] or [90]).

By the condition 1 in Definition 3.1, we can define the convex hull of a subset of a convexity space as follows.

Definition 3.4 ([24, 93])**.** Let $(S, \mathcal{C})$ be a convexity space. For $A \subset S$, define

$$\mathrm{ch}(A) = \bigcap \{C \in \mathcal{C} \ ; \ A \subset C\}.$$

Then, $\mathrm{ch}(A)$ is called the convex hull of A.

As usual we define a morphism of convexity spaces as follows.

Definition 3.5 ([93, 58])**.** Let $(S, \mathcal{C})$, $(S', \mathcal{D})$ be convexity spaces. A map $f : S \to S'$ is a convexity preserving map iff, for any $D \in \mathcal{D}$, we have $f^{-1}(D) \in \mathcal{C}$.

This definition of morphism of convexity spaces seems to be most popular, though other definitions may be possible. Even if a stronger definition of morphism of convexity spaces is employed, our duality results still work by restricting the morphisms parts of the related categories.

Note that the inverse map of a bijective convexity preserving map is not necessarily convexity preserving.

By the following proposition, we can consider the set of convex sets in a convexity space as the hom-set from the convexity space to $\mathbf{2}$.

Proposition 3.6. *Let $(S, \mathcal{C})$ be a convexity space. Then, there is a natural bijection between the set $\mathcal{C}$ of all convex sets in S and the set of all convexity preserving maps from S to $\mathbf{2}$.*

Proof. For a convex set C in S, define $f_C : S \to \mathbf{2}$ by

$$f_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that f_C is a convexity preserving map and that if $C \neq D$ for convex sets C and D in S then we have $f_C \neq f_D$. Thus, the map $C \mapsto f_C$ is injective. To show the surjectivity, let g be a convexity preserving map from S to $\mathbf{2}$. Define $C = g^{-1}(\{1\})$. Then it is clear that C is a convex set in S and that $f_C = g$. This completes the proof. □

The notion of polytope is defined as follows.

Definition 3.7 ([93]). Let S be a convexity space. A non-empty subset P of S is called a polytope in S iff P is the convex hull of a finite subset of S.

By [93, Theorem 1.6], we have the following proposition, which characterize a polytope in a convexity space as a subset satisfying a certain condition similar to compactness in topology.

Proposition 3.8. *For a convexity space $(S, \mathcal{C})$, the following are equivalent.*

1. *$P \in \mathcal{C}$ is a polytope in S;*

2. *if $P = \bigcup_{i \in I} C_i$ for a directed set $\{C_i \in \mathcal{C} \,;\, i \in I\}$, then there is $i \in I$ such that $P = C_i$.*

By [93, Proposition 1.7.1], we have the following proposition.

Proposition 3.9. *Let S be a convexity space. Any convex set in S is the union of a directed set of polytopes in S.*

Thus, the set of all polytopes in a convexity space forms the canonical base of the convexity space. There is no such canonical base of a topological space in general. This is a striking difference between topology and convex geometry.

The notion of polytope shall play a crucial role in our investigation. For instance, the polytopes in a convexity space can be used for "sobrification" of the convexity space as we shall see later.

3.1.2 Continuous Lattices

We introduce the notion of continuous lattice in a way different from the usual one. This is because we consider continuous lattices as pointfree convexity spaces and as being analogous to frames, which are pointfree topological spaces, in locale theory (for basic concepts of lattice theory, see [29]).

Definition 3.10. A poset L is a continuous lattice iff it satisfies the following properties:

1. L has arbitrary meets;

2. if $\{x_i \in L \,;\, i \in I\}$ is directed in L, then $\{x_i \,;\, i \in I\}$ has a join in L;

3. for any doubly indexed family $\{x_{i,j} \in L \,;\, i \in I \text{ and } j \in J_i\}$, if $\{x_{i,j} \,;\, j \in J_i\}$ is directed for every $i \in I$ and if $\{\bigwedge_{i \in I} x_{i,f(i)} \,;\, f \in F\}$ is directed, then

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigvee_{f \in F} \bigwedge_{i \in I} x_{i,f(i)}$$

   where $F = \prod_{i \in I} J_i$ ($= \{f : I \to \bigcup_{i \in I} J_i \,;\, \forall i \in I \; f(i) \in J_i\}$).

Note that a continuous lattice has the least element 0 and the greatest element 1 by letting the index sets be empty in the conditions 1 and 2 above.

We call the condition 3 in the above definition the directed-completely distributive law.

Remark 3.11. The above definition of continuous lattice is equivalent to the usual one via the notion of way-below relation (see [39, Theorem I-2.7]).

Definition 3.12. Let L_1 and L_2 be continuous lattices. A function $f : L_1 \to L_2$ is a homomorphism from L_1 to L_2 iff it satisfies the following properties:

1. $f(\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I} f(a_i)$ for any $\{a_i\,;\, i \in I\} \subset L_1$;
2. if $\{a_i \in L_1\,;\, i \in I\}$ is directed, then $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$.

Note that for a homomorphism f of continuous lattices, we have $f(0) = 0$ and $f(1) = 1$ by letting the index sets be empty in the above conditions.

We can easily verify the following proposition.

Proposition 3.13. *Let $(S, \mathcal{C})$ be a convexity space. Then, $\mathcal{C}$ forms a continuous lattice (when equipped with set-theoretical operations).*

Next we define the concepts of meet-complete filter and Scott-open meet-complete filter (so-mc filter for short), which correspond to filter and completely prime filter respectively in locale theory.

Definition 3.14. Let L be a continuous lattice. A subset F of L is called a meet-complete filter of L iff the following hold:

1. if $a \in F$ and $a \leq x$ then $x \in F$;
2. if $\{a_i\,;\, i \in I\} \subset F$, then $\bigwedge_{i \in I} a_i \in F$.

Note that a meet-complete filter F is non-empty, since we have $1 \in F$ by the condition 2 applied to the empty family.

A meet-complete filter may also be called a Moore filter, since the related notion of Moore family is well known.

Definition 3.15. Let L be a continuous lattice. A subset P of L is called a Scott-open meet-complete filter (so-mc filter for short) of L iff the following hold:

1. P is a meet-complete filter of L;
2. if $\{a_i \in L\,;\, i \in I\}$ is directed and $\bigvee_{i \in I} a_i \in P$, then there is $i \in I$ with $a_i \in P$.

Note that any so-mc filter P of L is not L, since we have $0 \notin P$ by the condition 2 applied to the empty family.

For a convexity space $(S, \mathcal{C})$ and $x \in S$, $\{C \in \mathcal{C}\,;\, x \in C\}$ is a so-mc filter of the continuous lattice $\mathcal{C}$.

The notion of algebraicity is defined as follows.

Definition 3.16. Let L be a continuous lattice.

For $a \in L$, a is said to be a compact element in L iff if $a \leq \bigvee_{i \in I} a_i$ for a directed subset $\{a_i\,;\, i \in I\}$ of L then there exists $i \in I$ such that $a \leq a_i$.

L is called algebraic iff for any $a \in L$ there is a directed set $\{a_i\,;\, i \in I\}$ of compact elements in L such that $a = \bigvee\{a_i\,;\, i \in I\}$.

Note that any compact element is not the least element, which is shown by letting $I = \emptyset$ in its definition.

By Proposition 3.8, we have the following.

