An optimization modeling of coordinated traffic signal control based on the variational theory and its stochastic extension

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Background

- Necessity of continuous review of signal setting
  - keeping a good performance of signal control
  - # of fixed time controls in Japan ≈ 135,000
    » # of professionals in Japan ≈ 250 (≈ 540/person)
    ✓ almost impossible to review these frequently

- Traffic monitoring system
  - various sensing information (detector, probe etc.)
  - can know delays at intersections more precisely
  ✓ Next step: signal setting optimization
Data Fusion based on variational theory (VT)

- Reconstruction of vehicle trajectories

*Mehran, Kuwahara and Nanzin, ISTTT19, 2011.*
Another motivation

- Fixed time coordinated signal control *in practice*
  1. Common cycle length
     » apply Webster’s formula to a critical intersection
  2. Green splits are *locally (separately)* determined
  3. Offsets are determined for given cycle length/green splits

✓ Objectives of stages aren’t usually consistent
  - expected and deterministic delay, throughput, bandwidth
✓ Interrelations btw signal parameters aren’t systematically understood
Existing theories

- Some fundamentals of coordinated signal controls
  - zigzag relations among delays, cycle and offset

Existing theories (cont'd)

- Applications of analytical MFD
  - stationary throughput is function of density & signal setting

\[ q = \frac{C}{k} \left[ 1 + \left( \frac{v}{w} \right) \left( 1 - \frac{C}{C} \right) \right] \]

\[ k_f = k_0 \left( 1 + \frac{c}{w} \right) \]


\[ \text{Throughput (proportional to inverse of delay)} \]

Jin & Yu, arxiv, 2015

✓ a simultaneous optimization method that can reflect these complex relations is need for evaluating current systems

cycle length
**Purpose of this study**

- Propose a MILP for **fixed-time** coordinated signal control that can optimize signal parameters simultaneously
  - traffic dynamics is modeled using the VT
  - signal constraints are modeled using a certain network

- Stochastic extension
  - incorporate **random arrivals** into the VT
  - expected delay calculation method is developed
    - consider **residual queues could largely affect delays**
      (almost all existing studies are deterministic)
Features of proposed methods

- Proposed formulations have
  - **fewer binary variables and constraints** than past KW-MILPs
    - link-based & free from vehicle holding (e.g., Han et al., 2016)
      - cell-based (Lo, 1999, 2001)
    - linearly formulate the fixed-time signal constraints
      - if-then (nonlinear) formulation (Lo, 1999, 2000)
      - ignore the lost times (Lin & Wang, 2004)
  - clear network structures
    - expected delay is calculated using **SP structure of the VT**
    - CE method for stochastic problem uses **Markov chain on signal constraints network** to generate samples efficiently
Deterministic optimization (Mixed Integer Linear Program)

\[
\min \sum_r D^r(s) \quad \text{... Calculate total delay } D_r(s) \text{ of road } r \text{ by VT}
\]

subject to \( s \in S \) \quad \text{... Network modeling of signal constraints}

Stochastic optimization under random arrivals

\[
\min_{s \in S} \sum_r \mathbb{E}[D^r(s)]
\]

... Calculate expected value of stochastic total delay \( D_r(s) \)

... Construct a heuristic solution method (omitted)
**Assumptions**

- Homogeneous arterial corridor with signalized junctions
  - triangular fundamental diagram is assumed
  - initial condition and (deterministic) demand arrival pattern at the upstream end of road are given
  - control time period $[0, T]$ is divided by small interval $\Delta t$
  - turning movements are prohibited (i.e., two-phase control)
Calculating total delay

- Total delay for a given signal pattern $s$

\[ D^r(s) = \sum_{t' = -L/u}^{T-L/u} N(t', 0) \Delta t - \sum_{t=0}^{T} N(t, L) \Delta t \]

- Variational Theory (Daganzo, 2005a,b)
  - Hamilton-Jacobi equation (FD)
    \[ \partial N(t, x)/\partial t = Q(-\partial N(t, x)/\partial x) \]
  - Variational problem
    \[ N_P = \min_{B \in \mathcal{B}} \left[ N_B + R_{BP} \right] \]

