From Operational Chu Duality to Coalgebraic Quantum Symmetry

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Abstract. We pursue the principles of duality and symmetry building upon Pratt's idea of the Stone Gamut and Abramsky's representations of quantum systems. In the first part of the paper, we first observe that the Chu space representation of quantum systems leads us to an operational form of state-observable duality, and then show via the Chu space formalism enriched with a generic concept of closure conditions that such operational dualities (which we call "T_1-type" as opposed to "sober-type") actually exist in fairly diverse contexts (topology, measurable spaces, and domain theory, to name but a few). The universal form of T_1-type dualities between point-set and point-free spaces is described in terms of Chu spaces and closure conditions. From the duality-theoretical perspective, in the second part, we improve upon Abramsky's "fibred" coalgebraic representation of quantum symmetries, thereby obtaining a finer, "purely" coalgebraic representation: our representing category is properly smaller than Abramsky's, but still large enough to accommodate the quantum symmetry groupoid. Among several features, our representation reduces Abramsky's two-step construction of his representing category into a simpler one-step one, thus rendering the Grothendieck construction redundant. Our purely coalgebraic representation stems from replacing the category of sets in Abramsky's representation with the category of closure spaces in the light of the state-observable duality telling us that closure is a right perspective on quantum state spaces.

1 Introduction

It is not uncommon these days to hear of applications of (the methods of) theoretical computer science to foundations of quantum physics; broadly speaking, theoretical computer science seems to be taking steps towards a new kind of the unified science (not that in the sense of logical positivism) via the language and methodology of category theory. Among them, Abramsky [1, 2] represents quantum systems as Chu spaces and as coalgebras, giving striking characterisations of quantum symmetries based upon the classic Wigner Theorem. Revisiting his work, in the present paper, we develop a Chu-space-based theory of dualities encompassing a form of state-observable duality in quantum physics, and thereafter improve upon his coalgebraic characterisation of quantum symmetries from our duality-theoretical perspective, in order to exhibit the meaning of duality.

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In Pratt's Stone Gamut paper [16], he analyses Stone-type dualities in the language of Chu spaces, saying boldly, but with good reasons, "the notoriously difficult notion of Stone duality reduces simply to matrix transposition." The concept of Chu spaces has played significant roles in fairly broad contexts, including concurrency and semantics of linear logic; similar concepts have been used in even more diverse disciplines, like Barwise-Seligman's classifications, Sambin's formal topology and basic pairs, Scott's information systems, and state-property systems in quantum foundations. This work is inspired by Pratt's perspective on Chu spaces, extending the realm of duality theory built upon the language of Chu spaces by enriching it with a generic concept of closure conditions.

In general, we have two types of dualities, namely sober-type and T_1-type ones, between set-theoretical concepts of space and their point-free, algebraic abstractions, which shall be called point-set spaces and point-free spaces respectively. The difference between the two types of dualities in fact lies in the difference between maximal and primal spectra. Our duality theory in this paper focuses upon T_1-type dualities between point-set and point-free spaces. The logical concept of closure conditions is contrived to the end of treating different sorts of point-set and point-free spaces in a unified manner, allowing us to discuss at once topological spaces, measurable spaces, closure spaces, convexity spaces, and so fourth. In a nutshell, the concept of closure conditions prescribes the notion of space. Whilst a typical example of sober-type duality is the well-known duality between sober spaces and spatial frames, an example of T_1-type duality is a duality between T_1 closure spaces and atomistic meet-complete lattices, including as particular instances state-observable dualities between quantum state spaces (with double negation closures) and projection operator lattices in the style of operational quantum mechanics (see Coecke and Moore [5] or Moore [14]).

Our theory of T_1-type dualities enables us to derive a number of concrete T_1-type dualities in various contexts, which include T_1-type dualities between Scott's continuous lattices and convexity spaces, between σ-complete boolean algebras and measurable spaces, and between topological spaces and frames, to name but a few. Let us illustrate by a topological example a striking difference between sober-type and T_1-type dualities. The T_1-type duality in topology is a duality between T_1 spaces and coatomistic frames in which continuous maps correspond not to frame homomorphisms but to maximal homomorphisms, which are frame homomorphisms $f : L \to L'$ such that, given a maximal join-complete ideal $M \subset L'$, $f^{-1}(M)$ is again a maximal join-complete ideal. Although the duality for T_1 spaces is not mentioned in standard references such as Johnstone [9], nevertheless, we consider it important for the reason that some spaces of interest are not sober but T_1: e.g., affine varieties in k^n with k an ACF (i.e., algebraically closed field) are non-sober T_1 spaces (if they are not singletons). Note that Bonsangue et al. [4] shows a duality for T_1 spaces via what they call observation frames, which are frames with additional structures, yet the T_1-type duality above only relies upon plain frame structures.

Whilst sober-type dualities are based upon prime spectrum "Spec", T_1-type dualities are based upon maximal spectrum "Spm". Different choices of spectrum lead to different Chu representations of algebras A concerned: maximal spectrum gives $(A, \mathrm{Spec}(A), e)$ and prime spectrum gives $(A, \mathrm{Spm}(A), e)$ where e is two-valued and defined in both cases by: $e(a, M) = 1$ iff $a \in M$. Accordingly, the corresponding classes of Chu morphisms are distinctively different: e.g., in locale theory, the Spec-based representation characterises frame homomorphisms as Chu morphisms (as shown in Pratt [16]), and the Spm-based representation characterises maximal homomorphisms as Chu morphisms (as shown in this paper for general point-free spaces encompassing frames as just a particular instance). In this way, the Chu space formalism yields a natural account of why different concepts of homomorphisms appear in sober-type and T_1-type dualities.

As in the case above, Chu morphisms can capture different sorts of homomorphisms by choosing different Chu representations. This is true even in quantum contexts, and in particular we can represent quantum symmetries as Chu morphisms by a suitable Chu representation. Coalgebras are Chu spaces with dynamics, and we have a coalgebraic representation of quantum symmetries as well. To be precise, in moving from Chu space to coalgebras, Abramsky [2] relies upon a fibred category $\int \mathbf{F}$ obtained by gluing categories $\mathbf{Coalg}(F^Q)$'s for every $Q \in \mathbf{Set}$ where F^Q is an endofunctor on $\mathbf{Set}$. He uses the Grothendieck construction to "accommodate contravariance" within a coalgebraic framework, fully embedding the groupoid of symmetries into the fibred category $\int \mathbf{F}$.

Looking at the $\int \mathbf{F}$ representation from a duality-theoretical perspective, we consider it odd that there is no structural relationship taken into account between quantum state spaces and projection operator lattices: both are seen as mere sets. For the very reason, Q (which is a projection lattice in a quantum context) first have to be fixed in the endofunctor F^Q on $\mathbf{Set}$ (objects of which are state spaces in a quantum context), and thereafter $\mathbf{Coalg}(F^Q)$'s are glued together to accommodate contravariance regarding $Q \in \mathbf{Set}$. This two-step construction is reduced in the present paper into a simpler, one-step one as follows.