Proposition 3.17. *Let $(S, \mathcal{C})$ be a convexity space. Then, $P \in \mathcal{C}$ is a polytope in S iff P is a compact element in the continuous lattice $\mathcal{C}$.*

It is interesting that polytopes in convex-geometric terms are compact elements in domain-theoretic terms.

Proposition 3.18. *Let L be a continuous lattice. Then, there is a natural bijection between the set of all so-mc filters of L and the set of all compact elements in L.*

Proof. For a so-mc filter P of L, $\bigwedge P$ is a compact element in L and, for so-mc filters P, Q of L with $P \neq Q$, we have $\bigwedge P \neq \bigwedge Q$. Thus, the map $P \mapsto \bigwedge P$ is injective. In order to show the surjectivity, let a be a compact element in L. Then, $\{x \in L \,;\, a \leq x\}$ is a so-mc filter and also we have $\bigwedge\{x \in L \,;\, a \leq x\} = a$. This completes the proof. □

Actually, any meet-complete filter F is principal (i.e., it is generated by an element, namely $\bigwedge F$), though $\bigwedge F$ is not necessarily compact.

Proposition 3.19. *Let L be a continuous lattice. Then, there is a natural bijection between the set of all so-mc filters of L and the set of all homomorphisms from L to $\mathbf{2}$.*

Proof. For a so-mc filter P, define $v_P : L \to \mathbf{2}$ by

$$v_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that v_P is a homomorphism and that if $P \neq Q$ for so-mc filters P, Q then we have $v_P \neq v_Q$. Therefore, the map $P \mapsto v_P$ is injective. To show the surjectivity, let u be a homomorphism from L to $\mathbf{2}$. Define $P = u^{-1}(\{1\})$. Then it is straightforward to verify that P is a so-mc filter and that $v_P = u$. This completes the proof. □

By Proposition 3.19 and Proposition 3.18, we do not distinguish between so-mc filters of L, compact elements in L, and homomorphisms from L to $\mathbf{2}$ for a continuous lattice L.

3.1.3 Dual Adjunction between Convexity Spaces and Continuous Lattices

Based on our general theory in Section 2, we can show a dual adjunction between categories **ContLat** and **Conv** defined as follows.

Definition 3.20. **ContLat** denotes the category of continuous lattices and homomorphisms.
Conv denotes the category of convexity spaces and convexity preserving maps.

We then introduce contravariant Hom functors Spec and Conv as follows.

Definition 3.21. We define a contravariant functor Spec from **ContLat** to **Conv** as follows:

1. For an object L in **ContLat**, Spec(L) is defined as the set of all homomorphisms from L to $\mathbf{2}$ equipped with the convexity generated by $\{\langle a \rangle \,;\, a \in L\}$ where

$$\langle a \rangle = \{v \,;\, v(a) = 1 \text{ and } v : L \to \mathbf{2} \text{ is a homomorphism }\}.$$

2. For an arrow $f : L_1 \to L_2$ in **ContLat**, $\mathrm{Spec}(f) : \mathrm{Spec}(L_2) \to \mathrm{Spec}(L_1)$ is defined by

$$\mathrm{Spec}(f)(v) = v \circ f$$

for $v \in \mathrm{Spec}(L_2)$.

By Proposition 3.19, we can consider $\mathrm{Spec}(L)$ as the set of all so-mc filters of L equipped with the convexity generated by the $\{P \in \mathrm{Spec}(L) \,;\, a \in P\}$'s for $a \in L$.

The following lemma tells us that $\langle\text{-}\rangle$ preserves the operations of continuous lattices.

Lemma 3.22. *Let L be a continuous lattice. For $\{a_i \,;\, i \in I\} \subset L$, $\langle \bigwedge_{i\in I} a_i\rangle = \bigcap_{i\in I}\langle a_i\rangle$. For a directed subset $\{a_i \,;\, i \in I\}$ of L, $\langle \bigvee_{i\in I} a_i\rangle = \bigcup_{i\in I}\langle a_i\rangle$.*

Proof. This follows immediately from the fact that a homomorphism of continuous lattices preserves arbitrary meets and joins of directed sets. □

Definition 3.23. We define a contravariant functor Conv from **Conv** to **ContLat** as follows:

1. For an object S in **Conv**, $\mathrm{Conv}(S)$ is defined as the set of all convexity preserving maps from S to $\mathbf{2}$ equipped with the pointwise operations. For instance, given $f_i \in \mathrm{Conv}(S)$ for $i \in I$, $\bigwedge_{i\in I} f_i \in \mathrm{Conv}(S)$ is defined by

$$(\bigwedge_{i\in I} f_i)(x) = \bigwedge_{i\in I} f_i(x).$$

2. For an arrow $f : S \to S'$ in **Conv**, $\mathrm{Conv}(f) : \mathrm{Conv}(S') \to \mathrm{Conv}(S)$ is defined by

$$\mathrm{Conv}(f)(g) = g \circ f$$

for $g \in \mathrm{Conv}(S')$.

For a convexity space S, we can consider $\mathrm{Conv}(S)$ as the set of all convex sets equipped with set-theoretical operations by Proposition 3.6. Note that in this case, the arrow part of the functor Conv can be defined by

$$\mathrm{Conv}(f)(C) = f^{-1}(C)$$

for $C \in \mathrm{Conv}(S')$. The two definitions of Conv are essentially equivalent and we do not have to distinguish between them.

To apply our general theory in this case, we have to check the following:

- **CA** is the Eilenberg-Moore category of a monad (see Proposition 2.7).
- **Conv** is a full subcategory of $\mathbf{Spa}(Q)^{\mathrm{op}}$ that is definable by Boolean topological coaxioms (see Proposition 2.14).
- We can take $\mathbf{2}$ in this subsection as Ω in Section 2, i.e., $\mathbf{2}$ is both a continuous lattice and a convexity space.
- The harmony condition holds, since $\mathrm{Conv}(S)$ forms a continuous lattice by Proposition 3.13 and Proposition 3.6.

Thus, Lemma 2.20 and Lemma 2.21 imply that functors Spec and Conv are well defined. Theorem 2.19 gives us the following dual adjunction between **ContLat** and **Conv**.

Theorem 3.24. Spec *is left adjoint to* $\mathrm{Conv}^{\mathrm{op}}$*.*

Since a left adjoint functor preserves colimits and a right adjoint functor preserves limits (see [8]), such categorical constructions in one category can be transferred into the other category via the above adjunction.

For later developments, we recall the unit and the counit of the adjunction.

Definition 3.25. We define a natural transformation $\Phi : 1_{\mathbf{ContLat}} \to \mathrm{Conv} \circ \mathrm{Spec}$ as follows. For a continuous lattice L, define $\Phi_L : L \to \mathrm{Conv} \circ \mathrm{Spec}(L)$ by

$$\Phi_L(a)(v) = v(a)$$

for $a \in L$ and $v \in \mathrm{Spec}(L)$.

Definition 3.26. We define a natural transformation $\Psi : 1_{\mathbf{Conv}} \to \mathrm{Spec} \circ \mathrm{Conv}$ as follows. For a convexity space S, define $\Psi_S : S \to \mathrm{Spec} \circ \mathrm{Conv}(S)$ by

$$\Psi_S(x)(f) = f(x)$$

for $x \in S$ and $f \in \mathrm{Conv}(S)$.