Cumulative flow at $B$ (boundary data)

Maximum change in cumulative flow along a time-space path between $B$ and $P$ ("cost")
VT network

- slope of link = wave speed
- link cost = maximum change in the cumulative flow
**VT network (cont'd)**

- **space**
- **exit nodes**
- **initial data**
- **boundary data (demand)**
- **dummy node**
- **signal links**
- **ordinary links**
- **green phase**
- **red phase**

- $u$
- $-\omega$
One-to-many shortest path problem

- discrete variational problem: finding optimal (valid) paths

\[
D^r(s) = U - \min_{y \geq 0} \cdot \sum_{(i,j) \in L_o \cap L_s} c_{ij} y_{ij}
\]

s.t. \[\sum_i y_{ij} - \sum_i y_{ji} = \delta_{i\text{d}} \quad \forall i \in \mathcal{V}\]

\[
\text{cost}_{ij} \text{ in unit time } \Delta t = \begin{cases} 
0 & \text{if slope of } ij = u \\
q_{\text{max}} \Delta t & \text{if slope of } ij = -w \\
q_{\text{max}} \Delta t \cdot s & \text{if slope of } ij = 0 \\
N_j \text{ (data)} & \text{if link } oj \text{ (dummy link)}
\end{cases}
\]

\[
s = \begin{cases} 
0 & \text{if link } ij \text{ is red phase} \\
1 & \text{if link } ij \text{ is green phase}
\end{cases}
\]

\[
\delta_{i\text{d}} = \begin{cases} 
1 & \text{if node } i \text{ is exit node} \\
0 & \text{otherwise}
\end{cases}
\]
Linear programming formulation (cont'd)

- One-to-many shortest path problem
  - dual problem: finding optimal costs (cumulative flows)
    \[ D^r(s) = U - \max_N \cdot \sum_{j \in V_{exit}} N_j \Delta t \]
    \[ \text{s.t. } N_j \Delta t \leq N_i \Delta t + c_{ij} \quad \forall (i, j) \in \mathcal{L}_0 \cap \mathcal{L}_s \]

- Vector-matrix forms (w.l.o.g. \( \Delta t = 1 \))
  - primal:
    \[ D^r(s) = U - \min_{y \geq 0} \cdot c^T y \quad \text{s.t. } Ay = b \]
  - dual:
    \[ D^r(s) = U - \max_N \cdot b^T N \quad \text{s.t. } A^T N + c \geq 0. \]
Deterministic optimization (Mixed Integer Linear Program)

\[
\min \sum_r D^r(s) \quad \text{... Calculate total delay } D_r(s) \text{ of road } r \text{ by VT}
\]
subject to \( s \in S \quad \text{... Network modeling of signal constraints} \)

Stochastic optimization under random arrivals

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**Signal constraints modeling**

- **Decision variables** $s$
  - on-and-off pattern *(green-and-red phase sequence)*
  
    cf. green splits, offsets *(Lo, 1999, 2001)*

- **Minimal physical constraints**
  - the signals of both **main and cross roads** must not have **green phase at the same time**
  - there is the **lost times** *(all red times)* when signal switches
Signal constraint network

- consists of signal links of VT networks for main and cross roads
Signal constraint network (cont'd)

- signal links are connected by dummy links for ensuring lost times
any path btw OD pair satisfies minimal constraints
if signal links are set to green phase when path passes through.
Signal constraint network (cont'd)

- **Mathematical form**
  - conservation of *binary* flow $z$ at each node
    $$\bar{A}z = \bar{b}$$
    link-node incidence matrix of signal constraint network
  - linear transformation from a path to the signal pattern
    $$\begin{bmatrix} s^{main} \\ s^{cross} \end{bmatrix} = Tz$$
    transformation matrix
Minimum green time constraint