First of all, there is actually a dual, structural relationship between quantum state spaces and projection lattices with the latter re-emerging as the fixpoints (or algebras) of double negation closures (or monads) on the former. This means that Q above can be derived, rather than independently assumed, from a closure structure, if one works on the base category of closure spaces, rather than mere sets. The closure-based reformulation of the $\int \mathbf{F}$ representation leads us to a "Born" endofunctor $\mathbf{B}$ on closure spaces, and to its coalgebra category $\mathbf{Coalg}(\mathbf{B})$, which turns out to be strictly smaller than fairly huge $\int \mathbf{F}$, but still large enough to represent the quantum symmetry groupoid, thus yielding a purely coalgebraic representation and enabling to accommodate contravariance within the single colagebra category $\mathbf{Coalg}(\mathbf{B})$ rather than the fibred $\int \mathbf{F}$ glueing different $\mathbf{Coalg}(F^Q)$'s for all sets Q; notice that contravariance is incorporated into the dualisation process of taking the fixpoints (or algebras) of closures.

2 Duality and Chu Space Representation

We first review basic concepts and notations on Chu spaces and closure spaces.

Chu Spaces Let us fix a set Ω. A Chu space over Ω is a triple (S, A, e) where S and A are sets, and e is a map from $S \times A$ to Ω. Ω is called the value set, and e the evaluation map. A Chu morphism from (S, A, e) to (S', A', e') is a tuple (f^*, f_*) of two maps $f^* : S \to S'$ and $f_* : A' \to A$ such that $e(x, f_*(a')) = e'(f^*(x), a')$. The category of Chu spaces and Chu morphisms is self-dual, and forms a $*$-autonomous category, giving a fully complete model of linear logic.

For a Chu space $(S, A, e : S \times A \to \Omega)$ and $a \in A$, $e(\text{-}, a) : S \to \Omega$ is called a column of (S, A, e). We denote the set of all columns of (S, A, e) by $\mathrm{Col}(S, A, e)$. On the other hand, $e(x, \text{-}) : A \to \Omega$ is called a row of (S, A, e). We denote the set of all rows of (S, A, e) by $\mathrm{Row}(S, A, e)$. If Ω is ordered, then we equip $\mathrm{Col}(S, A, e)$ and $\mathrm{Row}(S, A, e)$ with the pointwise orderings: e.g., in the case of $\mathrm{Col}(S, A, e)$, this means that, for $a, b \in A$, $e(\text{-}, a) \leq e(\text{-}, b)$ iff $e(x, a) \leq e(x, b)$ for any $x \in S$.

A Chu space (S, A, e) is called extensional iff all the columns are mutually different, i.e., if $e(x, a) = e(x, b)$ for any $x \in S$ then $a = b$. On the other hand, a Chu space (S, A, e) is called separated iff all the rows are mutually different, i.e., if $e(x, a) = e(y, a)$ for any $a \in A$ then $x = y$.

Closure Spaces Closure spaces may be seen as either a set with a closure operator or a set with a family of subsets that is closed under arbitrary intersections. We denote by $\mathcal{C}(S)$ the set of closed subsets of a closure space S, and by $\mathrm{cl}(\text{-})$ the closure operator of S. In this paper we always assume $\emptyset \in \mathcal{C}(S)$ or equivalently $\mathrm{cl}(\emptyset) = \emptyset$. Note then that there is a unique closure structure on a singleton. A map $f : S \to S'$ is called closure-preserving iff $f^{-1}(C) \in \mathcal{C}(S)$ for any $C \in \mathcal{C}(S')$ iff $f(\mathrm{cl}(A)) \subset \mathrm{cl}(f(A))$. We denote by **Clos** the category of closure spaces and closure-preserving maps, which has products and coproducts. A closure space is called T_1 iff any singleton is closed.

2.1 Chu Representation of Quantum Systems

Abramsky [2] represents a quantum system as a Chu space defined via the Born rule, which provides the predictive content of quantum mechanics. Given a Hilbert space H, he constructs the following Chu space over the unit interval $[0, 1]$: $(\mathrm{P}(H), \mathrm{L}(H), e_H : \mathrm{P}(H) \times \mathrm{L}(H) \to [0, 1])$ where $\mathrm{P}(H)$ denotes the set of quantum states as rays (i.e., one-dimesional subspaces) in H, $\mathrm{L}(H)$ denotes the the set of projection operators (or projectors) on H, and finally the evaluation map e_H is defined as follows (let $[\varphi] = \{\alpha\varphi \mid \alpha \in \mathbb{C}\}$): $e_H([\varphi], P) = \frac{\langle \varphi | P\varphi \rangle}{\langle \varphi | \varphi \rangle}$.

We consider that Chu spaces have built-in dualities, or they are dualities without structures: whilst S and A have no structure, e still specifies duality. The category of Chu spaces has duals in terms of monoidal categories; this is internal duality. Can we externalise internal duality in Chu spaces by restoring structures on S and A through e? It is an inverse problem as it were. In the quantum context, it amounts to explicating the structures of $\mathrm{P}(H)$ and $\mathrm{L}(H)$ that give (external) duality.

The first observation is the bijective correspondences: $\mathrm{P}(H) \simeq \{e(\varphi, \text{-}) \mid \varphi \in \mathrm{P}(H)\} \simeq \{c \in \mathrm{Col}(\mathrm{P}(H), \mathrm{L}(H), e_H) \mid$ the precisely one 1 appears in $c\}$. So, the states are the atoms of L(H): in this way we can recover P(H) from L(H). This means L(H) should be equipped with the lattice structure as in Birkhoff-von-Neumann's quantum logic. Although we have $\mathrm{L}(H) \simeq \{e(\text{-}, P) \mid P \in \mathrm{L}(H)\}$, it is not clear at this stage what intrinsic structure of P(H) enables to recover L(H) from P(H). Let us see that a double negation operator on P(H) does the job.

Define $(\text{-})^{\perp} : \mathcal{P}(\mathrm{P}(H)) \to \mathcal{P}(\mathrm{P}(H))$ as follows: for $X \subset \mathrm{P}(H)$, let $X^{\perp} = \{[\varphi] \in \mathrm{P}(H) \mid \forall [\psi] \in \mathrm{P}(H)\ \langle \varphi | \psi \rangle = 0\}$. It is straightforward to see that $(\text{-})^{\perp\perp}$ is a closure operator on P(H). Categorically, $(\text{-})^{\perp\perp}$ is a sort of double negation monad. Taking the closed sets or algebras of $(\text{-})^{\perp\perp}$ enables us to recover L(H):

Proposition 1 *The lattice of closed subsets of* P(H)*, i.e.,* $\{X \subset \mathrm{P}(H) \mid X^{\perp\perp} = X\}$*, is isomorphic to* L($H$)*. Schematically,* $\mathcal{C}(\mathrm{P}(H)) \simeq \mathrm{L}(H)$.

We thus have a duality between P(H) qua closure space and L(H) qua lattice. We can reconstruct P(H) from L(H) by taking the atoms on the one hand, and L(H) from P(H) by taking the closed sets (or algebras) of $(\text{-})^{\perp\perp}$ on the other. This dualising construction generally works for T_1 closure spaces and atomistic meet-complete lattices, in particular including P(H) and L(H) respectively; orthocomplements can be added to this duality.