Φ and Ψ are well defined by Lemma 2.23, Lemma 2.24, Lemma 2.25, and Lemma 2.26.

3.2 Sober-Type Duality between Convexity Spaces and Continuous Lattices

In this subsection, we introduce the notions of spatial continuous lattice and sober convexity space. It turns out that a continuous lattice is spatial iff it is algebraic. Then, by restricting the dual adjunction between **ContLat** and **Conv**, we shall show a dual equivalence between the category **SpContLat** of spatial continuous lattices (= algebraic lattices) and the category **SobConv** of sober convexity spaces.

3.2.1 Spatiality

We define the notion of spatiality as the existence of "enough" so-mc filters:

Definition 3.27. For a continuous lattice L, L is spatial iff, for any $a, b \in L$ with $a \nleq b$, there is a so-mc filter P of L such that $a \in P$ and $b \notin P$.

We can characterize the spatiality of a continuous lattice L as the injectivity of Φ_L.

Lemma 3.28. *Let L be a continuous lattice. The following are equivalent:*

1. *L is spatial;*
2. *Φ_L is injective, i.e., for any $a, b \in L$ with $a \neq b$ there is $v \in \mathrm{Spec}(L)$ with $v(a) \neq v(b)$;*
3. *if $\langle a \rangle \subset \langle b \rangle$ for $a, b \in L$, then $a \leq b$.*

Proof. By Proposition 3.19, it is straightforward to show that 1 implies 2 and that 3 implies 1. We show that 2 implies 3. Assume 2. To show the contrapositive of 3, assume $a \not\leq b$. Then, since Φ_L is an injective homomorphism, we have $\Phi_L(a) \not\leq \Phi_L(b)$. Therefore, there is $v \in \mathrm{Spec}(L)$ such that

$$v(a) = \Phi_L(a)(v) > \Phi_L(b)(v) = v(b).$$

Hence we have $v \in \langle a \rangle$ and $v \notin \langle b \rangle$. This completes the proof. □

The following proposition provides many natural examples of spatial continuous lattices.

Proposition 3.29. *Let S be a convexity space. Then,* $\mathrm{Conv}(S)$ *is a spatial continuous lattice.*

Proof. Let $f, g \in \mathrm{Conv}(S)$ with $f \neq g$. Then, we have $f(x) \neq g(x)$ for some $x \in S$. Let $v = \Psi_S(x)$. Then, we have

$$\Phi_{\mathrm{Conv}(S)}(f)(v) = v(f) = f(x).$$

We also have

$$\Phi_{\mathrm{Conv}(S)}(g)(v) = v(g) = g(x).$$

Thus, $\Phi_{\mathrm{Conv}(S)}$ is injective and so $\mathrm{Conv}(S)$ is spatial by Lemma 3.28. □

The following lemma plays an important role in our duality theory for pointfree convex geometry.

Lemma 3.30. *For a continuous lattice L, the convexity of* $\mathrm{Spec}(L)$ *coincides with* $\{\langle a \rangle \,;\, a \in L\}$.

Proof. By the definition of the convexity of $\mathrm{Spec}(L)$, it suffices to prove that $\{\langle a \rangle \,;\, a \in L\}$ satisfies the two conditions in Definition 3.1.

First, for any subset $\{\langle a_i \rangle \,;\, i \in I\}$ of $\{\langle a \rangle \,;\, a \in L\}$, it follows from Lemma 3.22 that

$$\bigcap_{i \in I} \langle a_i \rangle = \langle \bigwedge_{i \in I} a_i \rangle \in \{\langle a \rangle \,;\, a \in L\},$$

whence $\{\langle a \rangle \,;\, a \in L\}$ is closed under arbitrary intersections.

Second, assume that a subset $\{\langle a_i \rangle \,;\, i \in I\}$ of $\{\langle a \rangle \,;\, a \in L\}$ is directed with respect to inclusion. We show that $\bigcup_{i \in I} \langle a_i \rangle \in \{\langle a \rangle \,;\, a \in L\}$. For each $i \in I$, define

$$b_i = \bigwedge \{a_j \,;\, \langle a_i \rangle \subset \langle a_j \rangle \text{ and } j \in I\}.$$

Note that $\langle b_i \rangle = \langle a_i \rangle$. If $\langle a_k \rangle \subset \langle a_l \rangle$ for $k, l \in I$ then we have $b_k \leq b_l$. Thus, since $\{\langle a_i \rangle \,;\, i \in I\}$ is directed with respect to inclusion, $\{b_i \,;\, i \in I\}$ is directed with respect to the partial order of L. Therefore, by Lemma 3.22, we have

$$\bigcup_{i \in I} \langle a_i \rangle = \bigcup_{i \in I} \langle b_i \rangle = \langle \bigvee_{i \in I} b_i \rangle \in \{\langle a \rangle \,;\, a \in L\}.$$

Hence, $\{\langle a \rangle \,;\, a \in L\}$ is closed under unions of directed subsets. This completes the proof. □

In fact, Φ_L is always surjective.

Lemma 3.31. *Let L be a continuous lattice. Then, Φ_L is surjective.*

Proof. Based on Proposition 3.6, we can consider $\mathrm{Conv} \circ \mathrm{Spec}(L)$ as the set of all convex sets in $\mathrm{Spec}(L)$. Thus, since $\Phi_L(a)^{-1}(\{1\}) = \langle a \rangle$, we can consider $\Phi_L(a) = \langle a \rangle$. Then, it follows from Lemma 3.30 that

$$\{\langle a \rangle \,;\, a \in L\} = \mathrm{Conv} \circ \mathrm{Spec}(L).$$

Hence, Φ_L is surjective by $\Phi_L(a) = \langle a \rangle$. □

By Lemma 3.28, Proposition 3.29 and Lemma 3.31, we obtain the following proposition (recall that Φ_L is a homomorphism for any continuous lattice L).

Proposition 3.32. *For a continuous lattice L, L is spatial iff $\Phi_L : L \to \mathrm{Conv} \circ \mathrm{Spec}(L)$ is an isomorphism.*

This proposition implies that any spatial continuous lattice can be represented as the continuous lattice of convex sets in a convexity space.

We can provide an algebraic characterization of spatiality as follows.

Proposition 3.33. *Let L be a continuous lattice. Then, L is spatial iff L is algebraic.*

Proof. Assume that L is spatial. By Proposition 3.32, L is isomorphic to the continuous lattice of all convex sets in a convexity space, which is shown to be algebraic by combining Proposition 3.17 and Proposition 3.9.

Assume that L is algebraic. Let $a, b \in L$ with $a \nleq b$. Let A be the set of compact elements that are less than or equal to a and B the set of compact elements that are less than or equal to b. Since L is algebraic, we have $a = \bigvee A$ and $b = \bigvee B$. Therefore, it follows from $a \nleq b$ that there is $c \in A$ such that $c \notin B$. Define $P = \{x \in L \,;\, c \leq x\}$. Then, since c is a compact element, P is a so-mc filter by Proposition 3.18 and also we have both $a \in P$ and $b \notin P$. Thus, L is spatial. □

Definition 3.34. Let L be a continuous lattice and $\mathrm{MCF}(L)$ the set of all meet-complete filters of L. Then, L is filter-closed iff for any non-empty directed subset $\{X_i \,;\, i \in I\}$ of $\mathrm{MCF}(L)$, $\bigcup_{i \in I} X_i$ is a meet-complete filter.