- Modified signal constraint network
  - dummy link connects nodes so that their time difference is the sum of lost and minimum green times
  - signal link corresponding to min. green time should be green

\[ \bar{A}z = b \quad [ s_{cross} ] = Tz \]
Constraint for fixed-time control

- Linear constraint on dummy links
  - common cycle length must be given in advance
  - flow states of two dummy links that the time difference of their destinations is common cycle length are same: \( z_{ii'} = z_{jj'} \)

\( \checkmark \) yielding the offset endogenously
Overall problem

- Bilevel problems
  \[
  \min_{s \in S} \cdot \sum_r D^r(s) \equiv \sum_r [U^r - \min_{y^r \in \Omega^r(y)} \cdot (c^r)^T y^r] \quad \text{or} \quad \min_{s \in S} \cdot \sum_r D^r(s) \equiv \sum_r [U^r - \max_{N^r \in \Omega^r \cdot N^r} \cdot (b^r)^T N^r]
  \]
  - upper level: signal pattern \(s\) to minimize the total delay \(D(s)\)
  - lower level: total delay \(D(s)\) is calculated by a LP (VT)

- Single-level MILP
  \[
  \max_{s \in S, N^r \in \Omega^r \cdot (N)} \cdot \sum_r (b^r)^T N^r
  \]
  (cf. \(\max_{s \in S} \cdot \min_{y^r \in \Omega^r(y)} \cdot \sum_r (c^r)^T y^r\) \(\leftrightarrow\))
Overall problem (cont'd)

- Single-level MILP

\[
\begin{align*}
\max_{z^m \in \mathbb{Z}^m, \ N^r} & \quad N^r \cdot \sum_{r \in \mathcal{R}} (b^r)^T N^r \\
\text{s.t.} & \quad (A^r_0)^T N^r + c^r_0 \geq 0 \quad \forall r \in \mathcal{R} \\
& \quad (A^r_s)^T N^r + q^r_{\max} s^r \geq 0 \quad \forall r \in \mathcal{R} \\
& \quad \bar{A}^m z^m = \bar{b}^m \quad \forall m \in \mathcal{M} \\
& \quad C^m z^m = 0 \quad \forall m \in \mathcal{M} \\
& \quad \bar{s}^m = T^m z^m \quad \forall m \in \mathcal{M}
\end{align*}
\]

- problem size is quite smaller than existing KW-based fixed-time signal control problems (e.g., Lo, 2001)
Overall problem (cont'd)

- Single-level MILP

\[
\max_{z^m \in \mathbb{Z}^m, \sum_{r \in \mathcal{R}} (b^r)^T N^r} \quad \text{VT (Dual)} \\
\text{s.t.} \\
(A_o^r)^T N^r + c_o^r \geq 0 \quad \forall r \in \mathcal{R} \\
(A_s^r)^T N^r + q_{\max}^r s^r \geq 0 \quad \forall r \in \mathcal{R} \\
\bar{A}^m z^m = \bar{b}^m \quad \forall m \in \mathcal{M} \\
C^m z_d^m = 0 \quad \forall m \in \mathcal{M} \\
\bar{s}^m = T^m z^m \quad \forall m \in \mathcal{M}
\]

- cycle, green splits, offsets can be optimized \textit{simultaneously}
  - given cycle, the problem optimizes green splits & offsets
  - solving problems with different cycles (\textbf{1-dimensional search})
Deterministic optimization (Mixed Integer Linear Program)

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\min_{s \in S} \sum_r \mathbb{E}[D_r(s)]
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... Construct a heuristic solution method (omitted)
Expected total delay

\[ \mathbb{E}[D^r(s)] = \sum_{t'=-L/u}^{T-L/u} \mathbb{E}[N(t',0)] \Delta t - \sum_{t=0}^{T} \mathbb{E}[N(t,L)] \Delta t \]

- expected virtual arrivals (Poisson arrivals, given)
- expected departures (unknown) at downstream end

Variational Theory with stochastic boundary data

- stochastic variational problem

\[ \mathbb{E}[N_P] = \mathbb{E}[\min_{B \in B} \cdot \{N_B + R_{BP}\}] \]