Categorically, we have a dual equivalence between the category of T_1 closure spaces with closure-preserving maps and the category of atomistic meet-complete lattices with maximal homomorphisms (defined below). This duality is basically known at the object level in operational quantum mechanics (see Moore [14] or Coecke and Moore [5]); nevertheless, our dualisation of arrows, i.e., the concept of maximality, may be new. In this section we aim at developing a theory of such T_1-type dualities in full generality, thereby deriving T_1-type dualities in various concrete contexts as immediate corollaries (which include the state-projector duality). We embark upon this enterprise in the next subsection.

2.2 Chu Theory of T_1-Type Dualities via Closure Conditions

In the following part of this section, we consider two-valued Chu spaces $(S, A, e : S \times A \to \mathbf{2})$ only, where $\mathbf{2}$ denotes $\{0, 1\}$ (with ordering $0 < 1$). This is because in the duality between states P(H) and property observables L(H) we do not need other intermediate values in $[0, 1]$; when considering duality, it suffices to take into account whether a value equals 1 or not. On the other hand, intermediate values in $[0, 1]$ play an essential role in characterising quantum symmetries coalgebraically; we need at least three values (i.e., 1, 0, and "neither 0 nor 1"). In a nutshell, duality is possibilistic, whilst symmetry is probabilistic.

In this subsection, we think of (Chu representations of) "point-set" spaces $(S, \mathcal{F})$ where $\mathcal{F} \subset \mathcal{P}(S)$, and of their "point-free" abstractions L which do not have an underlying set S whilst keeping algebraic structures corresponding to closure properties of $\mathcal{F}$. Especially, we discuss **Top**, **Set**, **Clos**, **Conv**, and **Meas** where **Conv** denotes the category of convexity spaces, which are sets S with $\mathcal{C} \subset \mathcal{P}(S)$ closed under arbitrary intersections and directed unions (quite some convex geometry can be developed based upon such abstract structures; see, e.g., van de Vel [18]); **Meas** denotes the category of measurable spaces, which are sets with $\mathcal{B} \subset \mathcal{P}(S)$ closed under complements and countable intersections.

Morphisms in all of these categories of point-set spaces are defined in the same way as continuous maps, closure-preserving maps, and measurable maps (a.k.a. Borel functions): i.e., they are $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ such that $f^{-1}(X) \in \mathcal{F}$ for any $X \in \mathcal{F}'$. Note that **Set** may be seen as the category of $(S, \mathcal{F})$ such that $\mathcal{F}$ is maximally closed, i.e., $\mathcal{F} = \mathcal{P}(S)$, with "continuous" maps as morphisms; in such a situation, any map satisfies the condition that $f^{-1}(X) \in \mathcal{F}$ for $X \in \mathcal{F}'$.

Their point-free counterparts are respectively: **Frm** (frames), **CABA** (complete atomic boolean algebras), **MCLat** (meet-complete lattices), **ContLat** (Scott's continuous lattices), and σ**BA** (σ-complete boolean algebras). Continuous lattices may be defined as meet-complete lattices with directed joins distributing over arbitrary meets (this is equivalent to the standard definition via way-below relations; see [6, Theorem I-2.7]); in the light of this, we see continuous lattices as point-free convexity spaces; later, duality justifies this view.

We emphasise that closure conditions on each type of point-set structures correspond to (possibly infinitary) algebraic operations on each type of point-free structures. An insight from our theory is that such a relationship between point-set and point-free spaces always leads us to duality; indeed, we shall show T_1-type dualities between **Top** and **Frm**; **Set** and **CABA**; **Clos** and **MCLat**; **Conv** and **ContLat**; **Meas** and σ**BA**; and even more (e.g., dcpos).

In order to treat different sorts of point-set spaces in a unified manner, we introduce a concept of closure conditions. A closure condition on $\mathcal{F} \subset \mathcal{P}(S)$ is a formula of the following form:

$$\forall \mathcal{X} \subset \mathcal{F}\ (\varphi(\mathcal{X}) \Rightarrow \mathrm{BC}(\mathcal{X}) \in \mathcal{F})$$

where $\mathrm{BC}(\mathcal{X})$ is a (possibly infinitary) boolean combination of elements of $\mathcal{X}$ and $\varphi(\mathcal{X})$ is a closed formula in the language of propositional connectives, quantifiers, equality, a binary, inclusion predicate $\subset$, and nullary, cardinality predicates[1], $\mathrm{card}_{\leq\kappa}(\mathcal{X})$ and $\mathrm{card}_{\geq\kappa}(\mathcal{X})$, for each countable cardinal κ; you may include arbitrary cardinals, though the language becomes uncountable. The domain of the intended interpretation of this language is $\mathcal{X}$, and predicates are to be interpreted in the obvious way: $X \subset Y$ with $X, Y \in \mathcal{X}$ is interpreted as saying that X is a subset of Y, $\mathrm{card}_{\leq\kappa}(\mathcal{X})$ as saying that the cardinality of $\mathcal{X}$ is less than or equal to κ, and so fourth. Note that predicates $\mathrm{card}_{=\kappa}(\mathcal{X})$, $\mathrm{card}_{<\kappa}(\mathcal{X})$, and $\mathrm{card}_{>\kappa}(\mathcal{X})$ are definable in the above language.

In this setting, for example, measurable spaces are $(S, \mathcal{F})$ such that $\mathcal{F} \subset \mathcal{P}(S)$ satisfies the following closure conditions: $\forall \mathcal{X} \subset \mathcal{F}\ (\mathrm{card}_{\leq\omega}(\mathcal{X}) \Rightarrow \bigcap \mathcal{X} \in \mathcal{F})$ and $\forall \mathcal{X} \subset \mathcal{F}\ (\mathrm{card}(\mathcal{X}) = 1 \Rightarrow \mathcal{X}^c \in \mathcal{F})$ where $\mathcal{X}^c$ denotes the complement of the unique element of $\mathcal{X}$. and notice that by letting $\mathcal{X} = \emptyset$ we have $\bigcap \emptyset = S \in \mathcal{F}$. Likewise, convexity spaces are $(S, \mathcal{F})$ with $\mathcal{F}$ satisfying the following: $\forall \mathcal{X} \subset$

[1] First-order logic allows us to express "there are n many elements" for each positive integer n, but cannot express certain cardinality statements (e.g., "there are at most countably many elements"; we need this when defining measurable spaces). For the very reason, we expand the language with the afore-mentioned cardinality predicates.

$\mathcal{F}$ ($\top \Rightarrow \bigcap \mathcal{X} \in \mathcal{F}$) and $\forall \mathcal{X} \subset \mathcal{F}$ ("$\mathcal{X}$ is directed w.r.t. $\subset$" $\Rightarrow \bigcup \mathcal{X} \in \mathcal{F}$) where $\top$ is any tautology and "$\mathcal{X}$ is directed w.r.t. $\subset$" is expressed as "$\forall X\, \forall Y\, \exists Z\, (X \subset Z \wedge Y \subset Z)$". It is straightforward to find closure conditions for other sorts of point-set spaces. We denote by $\mathfrak{X}_{\mathrm{top}}$ the closure conditions for **Top**, by $\mathfrak{X}_{\mathrm{meas}}$ those for **Meas**, by $\mathfrak{X}_{\mathrm{clos}}$ those for **Clos**, and by $\mathfrak{X}_{\mathrm{conv}}$ those for **Conv**.