For instance, every successor ordinal is a filter-closed continuous lattice and also the finite product of successor ordinals is a filter-closed continuous lattice. More generally, we have the following characterization of filter-closed continuous lattice.

Proposition 3.35. *For a continuous lattice L, L is filter-closed iff there is no infinite descending chain in L.*

Proof. We first show that filter-closedness implies the non-existence of an infinite descending chain. In order to prove the contrapositive, assume that there is an infinite descending chain $\{a_i \in L \,;\, i \in I\}$. Then, we have $\bigwedge_{i \in I} a_i < a_k$ for any $k \in I$, since if not, then there is $k \in I$ such that $a_k \leq a_i$ for any $i \in I$, i.e., $\{a_i \in L \,;\, i \in I\}$ is not an infinite descending chain. Define

$$A_i = \{x \in L \,;\, a_i \leq x\},$$

which is a meet-complete filter. Clearly, $\{A_i \,;\, i \in I\}$ is directed. Moreover, $\bigcup_{i \in I} A_i$ is not a meet-complete filter, since we have both $a_i \in \bigcup_{i \in I} A_i$ for any $i \in I$ and $\bigwedge_{i \in I} a_i \notin \bigcup_{i \in I} A_i$ by the fact that $\bigwedge_{i \in I} a_i < a_k$ for any $k \in I$. Therefore, L is not filter-closed.

To show the converse, assume that there is no infinite descending chain in L. Let $\{X_i \,;\, i \in I\}$ be a non-empty directed subset of $\mathrm{MCF}(L)$. Since any meet-complete filter X is generated by $\bigwedge X$, $\{\bigwedge X_i \,;\, i \in I\}$ is directed in L. However, it follows from assumption that $\{\bigwedge X_i \,;\, i \in I\}$ is not an infinite descending chain. Thus, $\{X_i \,;\, i \in I\}$ is not an infinite ascending chain. Then there is $j \in I$ such that

$$X_j = \bigcup_{i \in I} X_i.$$

Hence, $\bigcup_{i \in I} X_i$ is a meet-complete filter. Thus, L is filter-closed. □

An analogue of the prime filter theorem for distributive lattices holds for filter-closed continuous lattices.

Proposition 3.36. *Let L be a filter-closed continuous lattice. Then, L is spatial.*

Proof. Let $a, b \in L$ with $a \not\leq b$. Let $\mathcal{H}$ be the set of meet-complete filters F of L such that $a \in F$ and $b \notin F$. Since $\{x \in L \,;\, a \leq x\} \in \mathcal{H}$, $\mathcal{H}$ is not empty. Since L is filter-closed, every directed subset $\{F_i \,;\, i \in I\}$ of $\mathcal{H}$ has an upper bound $\bigcup_{i \in I} F_i$ in $\mathcal{H}$. Thus, by Zorn's lemma, we have a maximal element P in $\mathcal{H}$. Clearly, $a \in P$ and $b \notin P$.

In order to complete the proof, we show that P is a so-mc filter of L. Let $\{a_i \,;\, i \in I\}$ be a directed subset of L and $\bigvee_{i \in I} a_i \in P$. Suppose for contradiction that $a_i \notin P$ for any $i \in I$. Then, it follows from the maximality of M that for every $i \in I$ there exists $p_i \in P$ such that $a_i \wedge p_i \leq b$. Let

$$p = \bigwedge_{i \in I} p_i.$$

Since P is a meet-complete filter, we have $p \in P$. Clearly, $a_i \wedge p \leq b$. Hence, we have

$$\bigvee_{i \in I} (a_i \wedge p) \leq b.$$

It follows from the chain-completely distributive law (i.e., the item 3 in Definition 3.10) that

$$\bigvee_{i \in I} (a_i \wedge p) = (\bigvee_{i \in I} a_i) \wedge p.$$

Since $\bigvee_{i \in I} a_i \in P$ and $p \in P$, we have $(\bigvee_{i \in I} a_i) \wedge p \in P$ and so $\bigvee_{i \in I} (a_i \wedge p) \in P$. By $\bigvee_{i \in I} (a_i \wedge p) \leq b$, we have $b \in P$, which is a contradiction. Thus, P is a so-mc filter. Hence, L is spatial. □

3.2.2 Sobriety

In order to define the concept of sober convexity space, we first define directed-irreducible convex set, whose role in our duality theory is analogous to that of irreducible closed set in Isbell duality.

Definition 3.37. Let $(S, \mathcal{C})$ be a convexity space. A convex set C in $\mathcal{C}$ is said to be directed-irreducible iff if $C = \bigcup_{i \in I} C_i$ for a directed subset $\{C_i \,;\, i \in I\}$ of $\mathcal{C}$ then there exists $i \in I$ such that $C = C_i$.

Note that a convex set in a convexity space $(S, \mathcal{C})$ is directed-irreducible iff it is a compact element in the continuous lattice $\mathcal{C}$. By Proposition 3.8, we have the following lemma.

Lemma 3.38. *Let S be a convexity space. Then, a convex subset of S is directed-irreducible iff it is a polytope in the convexity space.*

Now we introduce the notion of sober convexity space.

Definition 3.39. A convexity space S is said to be sober iff, for every directed-irreducible convex set C in S, there is a unique point $x \in S$ such that $C = \mathrm{ch}(\{x\})$.

By Lemma 3.38, we obtain the following alternative definition of sobriety, which clarifies the convexity theoretical meaning of sobriety.

Proposition 3.40. *A convexity space is sober iff every polytope in it is the convex hull of a unique point.*

We remark that not all natural examples of convexity spaces are sober. For example, by the above proposition, $\mathbb{R}^n$ with the usual convexity (see Example 3.3) is not a sober convexity space, though it is a sober topological space.

Example 3.41. Consider $\mathbf{2}^\omega$, i.e., the set of all functions from the set ω of all non-negative integers to $\mathbf{2}$ $(= \{0, 1\})$. Let $C_0 = \mathbf{2}^\omega$. For $k \in \omega$ with $k \geq 1$ and $n_1, ..., n_k \in \omega$, let

$$C_k(n_1, ..., n_k) = \{f \in \mathbf{2}^\omega \ ; \ f(n_1) = f(n_2) = ... = f(n_k) = 1\}.$$

Equip $\mathbf{2}^\omega$ with the convexity generated by

$$\{C_k(n_1, ..., n_k) \ ; \ k \in \omega \text{ and } n_1, ..., n_k \in \omega\}.$$

Then, $\mathbf{2}^\omega$ forms a sober convexity space.

The next proposition provides many natural examples of sober convexity spaces.