- stochastic cumulative count at B (stochastic term)
- minimum cost btw B and P (deterministic term)

✓ looks like a probit-type SPP (or discrete choice model)
Problem structure

- Deterministic case
  - initially empty road
  - min. path consists of
    » FW links
    » signal links
Problem structure: Deterministic case

- Deterministic case
  - observation: local minima occur at valleys of $R$ graph
    - monotonic increase of cum. arrivals & capacity constraint

\[ R_{k,j} \]
\[ N_k + R_{kj} \]
\[ N_k : \text{cumulative arrivals (given)} \]
\[ N_j = \min_k \{N_k + R_{kj}\} \]
\[ -q_{\text{max}} \]

candidate boundary points

time at the entrance
k-th entrance node
Problem structure: Stochastic case

- how to calculate $\mathbb{E}[N_j] = \mathbb{E}[\min_k \{N_k + R_{kj}\}]$ ??

$\mathbb{E}[N_k] + 3\sigma$

$\mathbb{E}[N_k] + R_{kj}$

$\mathbb{E}[N_k] - 3\sigma$

$\mathbb{E}[N_k]$ : expected cumulative arrivals

$R_{k,j}$

$-q_{max}$

candidate boundary points

time at the entrance
k-th entrance node
Difficulties & Solutions

- # of candidate boundary points increases with time
  - computationally inefficient
  - "local minima occur at valleys of $R$ graph" is still true
    - can reduce # of candidates drastically and deterministically

- Correlation among "cumulative arrivals" $N_k = \sum_{k'=0}^{k} \epsilon_{k'}$
  - exact calculation is costly and might be difficult
  - analytical approximation: Clark’s method
Clark’s approximation

- approximate the distribution of minimum values for two normal random variables as a new normal distribution

  » classical approx. method for probit-models (Daganzo et al., 1977)
  » recursively apply the method to the problem

\[
N_j = \min_{k=1,\ldots,K}.\{N_1 + R_{1j}, N_2 + R_{2j}, \ldots, N_K + R_{Kj}\}
\approx \min_k.\{\epsilon_{1,2}, N_3 + R_{3j}, \ldots, N_K + R_{Kj}\}
\approx \min_k.\{\epsilon_{1,3}, N_4 + R_{4j}, \ldots, N_K + R_{Kj}\}
\vdots
\approx \min_k.\{\epsilon_{1,K-1}, N_K + R_{Kj}\}
\approx \epsilon_{1,K} \sim \mathcal{N}(\cdot, \cdot)
\]

\(N_j \mapsto \mathcal{N}(\cdot, \cdot)\)  

stochastic costs at local minima

(Poisson distribution)

\(\text{Po}(\lambda) \to_{\lambda \to \infty} \mathcal{N}(\lambda, \sqrt{\lambda})\)
Accuracy of Clark’s approximation

- 3 intersections, demand \((t = 0 \sim 15 \text{ [min]}), \Delta t = 1 \text{ [sec]}\)

\[\begin{array}{c}
\text{(a) Expected cumulative departures} \\
\text{(b) Variations of cumulative departures} \\
\text{(only 5,000 sample paths are shown)}
\end{array}\]

✓ expected departures can be approximated with extremely high accuracy (within 1% error)
Applications

- Comparison of three types of signal setting strategies
  - **strategy S**: minimizing expected total delay (Poisson arrivals)
  - **strategy D**: minimizing deterministic delay (uniform arrivals)
  - **strategy P**: practical setting
    - Webster’s cycle, equisaturation (splits), optimization (offsets)

<table>
<thead>
<tr>
<th>Road No.</th>
<th>$r = 7$</th>
<th>$r = 8$</th>
<th>$r = 1$</th>
<th>$r = 2$</th>
<th>$r = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand rate [veh/h]</td>
<td>700</td>
<td>700</td>
<td>150</td>
<td>700</td>
<td>400</td>
</tr>
</tbody>
</table>

near saturated condition
Result: Optimal cycle length

- Expected total delay vs. common cycle length
  - strategy S produces the lowest expected delay
  - strategy D leads to poor performance for some cases

simultaneous optimization in a deterministic fashion may not work effectively unless cycle is carefully chosen.
Result: Optimal cycle length (cont'd)