Let us denote by $\mathfrak{X}$ a class of closure conditions, and $(S, \mathcal{F})$ with $\mathcal{F} \subset \mathcal{P}(S)$ satisfying $\mathfrak{X}$ is called a point-set $\mathfrak{X}$-space. We always assume that $\mathfrak{X}$ contains: $\forall \mathcal{X} \subset \mathcal{F}$ ($\mathrm{card}_{=0}(\mathcal{X}) \Rightarrow \bigcup \mathcal{X} \in \mathcal{F}$). This ensures that $\emptyset$ is in $\mathcal{F}$. We denote by $\mathbf{PtSp}_{\mathfrak{X}}$ the category of point-set $\mathfrak{X}$-spaces with $\mathfrak{X}$-preserving maps (i.e., maps $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ such that $f^{-1}(X) \in \mathcal{F}$ for any $X \in \mathcal{F}'$). If this setting looks too abstract, $\mathbf{PtSp}_{\mathfrak{X}}$ in the following discussion may be thought of as any of our primary examples: **Top**, **Clos**, **Conv**, and **Meas**.

It plays a crucial role in our duality theory that φ in a closure condition can be interpreted in a point-free setting: in other words, it only talks about the mutual relationships between elements of $\mathcal{X}$, and does not mention elements of elements of $\mathcal{X}$ or any point of an observable region $X \in \mathcal{X}$ (which may be an open set, convex set, measurable set, or the like), thus allowing us to interpret it in any abstract poset $(L, \leq)$ by interpreting the subset symbol $\subset$ as a partial order $\leq$, and lead to the concept of point-free $\mathfrak{X}$-spaces as opposed to point-set ones. We call this interpretation of φ in a poset $(L, \leq)$ the point-free interpretation of φ. Note that the above language for φ is actually nothing but the language of the first-order theory of posets enriched with the cardinality predicates.

A point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ can be regarded as a Chu space $(S, \mathcal{F}, e_{(S,\mathcal{F})} : S \times \mathcal{F} \to \mathbf{2})$ where e is defined by: $e_{(S,\mathcal{F})}(x, X) = 1$ iff $x \in X$.

A special focus of the paper is on T_1 point-set spaces: a point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ is T_1 iff any singleton is in $\mathcal{F}$. When applying this definition to topology, we see a topological space as a set with a family of closed sets rather than open sets. The T_1 property of a Chu space is defined as follows.

Definition 2 *A Chu space (S, A, e) is called T_1 iff for any $x \in S$, there is $a \in A$ such that $e(x, a) = 1$ and $e(y, a) = 0$ for any $y \neq x$.*

Intuitively, a above may be thought of as a region in which there is only one point, namely x, or a property that x does satisfy and any other $y \in S$ does not.

Lemma 3 *A point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ is T_1 iff the corresponding Chu space $(S, \mathcal{F}, e_{(S,\mathcal{F})})$ defined above is a T_1 Chu space.*

Lemma 4 *For point-set $\mathfrak{X}$-spaces $(S, \mathcal{F})$ and $(S', \mathcal{F}')$, a tuple of maps $(f, g) : (S, \mathcal{F}, e_{(S,\mathcal{F})}) \to (S', \mathcal{F}', e_{(S',\mathcal{F}')})$ is a Chu morphism iff $g = f^{-1} : \mathcal{F}' \to \mathcal{F}$ iff $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ is $\mathfrak{X}$-preserving.*

Lemma 5 *If a Chu space (S, A, e) is T_1 and extensional, then for any $x \in S$ there is a unique $a \in A$ such that $e(x, a) = 1$ and $e(y, a) = 0$ for any $y \neq x$.*

Each column $e(\text{-}, a)$ of a Chu space (S, A, e) can be regarded as a subset of S, i.e., as $\{x \in S \mid e(x, a) = 1\}$. We say that $\mathrm{Col}(S, A, e)$ satisfies closure conditions iff the corresponding family of subsets of S satisfies them. The same property can be defined for $\mathrm{Row}(S, A, e)$ as well. The following proposition shows that a broad variety of point-set spaces can be represented as Chu spaces.

Proposition 6 *The category* $\mathbf{PtSp}_{\mathfrak{X}}$ *is equivalent to the category of extensional Chu spaces* (S, A, e) *such that* $\mathrm{Col}(S, A, e)$ *satisfies the closure conditions* $\mathfrak{X}$*, denoted by* $\mathbf{ExtChu}_{\mathfrak{X}}$*. In particular, this can be instantiated for* $\mathfrak{X}_{\mathrm{top}}$*,* $\mathfrak{X}_{\mathrm{meas}}$*,* $\mathfrak{X}_{\mathrm{clos}}$*, and* $\mathfrak{X}_{\mathrm{conv}}$*.*

In the following, we focus on a more specific class of closure conditions. A closure condition $\forall \mathcal{X} \subset \mathcal{F}\ (\varphi(\mathcal{X}) \rightarrow \mathrm{BC}(\mathcal{X}) \in \mathcal{F})$ is called pure iff $\mathrm{BC}(\mathcal{X})$ contains precisely one of unions, intersections, and complements. A pure closure condition is monolithic, and does not blend different operations; this is true in any major example mentioned above.

In order to define point-free $\mathfrak{X}$-spaces, we let $\mathfrak{X}$ be a class of pure closure conditions satisfying the following: if a closure condition in $\mathfrak{X}$ contains complementation in its boolean combination part, then the following two closure conditions are in $\mathfrak{X}$: $\forall \mathcal{X} \subset \mathcal{F}\ (\mathrm{card}_{<\omega}(\mathcal{X}) \rightarrow \bigcap \mathcal{X} \in \mathcal{F})$ and $\forall \mathcal{X} \subset \mathcal{F}\ (\mathrm{card}_{<\omega}(\mathcal{X}) \rightarrow \bigcup \mathcal{X} \in \mathcal{F})$. These additional conditions ensure that once we have complementation on the point-set side we can define boolean negation on the point-free side. Note that, although complementation on sets is, and should be, interpreted as boolean negation on posets of subsets, nevertheless, we are not excluding intuitionistic negation (or interiors of complements of opens), which does not arise from complements in closure conditions (i.e., complements without interiors are boolean), but from unions and finite intersections in them, by which we can define intuitionistic implication, and so intuitionistic negation.

We then define a point-free $\mathfrak{X}$-space as a bounded poset $(L, \leq, 0, 1)$ satisfying the following. If a closure condition in $\mathfrak{X}$ have unions (intersections, complements) in its $\mathrm{BC}(\mathcal{X})$ under the condition φ, then we require L to have joins (meets, boolean negation) under the point-free interpretation of φ (i.e., the subset symbol $\subset$ is interpreted as $\leq$). If one closure condition in $\mathfrak{X}$ contains unions and another contains intersections under the conditions $\varphi(\mathcal{X})$ and $\psi(\mathcal{X})$ respectively, then we require L to satisfy the following (possibly infinitary) distributive law: for any doubly indexed family $\{x_{i,j} \mid i \in I,\ j \in J_i\} \subset L$ with $F := \prod_{i \in I} J_i$, if $\{x_{i,j} \mid j \in J_i\}$ denoted by L_1 and $\{\bigwedge_{i \in I} x_{i,f(i)} \mid f \in F\}$ denoted by L_2 satisfy $\varphi(L_1)$ and $\varphi(L_2)$ respectively, and if $\{x_{i,f(i)} \mid i \in I\}$ denoted by L_3 and $\{\bigvee_{j \in J_i} x_{i,j} \mid i \in I\}$ denoted by L_4 satisfy $\psi(L_3)$ and $\psi(L_4)$ respectively, then $\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigvee_{f \in F} \bigwedge_{i \in I} x_{i,f(i)}$. Note that this reduces to the ordinary infinite distributive law in the case of frames, and to distributivity between meets and directed joins in the case of continuous lattices.