Proposition 3.42. *Let L be a continuous lattice. Then,* $\mathrm{Spec}(L)$ *is a sober convexity space.*

Proof. Assume that C is a directed-irreducible convex set in $\mathrm{Spec}(L)$. Define

$$a = \bigwedge \{x \in L \ ; \ C = \langle x \rangle\},$$

where $\{x \in L \ ; C = \langle x \rangle\}$ is not empty, since any convex set in $\mathrm{Spec}(L)$ is of the form $\langle x \rangle$ for $x \in L$ by Lemma 3.30. Then, we have $C = \langle a \rangle$ by Lemma 3.22. We claim that a is a compact element in L. Suppose that $a \leq \bigvee_{i \in I} a_i$ for a directed subset $\{a_i \ ; \ i \in I\}$ of L. Then, by Lemma 3.22, we have

$$C = \langle a \rangle = \langle \bigvee_{i \in I} a_i \rangle \cap \langle a \rangle = \bigcup_{i \in I} \langle a_i \wedge a \rangle.$$

Since C is directed-irreducible and since $\{\langle a_i \wedge a \rangle \ ; \ i \in I\}$ is directed, there exists $i \in I$ such that

$$C = \langle a \rangle = \langle a \wedge a_i \rangle.$$

Thus, it follows from the definition of a that $a \leq a \wedge a_i$, whence we have $a \leq a_i$. Therefore, a is a compact element in L. Let $P_a = \{x \in L \ ; \ a \leq x\}$. Then, P_a is a so-mc filter. In the following, we do not distinguish between so-mc filters and homomorphisms into **2**, based on Proposition 3.19. Then we have

$$C = \langle a \rangle = \bigcap \{ \langle x \rangle \; ; \; x \in P_a \} = \bigcap \{ \langle x \rangle \; ; \; P_a \in \langle x \rangle \} = \mathrm{ch}(\{P_a\}).$$

To show the uniqueness, assume that, for $P, Q \in \mathrm{Spec}(L)$, $\mathrm{ch}(\{P\}) = C = \mathrm{ch}(\{Q\})$. Suppose for contradiction that $P \neq Q$. Then, we may assume that there is $b \in L$ such that $b \in P$ and $b \notin Q$. Therefore, we have $\mathrm{ch}(\{P\}) \subset \langle b \rangle$ and $\neg(\mathrm{ch}(\{Q\}) \subset \langle b \rangle)$, which is a contradiction. □

By letting L be the continuous lattice of convex sets in a convexity space, $\mathrm{Spec}(L)$ can be considered as the space of polytopes in the convexity space. Spaces of polytopes in convexity spaces seem to be natural examples of sober convexity spaces.

Proposition 3.43. *Let S be a sober convexity space and $\mathcal{C}$ its convexity. Then, $\Psi_S : S \to \mathrm{Spec} \circ \mathrm{Conv}(S)$ is an isomorphism in* **Conv**.

Proof. We first show that Ψ_S is injective. Assume that $\Psi_S(x) = \Psi_S(y)$ for $x, y \in S$. Then we have $\Psi_S(x)^{-1}(\{1\}) = \Psi_S(y)^{-1}(\{1\})$, i.e.,

$$\{f \in \mathrm{Conv}(S) \; ; \; f(x) = 1\} = \{f \in \mathrm{Conv}(S) \; ; \; f(y) = 1\}.$$

By Proposition 3.6, we have $\{C \in \mathcal{C} \; ; \; x \in C\} = \{C \in \mathcal{C} \; ; \; y \in C\}$. By taking the intersections, it follows from the definition of convex hull that

$$\mathrm{ch}(\{x\}) = \bigcap \{C \in \mathcal{C} \; ; \; x \in C\} = \bigcap \{C \in \mathcal{C} \; ; \; y \in C\} = \mathrm{ch}(\{y\}).$$

Since S is sober and since $\mathrm{ch}(\{x\})$ is a directed-irreducible convex set, we have $x = y$. Thus, Ψ_S is injective.

We next show that Ψ_S is surjective. Let $v \in \mathrm{Spec} \circ \mathrm{Conv}(S)$. By Proposition 3.19, $v^{-1}(\{1\})$ is a so-mc filter of $\mathrm{Conv}(S)$. By Proposition 3.18, $\bigwedge v^{-1}(\{1\})$ is a compact element in $\mathrm{Conv}(S)$. Since $\mathrm{Conv}(S)$ is isomorphic to the continuous lattice $\mathcal{C}$ via the map $f \mapsto f^{-1}(\{1\})$, it follows that

$$\bigcap \{f^{-1}(\{1\}) \; ; \; f \in v^{-1}(\{1\})\}$$

is a compact element in $\mathcal{C}$ and is thus a directed-irreducible convex set in S. Since S is sober, there is $x \in S$ such that

$$\bigcap \{f^{-1}(\{1\}) \; ; \; f \in v^{-1}(\{1\})\} = \mathrm{ch}(\{x\}).$$

We claim that $\Psi_S(x) = v$. Let $g \in \mathrm{Conv}(S)$. We first assume that $v(g) = 1$. Then we have $g \in v^{-1}(\{1\})$. By the choice of x, we have $x \in \mathrm{ch}(\{x\}) \subset g^{-1}(\{1\})$. Thus it follows that $\Psi_S(x)(g) = 1 = v(g)$. We next assume that $v(g) = 0$. Suppose for contradiction that $\Psi_S(x)(g) = 1$, i.e., $g(x) = 1$. Since $g^{-1}(\{1\})$ is a convex set in S and $x \in g^{-1}(\{1\})$, we have

$$\bigcap \{f^{-1}(\{1\}) \; ; \; f \in v^{-1}(\{1\})\} = \mathrm{ch}(\{x\}) \subset g^{-1}(\{1\}).$$

Thus we have $\bigwedge v^{-1}(\{1\}) \leq g$ in $\mathrm{Conv}(S)$. Since $v^{-1}(\{1\})$ is a so-mc filter and $\bigwedge v^{-1}(\{1\}) \in v^{-1}(\{1\})$, we have $g \in v^{-1}(\{1\})$, which contradicts $v(g) = 0$. Therefore we have $\Psi_S(x)(g) = 0 = v(g)$. Thus we obtain $\Psi_S(x) = v$. Hence, Ψ_S is surjective.

It has already been shown that Ψ_S is a convexity preserving map. To complete the proof, we show that Ψ_S^{-1} is a convexity preserving map. Let C be a convex set in S. Define $f_C : S \to \mathbf{2}$ as in the proof of Proposition 3.6. We claim that $\Psi_S(C) = \langle f_C \rangle$. Suppose $v \in \Psi_S(C)$. Then, $v = \Psi_S(x)$ for some $x \in C$, whence we have

$$v(f_C) = \Psi_S(x)(f_C) = f_C(x) = 1.$$

Hence, we have $v \in \langle f_C \rangle$. Conversely, suppose $v \in \langle f_C \rangle$. Since Ψ_S is surjective, there exists $x \in S$ such that $\Psi_S(x) = v$. By $v \in \langle f_C \rangle$, we have $\Psi_S(x)(f_C) = f_C(x) = 1$, i.e., $x \in C$. Hence, we have $v \in \Psi_S(C)$. □

In this way, we can recover the points of a sober convexity space from the continuous lattice of convex sets in it. The above proposition implies that any sober convexity space can be represented as Spec(L) for a continuous lattice L.

By Proposition 3.42 and Proposition 3.43, we have the following characterization of sobriety.

Proposition 3.44. *For a convexity space S, S is sober iff Ψ_S is an isomorphism in* **Conv**.

SpContLat denotes the category of spatial continuous lattices (= algebraic lattices) and homomorphisms. **SobConv** denotes the category of sober convexity spaces and convexity preserving maps. Finally we obtain the following duality between spatial continuous lattices and sober convexity spaces.