- Optimal common cycle length
  - strategy S is much shorter than others
    » fluctuations (random arrivals, effect of up. intersections)
    » zigzag relations among delay and cycle (Koshi, 1985)
Result: Expected vehicle trajectories

- Contour on 3D surface of expected cumulative flows
  - expected queue length information can be obtained
  - instantaneous deceleration at the back of queue is not always observed because queue length fluctuates
Result: Expected vehicle trajectories (cont'd)

- Strategy S (cycle length = 60 sec)
  - effective green times are almost fully utilized

**main road 1**

**main road 2**
Result: Expected vehicle trajectories (cont'd)

- Strategy D (cycle length = 90 sec)
  - those times of the most downstream intersection are wasted
**Result: Expected vehicle trajectories (cont'd)**

- Strategy P (cycle length = 110 sec)
  - those times of the most downstream intersection are wasted
Summary

- Proposed a MILP for fixed-time coordinated signal control
  - has a clear network structure and requires fewer binary variables and constraints than existing formulations

- Stochastic extension
  - expected delay calculation method is developed
  - examined optimal signal control parameters under both deterministic and stochastic arrivals
Future research directions

- Investigating basic properties of coordinated signal control by a systematic numerical experiments

- Considering turning movements
  - need to introduce CTM diverge/merge model
  - violate SP property of VT that makes problem more complex

- Comparisons with other stochastic KWMs
  - CTM-based models (Sumalee et al., 2011; Jabari & Liu, 2012)
  - LTM-based model (Osorio & Flötteröd, 2015)
Thank you!

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References

References

Appendix
Constraint for fixed-time control (cont.)

- Difference in the starting times of first green phase (blue link) on the main roads for the adjacent intersections
Accuracy of Clark’s approximation

- Result in different scenarios

<table>
<thead>
<tr>
<th>Cycle length [sec]</th>
<th>DoS [%]</th>
<th>Coordination</th>
<th>RMSE [veh]</th>
<th>Error [%]</th>
<th>Time (C) [sec]</th>
<th>Time (M) [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>85</td>
<td>good</td>
<td>0.17</td>
<td>0.19</td>
<td>0.60</td>
<td>50.1</td>
</tr>
<tr>
<td>90</td>
<td>85</td>
<td>bad</td>
<td>0.12</td>
<td>0.14</td>
<td>0.60</td>
<td>53.3</td>
</tr>
<tr>
<td>90</td>
<td>100</td>
<td>good</td>
<td>0.34</td>
<td>-0.25</td>
<td>0.62</td>
<td>51.8</td>
</tr>
<tr>
<td>90</td>
<td>115</td>
<td>good</td>
<td>0.32</td>
<td>-0.22</td>
<td>0.62</td>
<td>51.0</td>
</tr>
<tr>
<td>60</td>
<td>85</td>
<td>good</td>
<td>0.17</td>
<td>0.19</td>
<td>0.67</td>
<td>50.3</td>
</tr>
</tbody>
</table>
Why does the Clark’s method have a high accuracy?

- Existing knowledge (Horowitz et al., 1982)
  - The Clark’s approximation yields good results when
    » the **random components** of the utilities (or costs) of different alternatives are **independent or positively correlated**
    » moreover, they have **variances that are not greatly different from one another**.

✓ both properties are satisfied in our case
**Cross-Entropy method**

- **Basic concept**
  - probability distribution $f$ of solutions are adjusted so as to reach an ideal distribution using an importance sampling (i.e., an optimal solution is generated with probability 1)
  - a probability of a path on the signal constraint network can be represented as the product of the transition probabilities of the Markov chain on this network.
Result: Delay distribution

- Strategy S has highest variance
  - smaller available capacity is more susceptible to demand fluctuations
- Frequency of high delay under S is always less than others
  - If it’s true for many cases, S can be regarded as a robust strategy