There is a subtlety in defining maps f preserving possibly partial operations: e.g., even if $\bigwedge X$ is defined, $\bigwedge f(X)$ is not necessarily defined. In the case of directed joins of continuous lattices, however, this causes no problem, since directedness is preserved under monotone maps, i.e., if X is directed then $\bigwedge f(X)$ is directed as well. This is also true in the case of σ-complete boolean algebras, since $\mathrm{card}_{\leq\omega}$(-) is always preserved. With these in mind, we assume: φ in each closure condition in $\mathfrak{X}$ is preserved under monotone maps, i.e., for a monotone map $f : L \to L'$ between point-free $\mathfrak{X}$-spaces L and L', if $\varphi(X)$ holds for $X \subset L$ then $\varphi(f(X))$ holds as well. Homomorphisms of point-free $\mathfrak{X}$-spaces are defined as monotone maps preserving (in general partial) operations induced from the closure conditions in $\mathfrak{X}$. The category of point-free $\mathfrak{X}$-spaces and homomorphisms is denoted by $\mathbf{PfSp}_{\mathfrak{X}}$.

For a point-free $\mathfrak{X}$-space L, we denote the set of atoms in L by $\mathrm{Spm}(L)$, which is called the maximal spectrum of L for the following reason. In the cases of **Frm**, **ContLat**, and **MCLat**, $\mathrm{Spm}(L)$ is actually isomorphic to the maximal filters or ideals with suitable completeness conditions; furthermore, the maximal spectrum of the coordinate ring of an affine variety V in k^n with k an ACF is homeomorphic to $\mathrm{Spm}(L)$ by taking L to be the closed set lattice of V. To exemplify the meaning of "completeness conditions", let us consider **MCLat**. A meet-complete filter is defined as a filter that is closed under arbitrary meets. Since the meet-complete filters of $L \in \mathbf{MCLat}$ bijectively correspond to the principal filters of L, we have an isomorphism between $\mathrm{Spm}(L)$ and the maximal meet-complete filters of L, which holds even in the presence of natural closure structures on them. Alternatively, we may also define $\mathrm{Spm}(L) = \{\uparrow a \mid a \text{ is an atom}\}$ where $\uparrow a = \{x \in L \mid a \leq x\}$. This definition is sometimes more useful than the former.

The continuous maps between T_1 spaces (e.g., affine varieties in $\mathbb{C}^n$) do not correspond to the frame homomorphisms between their open set frames, but to a more restricted class of frame homomorphisms; this exhibits a sharp difference from the case of sober spaces. A maximal homomorphism of point-free $\mathfrak{X}$-spaces is a homomorphism $f : L \to L'$ of them satisfying the maximality condition: for any $b \in \mathrm{Spm}(L')$ there is $a \in \mathrm{Spm}(L)$ such that $\uparrow a = f^{-1}(\uparrow b)$, where note that such an $a \in \mathrm{Spm}(L)$ is necessarily unique. If $\mathrm{Spm}(L)$ is defined as $\{\uparrow a \mid a \text{ is an atom}\}$, then we may state maximality in a more familiar manner: $f^{-1}(M) \in \mathrm{Spm}(L)$ for any $M \in \mathrm{Spm}(L')$. The category of atomistic point-free $\mathfrak{X}$-spaces and maximal homomorphisms is denoted by $\mathbf{AtmsPfSp}_{\mathfrak{X}}$ where recall that a poset with the least element is called atomistic iff any element can be described as the join of a set of atoms. Note that atomic posets and atomistic posets are different in general.

The atomisticity of a Chu space is defined in the following way.

Definition 7 *A Chu space (A, S, e) is called atomistic iff there are $A' \subset A$ and a bijection $\eta : S \to A'$ such that*

1. *any two elements of* $\mathrm{Row}(A', S, e')$ *are incomparable (with respect to its point-wise ordering) where e' is defined by $e'(a, x) = e(a, x)$;*
2. *for any $x \in S$ and $a \in A$, $e(a, x) = 1$ iff $e(\eta(x), \text{-}) \leq e(a, \text{-})$.*

The intended meaning of A' above is $\mathrm{Spm}(A)$, or the set of atoms of A. In the context of quantum mechanics, item 1 above means that any two quantum states, when seen as one-dimensional subspaces or projectors onto them, are incomparable, and item 2 means that there is a canonical correspondence between the quantum state space $\mathrm{P}(H)$ and the projection lattice $\mathrm{L}(H)$, by mapping the quantum states to the atoms of the lattice.

Proposition 8 *A Chu space* (S, A, e) *is* T_1 *and extensional iff its dual* $(A, S, \hat{e})$ *is atomistic and separated where we define* $\hat{e}(a, x) = e(x, a)$*.*

It does not necessarily hold that (S, A, e) is T_1 iff $(A, S, \hat{e})$ is atomistic. As a corollary of the above proposition, we obtain:

Corollary 9 *If a Chu space* (A, S, e) *is atomistic and separated, and* $\mathrm{Row}(A, S, e)$ *has a least element, then* $\mathrm{Row}(A, S, e)$ *is an atomistic poset with its atoms given by* $\{e(\eta(x), \text{-}) \mid x \in S\}$*.*

Given a point-free $\mathfrak{X}$-space L, we can construct a Chu space $(L, \mathrm{Spm}(L), e_L)$ where e_L is defined by: $e_L(b, a) = 1$ iff $a \leq b$. If we define $\mathrm{Spm}(L) = \{\uparrow a \mid a \text{ is an atom}\}$, the corresponding e_L is specified by: $e_L(a, M) = 1$ iff $a \in M$.

Lemma 10 *A point-free* $\mathfrak{X}$*-space* L *is atomistic iff* $(L, \mathrm{Spm}(L), e_L)$ *is an atomistic Chu space.*

If we define $\mathrm{Spm}(L) = \{\uparrow a \mid a \text{ is an atom}\}$, we can take $\tilde{f}$ in the following lemma to be f^{-1}; in this case, the alternative definition of $\mathrm{Spm}(L)$ seems more transparent than the definition of it as the set of atoms themselves.