Theorem 3.45. **SpContLat** *and* **SobConv** *are dually equivalent via the functors* Spec *and* Conv*.*

Proof. By Proposition 3.42, (the restriction of) Spec is well-defined. By Proposition 3.29, (the restriction of) Conv is well-defined. By Proposition 3.32, $\Phi : 1_{\mathbf{SpContLat}} \to \mathrm{Conv} \circ \mathrm{Spec}$ is a natural isomorphism. By Proposition 3.43, $\Psi : 1_{\mathbf{SobConv}} \to \mathrm{Spec} \circ \mathrm{Conv}$ is a natural isomorphism. □

This is a convexity-theoretical analogue of Isbell duality between spatial frames and sober topological spaces. However, there is a big difference between the above duality and Isbell duality, especially between the notion of sobriety for convexity spaces and the notion of sobriety for topological spaces. That is, most of ordinary topological spaces such as $\mathbb{R}^n$ are sober and so fall into Isbell duality, while most of ordinary convexity spaces such as $\mathbb{R}^n$ are not sober and so do not fall into the above duality. In the next subsection, we consider another duality into which most of ordinary convexity spaces do fall.

3.3 T_1-Type Duality between Convexity Spaces and Continuous Lattices

In this subsection, by introducing the notions of m-spatiality, m-homomorphism and T_1 convexity space, we shall show a duality between the category of m-spatial continuous lattices and m-homomorphisms and the category of T_1 convexity spaces and convexity preserving maps.

For a continuous lattice L, we mean by an m-mc filter of L a maximal meet-complete filter of L where maximality means that with respect to inclusion.

Consider a convexity space $(S, \mathcal{C})$ such that $\{x\}$ is convex for any $x \in S$. Then, for $x \in S$, $\{C \in \mathcal{C}\ ;\ x \in C\}$ is an m-mc filter of the continuous lattice $\mathcal{C}$.

Lemma 3.46. *Let M be an m-mc filter of a continuous lattice L. Then, M is a so-mc filter of L.*

Proof. Assume that $\bigvee_{i\in I} a_i \in M$ for a directed subset $\{a_i\ ;\ i \in I\}$ of L. Suppose for contradiction that for any $i \in I$, $a_i \notin M$. Since M is an m-mc filter, we have: For any $i \in I$ there is $b_i \in M$ such that $a_i \wedge b_i = 0$. Then we have $\bigwedge_{i\in I} b_i \in M$ by $b_i \in M$. We also have $a_i \wedge (\bigwedge_{i\in I} b_i) = 0$, whence it follows that

$$\bigvee_{i\in I}(a_i \wedge (\bigwedge_{i\in I} b_i)) = 0.$$

Since $\{a_i\ ;\ i \in I\}$ is directed, it follows from the chain-completely distributive law (i.e., the item 3 in Definition 3.10) that

$$(\bigvee_{i\in I} a_i) \wedge (\bigwedge_{i\in I} b_i)) = 0.$$

Since $\bigwedge_{i\in I} b_i \in M$ and $\bigvee_{i\in I} a_i \in M$, we have $0 \in M$, which is a contradiction. Thus there is $i \in I$ such that $a_i \in M$. Hence, M is a so-mc filter of L. □

Then, m-spatiality is defined as follows.

Definition 3.47. A continuous lattice L is called m-spatial iff for any $a, b \in L$ with $a \not\leq b$ there is an m-mc filter M of L such that $a \in M$ and $b \notin M$.

By Lemma 3.46, we obtain the following proposition.

Proposition 3.48. *Let L be a continuous lattice. If L is m-spatial then L is spatial.*

We remark that, although m-spatiality implies spatiality, being T_1, which is defined below, does not imply being sober.

We next introduce the notion of m-homomorphism. A similar notion is used also in the context of duality theory for distributive semilattices (see [38, 44]).

Definition 3.49. An m-homomorphism $f : L_1 \to L_2$ between continuous lattices L_1 and L_2 is defined as a homomorphism of continuous lattices such that for any m-mc filter M of L_2, $f^{-1}(M)$ is an m-mc filter of L_1.

It shall be shown that the dual notion of convexity preserving map between T_1 (defined below) convexity spaces is m-homomorphism and is not homomorphism.

Let us review the concept of atomistic poset (see [29]). Recall that an atom in a poset P with a least element 0 is an element of P that is minimal in $P \setminus \{0\}$.

Definition 3.50. A poset with a least element 0 is called atomistic iff any element of the poset is the join of a set of atoms of the poset.

Note that in general, being atomistic is not equivalent to being atomic.

We can provide an algebraic characterization of m-spatiality as follows.

Proposition 3.51. *For a continuous lattice L, the following are equivalent:*

1. *L is m-spatial;*
2. *L is atomistic.*

Proof. We first show that 1 implies 2. Let $a \in L$. If $a = 0$ then a is the join of $\emptyset$. Assume $a > 0$. Let a' be the join of those atoms $x \in L$ such that $x \leq a$. It suffices to show that $a = a'$. Since if there is no atom $x \in L$ with $x \leq a$ then we have $a' = 0$, it follows from the choice of a' that $a \geq a'$. Suppose for contradiction that $a > a'$. Since L is m-spatial, there is an m-mc filter M of L such that $a \in M$ and $a' \notin M$. Since M is an m-mc filter of L, $\bigwedge M$ is an atom of L. By $a \in M$, we also have $\bigwedge M \leq a$. Then it follows from the definition of a' that

$$\bigwedge M \leq a'.$$

Since M is a meet-complete filter, we have $\bigwedge M \in M$ and so $a' \in M$, which contradicts $a' \notin M$. Hence $a = a'$.

We next show that 2 implies 1. Let $a, b \in L$ with $a \nleq b$. Since L is atomistic, there is a set A of atoms of L such that $\bigvee A = a$. Similarly, there is a set B of atoms of L such that $\bigvee B = b$. Then we may assume that B is the set of those atoms $x \in L$ such that $x \leq b$. By $a \nleq b$, there is $c \in A$ such that $c \notin B$. Define

$$M = \{x \in L\ ;\ c \leq x\}.$$

Then, we have both $a \in M$ and $b \notin M$, since B is the set of those atoms $x \in L$ such that $x \leq b$. Now it remains to show that M is an m-mc filter of L, which follows from the fact that c is an atom of L. □

We next introduce the notion of T_1 convexity space.

Definition 3.52. A convexity space S is called T_1 iff $\{x\}$ is convex for any $x \in S$.

Many ordinary convexity spaces are T_1, including those in Example 3.3.

An T_1 convexity space is not necessarily sober. For example, the n-dimensional Euclidean space $\mathbb{R}^n$ with the usual convexity (see Example 3.3) is not sober and is T_1. The same thing holds also for other convexity spaces such as those in Example 3.3. Thus we may consider that the notion of T_1 convexity is more natural than that of sobriety.

3.3.1 T_1-Type Duality between Convexity Spaces and Continuous Lattices

We show a dual equivalence between categories **mSpContLat** and **$\mathbf{T_1}$Conv**, which are defined as follows.

Definition 3.53. **$\mathbf{T_1}$Conv** denotes the category of T_1 convexity spaces and convexity preserving maps. **mSpContLat** denotes the category of m-spatial continuous lattices and m-homomorphisms.

We introduce a functor mSpec based on the view that a point is an m-mc filter.