Lemma 11 *Let* L *and* L' *be atomistic point-free* $\mathfrak{X}$*-spaces. A pair of maps,* $(f, g) : (L, \mathrm{Spm}(L), e_L) \to (L', \mathrm{Spm}(L'), e_{L'})$*, is a Chu morphism iff* f *is a maximal homomorphism and* $g = \tilde{f}$ *where* $\tilde{f} : \mathrm{Spm}(L') \to \mathrm{Spm}(L)$ *is such that, for any* $b \in \mathrm{Spm}(L')$*,* $f^{-1}(\uparrow b) = \uparrow \tilde{f}(b)$ *(note* $\tilde{f}$ *is well defined because* f *is maximal).*

Proposition 12 *The category* $\mathbf{AtmsPfSp}_{\mathfrak{X}}$ *is equivalent to the category of atomistic separated Chu spaces* (A, S, e) *such that* $\mathrm{Row}(A, S, e)$ *satisfies the closure conditions* $\mathfrak{X}$*, denoted by* $\mathbf{AtmsSepChu}_{\mathfrak{X}}$*.*

We finally lead to the main duality theorem, exposing and unifying T_1-type dualities in diverse contexts, including sets, topology, measurable spaces, closure spaces, domain theory, and convex geometry.

Theorem 13 $\mathrm{T}_\mathbf{1}\mathbf{ExtChu}_{\mathfrak{X}}$ *is dually equivalent to* $\mathbf{AtmsSepChu}_{\mathfrak{X}}$*; therefore,* $\mathrm{T}_\mathbf{1}\mathbf{PsSp}_{\mathfrak{X}}$ *is dually equivalent to* $\mathbf{AtmsPfSp}_{\mathfrak{X}}$*. In particular, this universal duality can be instantiated for* $\mathfrak{X}_{\mathrm{top}}$*,* $\mathfrak{X}_{\mathrm{meas}}$*,* $\mathfrak{X}_{\mathrm{clos}}$*, and* $\mathfrak{X}_{\mathrm{conv}}$*.*

Although many sorts of point-free spaces are complete, nevertheless, the case of $\mathfrak{X}_{\mathrm{meas}}$ is different, and only requires σ-completeness. In this case, the universal duality above yields a duality between atomistic σ-complete boolean algebras and T_1 measurable spaces. As noted above, **Set** may be seen as the category of $(S, \mathcal{P}(S))$'s with measurable maps (note any map is measurable on $(S, \mathcal{P}(S))$), so that the duality for measurable spaces turns out to restrict to the classic Stone duality between **Set** and **CABA** (note "atomic" and "atomistic" are equivalent in boolean algebras). It is thus a vast globalisation of the classic Stone duality.

Furthermore, we can apply the theorem above to dcpos (with 0), which is not complete in general, by considering closure under directed unions, which yields point-set spaces $(S, \mathcal{F})$ with $\mathcal{F}$ closed under directed unions; dcpos are their duals. Likewise, preframes fall into the picture as well. We are able to derive even more dualities in the same, simple way; although some general theories of dualities require much labour in deriving concrete dualities (this is a typical complaint on abstract duality theory from the practicing duality theorist), the universal duality above immediately gives us concrete dualities of T_1-type.

The duality obtained in the case of $\mathfrak{X}_{\mathrm{top}}$ is not subsumed by the orthodox duality between sober spaces and spatial frames, since "sober" does not imply "T_1"; there are important examples of non-sober T_1 spaces, including affine varieties in k^n with the Zariski topologies where k is an ACF. As discussed in the Introduction, furthermore, the morphism part of the T_1-type duality is distinctively different from that of the sober-type one.

In the case of $\mathfrak{X}_{\mathrm{conv}}$, we obtain a duality between atomistic continuous lattices and T_1 convexity spaces, exposing a new connection between domains and convex structures. Maruyama [12] also gives closely related dualities for convexity spaces. Jacobs [8] shows a dual adjunction between preframes and algebras of the distribution monad, which are abstract convex structures as well as convexity spaces. We can actually relate the two sorts of abstract convex structures, and thus dualities for them, by several adjunctions and equivalences, though here we do not have space to work out the details.

In the case of sober-type dualities, we first have dual adjunctions for general point-free spaces, which then restrict to dualities (i.e., dual equivalences). In the case of T_1-type dualities, however, we do not have dual adjunctions behind them because we use maximal spectrum Spm rather than prime spectrum Spec. This is the reason why in this paper we have concentrated on the Chu representation of atomistic point-free spaces, rather than point-free spaces in general. We leave it for future work to work out the dual adjunction between $\mathbf{PsSp}_{\mathfrak{X}}$ and $\mathbf{PfSp}_{\mathfrak{X}}$ which restricts to the corresponding sober-type duality.

3 Quantum Symmetries and Closure-Based Coalgebras

We first review the Grothendieck construction for later discussion.

Grothendieck Construction The Grothendieck construction enables us to glue different categories together into a single category, or turn an indexed category into a fibration. Given a functor $\mathbf{I} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$, we define a category

$$\int \mathbf{I} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

as follows ($\mathbf{CAT}$ denotes the category of (small) categories and functors). The objects of $\int \mathbf{I}$ consist of tuples (C, X) where $C \in \mathbf{C}$ and $X \in \mathbf{I}(C)$. An arrow from (C, X) to (D, Y) in $\int \mathbf{I}$ is defined as a pair (f, g) where $f : D \to C$ and $g : \mathbf{I}(f)(X) \to Y$. Finally, composition of $(f : D \to C, g : \mathbf{I}(f)(X) \to Y) : (C, X) \to (D, Y)$ and $(p : E \to D, q : \mathbf{I}(p)(Y) \to Z) : (D, Y) \to (E, Z)$ is defined as:

$$(f \circ p, q \circ \mathbf{I}(p)(g)) : (C, X) \to (E, Z).$$

Note that the type of $\mathbf{I}(p)(g)$ is $\mathbf{I}(p)(\mathbf{I}(f)(X)) \to \mathbf{I}(p)(Y)$, which in turn equals $\mathbf{I}(f \circ p)(X) \to \mathbf{I}(p)(Y)$. We call $\int \mathbf{I}$ the fibred category constructed from the indexed category $\mathbf{I}$. The obvious forgetful functor from the fibred category $\int \mathbf{I}$ to the base category $\mathbf{C}$ which maps (C, X) to C gives a fibration.

3.1 Born Coalgebras on Closure Spaces

Now, we define an endofunctor $\mathbf{B} : \mathbf{Clos} \to \mathbf{Clos}$ on the category of closure spaces. For a closure space X, let

$$\mathbf{B}(X) := (\{0\} + (0,1] \times X)^{\mathcal{C}(X)}$$

where $(\{0\}+(0,1]\times X)^{\mathcal{C}(X)}$ is the product of $\mathcal{C}(X)$-many copies of $\{0\}+(0,1]\times X$. For a closure-preserving map $f : X \to Y$, we define a map

$$\mathbf{B}(f) : (\{0\} + (0,1] \times X)^{\mathcal{C}(X)} \to (\{0\} + (0,1] \times Y)^{\mathcal{C}(Y)}$$

by

$$\mathbf{B}(f)(h)(C) = (id_{\{0\}} + id_{(0,1]} \times f) \circ h \circ f^{-1}(C)$$

where $h \in (\{0\} + (0,1] \times X)^{\mathcal{C}(X)}$ and $C \in \mathcal{C}(Y)$.

Lemma 14 *For a closure-preserving map $f : X \to Y$, $\mathbf{B}(f)$ is closure-preserving.*

Lemma 15 *Let X, Y, Z be closure spaces. (i) $\mathbf{B}(id_X) = id_{\mathbf{B}(X)}$. (ii) $\mathbf{B}(g \circ f) = \mathbf{B}(g) \circ \mathbf{B}(f)$ for closure-preserving maps $f : X \to Y$ and $g : Y \to Z$.*

Now, we describe primary examples of $\mathbf{B}$-coalgebras, which are of central importance in our investigation.