Definition 3.54. We define a contravariant functor mSpec from **mSpContLat** to **$\mathbf{T_1}$Conv** as follows:

1. For an object L in **mSpContLat**, mSpec(L) is defined as the set of all m-mc filters of L equipped with the convexity generated by $\{\langle a\rangle_m\ ;\ a \in L\}$ where

$$\langle a\rangle_m = \{M \in \mathrm{mSpec}(L)\ ;\ a \in M\}.$$

2. For an arrow $f : L_1 \to L_2$ in **mSpContLat**, $\mathrm{mSpec}(f) : \mathrm{mSpec}(L_2) \to \mathrm{mSpec}(L_1)$ is defined by
$$\mathrm{mSpec}(f)(M) = f^{-1}(M)$$
for $M \in \mathrm{mSpec}(L_2)$.

The well-definedness of the functor mSpec is shown by the following proposition.

Proposition 3.55. *For a continuous lattice L, $\mathrm{mSpec}(L)$ is T_1.*

Proof. Let $M \in \mathrm{mSpec}(L)$. We claim that $\langle \bigwedge M \rangle_m = \{M\}$. Since $\bigwedge M \in M$, we have
$$\{M\} \subset \langle \bigwedge M \rangle_m.$$
Assume that $\bigwedge M \in N$ for $N \in \mathrm{mSpec}(L)$. It follows from $M, N \in \mathrm{mSpec}(L)$ that $M \subset N$ and so $M = N$ by maximality. Thus we have
$$\langle \bigwedge M \rangle_m \subset \{M\}.$$
Therefore we have $\langle \bigwedge M \rangle_m = \{M\}$. Hence, $\{M\}$ is convex. □

Remark 3.56. Throughout this subsection, based on Proposition 3.6, we consider Conv as a functor from $\mathbf{T_1Conv}$ to **mSpContLat** defined as follows. For an object S in $\mathbf{T_1Conv}$, $\mathrm{Conv}(S)$ is defined as the continuous lattice of all convex subsets of S. For an arrow $f : S_1 \to S_2$ in $\mathbf{T_1Conv}$, $\mathrm{Conv}(f) : \mathrm{Conv}(S_2) \to \mathrm{Conv}(S_1)$ is defined by $\mathrm{Conv}(f)(C) = f^{-1}(C)$ for $C \in \mathrm{Conv}(S_2)$.

Then the well-definedness of the functor Conv : $\mathbf{T_1Conv} \to$ **mSpContLat** is shown by the following two propositions.

Proposition 3.57. *Let S be an object in $\mathbf{T_1Conv}$. Then, $\mathrm{Conv}(S)$ is an m-spatial continuous lattice.*

Proof. Let $C_1, C_2 \in \mathrm{Conv}(S)$ such that C_1 is not a subset of C_2. Then, there is $x \in C_1$ with $x \notin C_2$. Define
$$M = \{C \in \mathrm{Conv}(S)\,;\, x \in C\}.$$
Then we have both $C_1 \in M$ and $C_2 \notin M$. Now it suffices to show that M is an m-mc filter of $\mathrm{Conv}(S)$. It is straightforward to verify that M is a meet-complete filter. Since S is T_1, $\{x\}$ is convex and so $\{x\} \in M$. If $C \notin M$ for $C \in \mathrm{Conv}(S)$ then $C \cap \{x\} = \emptyset$. Thus, M is maximal. □

Since an arrow in **mSpContLat** is an m-homomorphism, not a homomorphism, it is important to verify that the arrow part of Conv is well-defined.

Proposition 3.58. *Let $f : S_1 \to S_2$ be an arrow in $\mathbf{T_1Conv}$. Then, $\mathrm{Conv}(f) : \mathrm{Conv}(S_2) \to \mathrm{Conv}(S_1)$ is an m-homomorphism.*

Proof. Clearly, $\mathrm{Conv}(f)$ is a homomorphism. Let M be an m-mc filter of $\mathrm{Conv}(S_1)$. Since $\bigcap M \in M$ and $M \neq \mathrm{Conv}(S_1)$, we have $\bigcap M \neq \emptyset$ and so there is $m \in \bigcap M$. Then, $M \subset \{C \in \mathrm{Conv}(S_1)\,;\, m \in C\}$. Since $\{C \in \mathrm{Conv}(S_1)\,;\, m \in C\}$ is a proper meet-complete filter, it follows from the maximality of M that
$$M = \{C \in \mathrm{Conv}(S_1)\,;\, m \in C\}.$$

Thus it follows that

$$\begin{aligned} \mathrm{Conv}(f)^{-1}(M) &= \{C \in \mathrm{Conv}(S_2)\,;\ f^{-1}(C) \in M\} \\ &= \{C \in \mathrm{Conv}(S_2)\,;\ m \in f^{-1}(C)\} \\ &= \{C \in \mathrm{Conv}(S_2)\,;\ f(m) \in C\}. \end{aligned}$$

Since S_2 is T_1, $\{f(m)\}$ is convex and so $\{C \in \mathrm{Conv}(S_2)\,;\ f(m) \in C\}$ is an m-mc filter of $\mathrm{Conv}(S_2)$. This completes the proof. □

We next define two natural transformations.

Let Id_1 denote the identity functor on **mSpContLat** and Id_2 the identity functor on $\mathbf{T_1Conv}$.

Definition 3.59. We define a natural transformation $\alpha : \mathrm{Id}_1 \to \mathrm{Conv} \circ \mathrm{mSpec}$ as follows. For an m-spatial continuous lattice L, define $\alpha_L : L \to \mathrm{Conv} \circ \mathrm{mSpec}(L)$ by

$$\alpha_L(a) = \{M \in \mathrm{mSpec}(L)\,;\ a \in M\} = \langle a \rangle_m.$$

It is straightforward to verify that α is actually a natural transformation.

Proposition 3.60. *For an m-spatial continuous lattice L, $\alpha_L : L \to \mathrm{Conv} \circ \mathrm{mSpec}(L)$ is an isomorphism in* **mSpContLat**.

Proof. Since an isomorphism in **ContLat** is always an isomorphism in **mSpContLat**, it suffices to show that α_L is an isomorphism in **ContLat**. We first show that α_L is a homomorphism. By Lemma 3.46, an m-mc filter is a so-mc filter. Thus we have

$$\langle \bigvee_{i\in I} a_i \rangle_m = \bigcup_{i\in I} \langle a_i \rangle_m$$

for a directed subset $\{a_i\,;\ i \in I\}$ of L. We also have

$$\langle \bigwedge_{i\in I} a_i \rangle_m = \bigcap_{i\in I} \langle a_i \rangle_m$$

for a subset $\{a_i\,;\ i \in I\}$ of L. Thus, α_L is a homomorphism. It is straightforward to see that α_L is injective by the m-spatiality of L. By arguing as in the proof of Lemma 3.30, it is shown that $\{\langle a \rangle_m\,;\ a \in L\}$ coincides with the convexity of $\mathrm{mSpec}(L)$. Thus, α_L is surjective. This completes the proof. □

Definition 3.61. We define a natural transformation $\beta : \mathrm{Id}_2 \to \mathrm{mSpec} \circ \mathrm{Conv}$ as follows. For an T_1 convexity space S, define $\beta_S : S \to \mathrm{mSpec} \circ \mathrm{Conv}(S)$ by

$$\beta_S(x) = \{C \in \mathrm{Conv}(S)\,;\ x \in C\}.$$

It is straightforward to verify that β is actually a natural transformation.