Example 16 *Given a Hilbert space H, we define a $\mathbf{B}$-coalgebra*

$$(\mathrm{P}(H), \alpha_H : \mathrm{P}(H) \to \mathbf{B}(\mathrm{P}(H)))$$

as follows. Let us define $\alpha_H : \mathrm{P}(H) \to (\{0\} + (0,1] \times \mathrm{P}(H))^{\mathcal{C}(\mathrm{P}(H))}$ by

$$\alpha_H([\varphi])(S) = \begin{cases} 0 & \textit{if } \langle\varphi|P_S\varphi\rangle = 0 \\ (\frac{\langle\varphi|P_S\varphi\rangle}{\langle\varphi|\varphi\rangle}, [P_S\varphi]) & \textit{otherwise} \end{cases}$$

where $[\varphi] \in \mathrm{P}(H)$, $S \in \mathrm{L}(H)$ ($\simeq \mathcal{C}(\mathrm{P}(H))$), and P_S is the projection operator corresponding to S.

The coalgebra $(\mathrm{P}(H), \alpha_H)$ expresses the dynamics of repeated Born-rule-based measurements of a quantum system represented by a Hilbert space H.

As in Abramsky [2], we define the groupoid of quantum symmetries as follows.

Definition 17 **QSym** *is the category whose objects are projective spaces of Hilbert spaces of dimension greater than 2 and whose arrows are semi-unitary maps identified up to a phase factor $e^{i\theta}$.*

Wigner's theorem (or Wigner-Bargmann's theorem) clarifies the physical meaning of **QSym** as follows. Note that "surjections" below are actually bijections, since injectivity follows by the other properties.

Theorem 18 **QSym** *is equivalent to the category whose objects are projective spaces of Hilbert spaces (i.e., quantum state spaces) and whose arrows are symmetry transformations (i.e., those surjections between projective spaces that preserve transition probabilities* $\frac{|\langle\varphi|\psi\rangle|^2}{|\varphi|^2|\psi|^2}$ *between quantum states* $[\varphi]$ *and* $[\psi]$*).*

Our aim is to establish a purely coalgebraic understanding of **QSym**. We remark that symmetries are of central importance in physics: they are higher laws of conservation of various physical quantities (Nöther's theorem); in quantum mechanics in particular, we can even derive the Schrödinger-equation-based dynamics of quantum systems from a continuous one-parameter group of symmetries (Stone's theorem).

3.2 Quantum Symmetries Are Purely Coalgebraic

For an endofunctor $G : \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$, let $\mathbf{Coalg}(G)$ denote the category of G-coalgebras.

Let us briefly review Abramsky's fibred category $\int \mathbf{F}$ of coalgebras in the following. For a fixed set Q, we define a functor $F^Q : \mathbf{Set} \to \mathbf{Set}$. Given a set X, let $F^Q(X) = (\{0\} + (0,1] \times X)^Q$. The arrow part is then defined canonically.

An indexed category

$$\mathbf{F} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{CAT}$$

is then defined as follows. Given $Q \in \mathbf{Set}$, let $\mathbf{F}(Q) = \mathbf{Coalg}(F^Q)$. For a map $f : Q' \to Q$, we define a functor $\mathbf{F}(f) : \mathbf{Coalg}(F^Q) \to \mathbf{Coalg}(F^{Q'})$ in the following way. Given an object $(X, \alpha : X \to F^Q(X))$ in $\mathbf{Coalg}(F^Q)$, let

$$\mathbf{F}(f)(X,\alpha) = (X, t^f_X \circ \alpha)$$

where $t^f_X : F^Q(X) \to F^{Q'}(X)$ is defined by $t^f_X(g) = g \circ f$. Given an arrow $g : (X,\alpha) \to (Y,\beta)$, let $\mathbf{F}(f)(g) = g : (X, t^f_X \circ \alpha) \to (Y, t^f_Y \circ \beta)$.

As Wigner's theorem above has the assumption of surjectivity, Abramsky [2] requires surjectivity on the first components f of morphisms (f,g) in $\int \mathbf{F}$. Let us denote by $\int \mathbf{F_s}$ the resulting category with the restricted class of morphisms. On the other hand, we require injectivity on the morphisms $f : (X,\alpha) \to (Y,\beta)$ of $\mathbf{Coalg}(\mathbf{B})$, and denote by $\mathbf{Coalg_i}(\mathbf{B})$ the resulting category with the restricted class of morphisms. The surjectivity/injectivity conditions ensure that **QSym** is not only faithfully but also fully represented in $\int \mathbf{F_s}$ and in $\mathbf{Coalg_i}(\mathbf{B})$.

In the following we observe that $\mathbf{Coalg}(\mathbf{B})$ is much smaller than $\int \mathbf{F}$, but still large enough to encompass the quantum symmetry groupoid **QSym**. To be precise, it shall be shown that $\mathbf{Coalg}(\mathbf{B})$ is a non-full proper subcategory of $\int \mathbf{F}$, and that **QSym** is a full subcategory of $\mathbf{Coalg_i}(\mathbf{B})$.

We then introduce a functor **BF** from $\mathbf{Coalg}(\mathbf{B})$ to $\int \mathbf{F}$, which will turn out to be a non-full embedding of categories.

Definition 19 *The object part of* $\mathbf{BF}:\mathbf{Coalg}(\mathbf{B})\to\int\mathbf{F}$ *is defined by*

$$\mathbf{BF}(X,\alpha:X\to\mathbf{B}(X))=(\mathcal{C}(X),(X,\alpha)\in\mathbf{Coalg}(F^{\mathcal{C}(X)})).$$

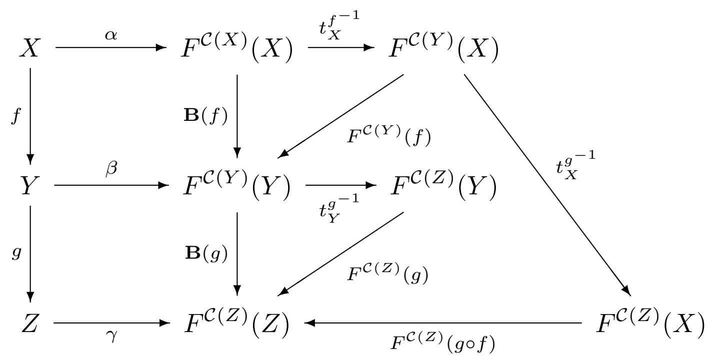
The arrow part of $\mathbf{BF}:\mathbf{Coalg}(\mathbf{B})\to\int\mathbf{F}$ *is defined by*

$$\mathbf{BF}(f:(X,\alpha)\to(Y,\beta))=(f^{-1}:\mathcal{C}(Y)\to\mathcal{C}(X),\tilde{f}:\mathbf{F}(f^{-1})(X,\alpha)\to(Y,\beta))$$

where $\tilde{f}$ *has the same underlying function as* f *(i.e.,* $\tilde{f}(x)=f(x)$ *for any* $x\in X$*; thus, the difference only lies in their types).*

In order to justify the definition above, we have to verify that $\tilde{f}$ is actually a morphism in $\mathbf{Colag}(F^{\mathcal{C}(Y)})$.