Proposition 3.62. *For an T_1 convexity space S, $\beta_S : S \to \mathrm{mSpec} \circ \mathrm{Conv}(S)$ is an isomorphism in* $\mathbf{T_1Conv}$.

Proof. Since S is T_1, $\beta_S(x)$ is an m-mc filter for $x \in S$ and so β_S is well-defined. Clearly, β_S is injective. Since S is T_1, an m-mc filter of $\mathrm{Conv}(S)$ is of the form

$$\{C \in \mathrm{Conv}(S) \,;\, x \in C\}$$

for some $x \in S$. Thus, β_S is surjective. Since $\beta_S^{-1}(\langle C \rangle_m) = C$ for $C \in \mathrm{Conv}(S)$, β_S is convexity preserving. It is easily verified that

$$\beta_S(C) = \langle C \rangle_m$$

for $C \in \mathrm{Conv}(S)$, whence β_S^{-1} is convexity preserving. Hence, β_S is an isomorphism in $\mathbf{T_1Conv}$. □

In this way, we can recover the points of an T_1 convexity space from the continuous lattice of convex sets in it. This proposition implies that any T_1 convexity space can be represented as $\mathrm{mSpec}(L)$ for a continuous lattice L, where note that most of ordinary convexity spaces are T_1.

By Proposition 3.60 and Proposition 3.62, α and β are natural isomorphisms and thus we obtain the following duality between m-spatial continuous lattices and T_1 convexity spaces.

Theorem 3.63. mSpContLat *and* $\mathbf{T_1Conv}$ *are dually equivalent via the functors* mSpec *and* Conv*.*

Most of ordinary convexity spaces are T_1 (recall that a singleton is usually convex) and thus fall into the above duality.

We remark that convexity preserving maps between T_1 convexity spaces correspond to m-homomorphisms between m-spatial continuous lattices and do not correspond to homomorphisms.

3.4 Remarks on Dualities for Topology and Measure Theory

In this subsection, we briefly discuss dualities between frames and topological spaces and dualities between σ-complete Boolean algebras and measurable spaces, though we omit all proofs because of limitation of space. We can consider two types (i.e., sober-type and T_1-type) of dualities also in these contexts. It seems that there has been no research on T_1-type dualities for frames and σ-complete Boolean algebras and thus our results would be new.

We first consider sober-type dualities. Theorem 2.19 gives us a dual adjunction between frames **Frm** and topological spaces **Top** and a dual adjunction between σ-complete Boolean algebras $\mathbf{BA}_\sigma$ and measurable spaces **Meas**. As is well known, the dual adjunction between **Frm** and **Top** can be made into a dual equivalence between spatial frames and sober spaces, which is what we call Isbell duality. Here, it is important that spatiality and sobriety provides nice description of Fix(**Frm**) and Fix(**Top**) (for the definition of Fix, see Definition 2.27). In the case of the dual adjunction between $\mathbf{BA}_\sigma$ and **Meas**, we have not yet found such nice description of Fix($\mathbf{BA}_\sigma$) and Fix(**Meas**).

We next consider T_1-type dualities. We have obtained T_1-type dualities for frames and for σ-complete Boolean algebras, which are briefly described in the following.

3.4.1 T_1-Type Duality between Frames and Topological Spaces

Usually, a point of a frame is defined as a completely prime filter of the frame. However, the points of some topological spaces such as algebraic varieties in $\mathbb{C}^n$ and $\mathbb{R}^n$ with Zariski topologies cannot be recovered by taking a completely prime filter as a point, while they can be recovered by taking a maximal join-complete ideal as a point. A maximal join-complete ideal is defined as follows.

Definition 3.64. A join-complete ideal of a frame is defined as a proper ideal of the frame which is closed under arbitrary joins.

A maximal join-complete ideal of a frame is defined as a join-complete ideal of the frame which is maximal with respect to inclusion.

We next introduce the notion of "spatiality" based on the view that a point is a maximal join-complete ideal. We define m-spatial frames as frames with enough maximal join-complete ideals as follows.

Definition 3.65. A frame L is called m-spatial iff for any $a, b \in L$ with $a \nleq b$ there is a maximal join-complete ideal M of L such that $a \notin M$ and $b \in M$.

A frame homomorphism $f : L \to L'$ between frames L and L' is called an m-homomorphism iff for any maximal join-complete ideal M of L', $f^{-1}(M)$ is a maximal join-complete ideal of L.

The view that a point is a maximal join-complete ideal gives us the following duality between m-spatial frames and T_1 spaces.

Theorem 3.66. *Let* **mSpFrm** *denote the category of m-spatial frames and m-homomorphisms and let* $\mathbf{T_1Top}$ *denote the category of* T_1 *topological spaces and continuous maps. Then, categories* **mSpFrm** *and* $\mathbf{T_1Top}$ *are dually equivalent.*

It is important here that the dual notion of continuous map between T_1 spaces is m-homomorphism and is not frame homomorphism in the usual sense.

In order to prove the above duality, we use two functors mSpec : **mSpFrm** $\to$ $\mathbf{T_1Top}$ and Open : $\mathbf{T_1Top}$ $\to$ **mSpFrm**. Open is the usual functor mapping a topological space into the frame of open sets of it. mSpec is different from the usual functor based on completely prime filters and is defined as a functor mapping a frame into the space of maximal join-complete ideals of it whose topology is defined in the usual way (recall Zariski topology or Stone topology).

We can also provide an algebraic characterization of m-spatiality: A frame is m-spatial iff it is co-atomistic.

3.4.2 T_1-Type Duality between σ-Complete Boolean Algebras and Measurable Spaces

We next consider σ-complete Boolean algebras and measurable spaces.

Definition 3.67. A measurable space (S, Σ) is T_1 iff $\{x\} \in \Sigma$ for any $x \in S$.

Then, $\mathbf{T_1Meas}$ denotes the category of T_1 measurable spaces and measurable maps.

Let us recall the concepts of σ-complete Boolean algebra and its homomorphism.

Definition 3.68. A σ-complete Boolean algebra is defined as a Boolean algebra with joins and meets of countable subsets of it.

A homomorphism of σ-complete Boolean algebras is defined as a homomorphism of Boolean algebras preserving joins and meets of countable subsets.

The σ-distributivity is defined as follows.

Definition 3.69. A σ-complete Boolean algebra A is σ-distributive iff for any $x \in A$ and $\{x_n \,|\, n \in \omega\} \subset A$, the following holds:

$$(\bigvee_{n\in\omega} x_n) \wedge x = \bigvee_{n\in\omega} (x_n \wedge x).$$

We then define the concept of atomic homomorphism.

Definition 3.70. For Boolean algebras A and A', a homomorphism $f : A \to A'$ is called an atomic homomorphism iff for any atom $a' \in A'$, there is a unique atom $a \in A$ such that

$$f(a) = a' \text{ and } f^{-1}(\uparrow a') = \uparrow a.$$

$\mathbf{AtmBA}_\sigma$ denotes the category of atomic σ-complete Boolean algebras with the σ-distributivity and atomic homomorphisms.

Now we have the following duality.

Theorem 3.71. *Categories* $\mathbf{AtmBA}_\sigma$ *and* $\mathbf{T_1MS}$ *are dually equivalent.*

It is important that measurable maps between T_1 measurable spaces correspond to atomic homomorphisms and do not correspond to homomorphisms in the usual sense.

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