The commutative diagram below would be useful to understand what is going on in the definition above and the two lemmata below.



where α,β,γ are $\mathbf{B}$-coalgebras, and f,g are morphisms of $\mathbf{B}$-coalgebras.

Lemma 20 $\tilde{f}:\mathbf{F}(f^{-1})(X,\alpha)\to(Y,\beta)$ *is an arrow in* $\mathbf{Colag}(F^{\mathcal{C}(Y)})$.

Lemma 21 *(i)* $\mathbf{BF}(id_{(X,\alpha)})=id_{\mathbf{BF}(X,\alpha)}$. *(ii) For* $f:(X,\alpha)\to(Y,\beta)$ *and* $g:(Y,\beta)\to(Z,\gamma)$ *in* $\mathbf{Coalg}(\mathbf{B})$*,* $\mathbf{BF}(g\circ f)=\mathbf{BF}(g)\circ\mathbf{BF}(f)$ *where the latter composition is that in* $\int\mathbf{F}$.

Proposition 22 $\mathbf{Coalg}(\mathbf{B})$ *can be embedded into* $\int\mathbf{F}$ *via the functor* $\mathbf{BF}$. *This is not a full embedding (i.e.,* $\mathbf{BF}$ *is not full).*

The non-fullness of $\mathbf{BF}$ implies that $\mathbf{Coalg}(\mathbf{B})$ is a smaller category than $\int\mathbf{F}$ with respect to arrows as well as objects.

We now introduce a functor $\mathbf{SC}$ from $\mathbf{QSym}$ to $\mathbf{Coalg_i}(\mathbf{B})$, which will turn out to be a full embedding of categories.

Definition 23 *The object part of* $\mathbf{SC}:\mathbf{QSym}\to\mathbf{Coalg_i}(\mathbf{B})$ *is defined by*

$$\mathbf{SC}(\mathrm{P}(H))=(\mathrm{P}(H),\alpha_H).$$

The arrow part of $\mathbf{SC}:\mathbf{QSym}\to\mathbf{Coalg_i}(\mathbf{B})$ *is defined by*

$$\mathbf{SC}(U)=U:(\mathrm{P}(H),\alpha_H)\to(\mathrm{P}(H'),\alpha_{H'})$$

where $U:\mathrm{P}(H)\to\mathrm{P}(H')$ *is a semi-unitary map from* H *to* H' *(up to a phase).*

Lemma 24 $\mathbf{SC}(U)$ *is a morphism of* $\mathbf{B}$*-coalgebras.*

W finally obtain the purely coalgebraic representation of quantum symmetries **QSym** via the non-fibred, single sort of coalgebra category $\mathbf{Coalg_i}(\mathbf{B})$ based upon closure spaces.

Theorem 25 *The quantum symmetry groupoid* **QSym** *can be fully embedded into the purely coalgebraic category* $\mathbf{Coalg_i}(\mathbf{B} : \mathbf{Clos} \to \mathbf{Clos})$.

Our closure-based coalgebraic approach to representation of quantum systems would allow us to develop "coalgebraic quantum logic" utilising existing work on coalgebraic logic over (duality between) general concrete categories (see, e.g., Kurz [11] or Klin [10]); this is left for future work.

References

1. S. Abramsky, Coalgebras, Chu spaces, and representations of physical systems, *Proc. of LICS 2010*, pp. 411-420.
2. S. Abramsky, Big toy models: representing physical systems as Chu spaces, *Synthese* 186 (2012) 697-718.
3. J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, Inc., 1990.
4. M. M. Bonsangue, B. Jacobs, and J. N. Kok, Duality beyond sober spaces: topological spaces and observation frames, *Theor. Comput. Sci.* 151 (1995) 79-124.
5. B. Coekce and D. J. Moore, Operational Galois adjunctions, *Current Research in Operational Quantum Logic*, pp. 195-218, 2000.
6. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003.
7. R. Hartshorne, *Algebraic Geometry*, Springer, 1977.
8. B. Jacobs, Convexity, duality, and effects, *Proc. of 6th IFIP International Conference on Theoretical Computer Science* (2010) 1-19.
9. P. T. Johnstone, *Stone Spaces*, Cambridge University Press, .
10. B. Klin, Coalgebraic modal logic beyond sets, *Electr. Notes Theor. Comput. Sci.* 173 (2007) 177-201.
11. A. Kurz, Coalgebras and their logics, *ACM SIGACT News* 37, 2006.
12. Y. Maruyama, Fundamental results for pointfree convex geometry, *Ann. Pure Appl. Logic* 161 (2010) 1486-1501.
13. Y. Maruyama, Natural duality, modality, and coalgebra, *Journal of Pure and Applied Algebra* (2012) 565-580.
14. D. J. Moore, On state spaces and property lattices, *Stud. Hist. Phil. Mod. Phys.* 30 (1999) 61-83.
15. H. E. Porst and W. Tholen, Concrete dualities, *Category Theory at Work*, pp. 111-136, 1991.
16. V. R. Pratt, The Stone Gamut: a coordinatization of mathematics, *Proc. of LICS 1995*, pp. 444-454.
17. J. J. M. M. Rutten, Universal coalgebra: a theory of systems, *Theor. Comput. Sci.* 249 (2000) 3-80.
18. M. L. J. van de Vel, *Theory of Convex Structures*, North-Holland, 1993.

Appendix

Proofs omitted in the main text are given here.

Proof of Lemma 4. This follows immediately from the observation that (f, g) is a Chu morphism iff $f(x) \in X$ is equivalent to $x \in g(X)$ for $x \in S$ and $X \in \mathcal{F}$.

Proof of Proposition 6. We define a functor $G : \mathbf{ExtChu}_{\mathfrak{X}} \to \mathbf{PtSp}_{\mathfrak{X}}$ as follows. Given a Chu space (S, A, e) with $\mathrm{Col}(S, A, e)$ satisfying $\mathfrak{X}$, we take $G(S, A, e)$ to be the following point-set $\mathfrak{X}$-space $(S, \mathcal{F}_S)$ where $\mathcal{F}_S$ denotes the set of those $X \subset S$ such that there is a column $e(\text{-}, a)$ of (S, A, e) with $X = \{x \in S \mid e(x, a) = 1\}$ where such a column is unique by the extensionality of (S, A, e). By the closure property of the original Chu space, $\mathcal{F}_S$ satisfies the closure conditions $\mathfrak{X}$.

Given a Chu morphism $(f, g) : (S, A, e) \to (S', A', e')$ (between two Chu spaces satisfying $\mathfrak{X}$), we define $G(f, g) = f : (S, \mathcal{F}_S) \to (S', \mathcal{F}_{S'})$. We must prove that f is $\mathfrak{X}$-preserving, i.e., $f^{-1}(Y) \in \mathcal{F}_S$ for $Y \in \mathcal{F}_{S'}$, which is equivalent to the following: there is $a \in A$ such that $f^{-1}(Y) = \{x \in S \mid e(x, a) = 1\}$. It follows from $Y \in \mathcal{F}_{S'}$ that there is $b \in A'$ such that $Y = \{y \in S' \mid e'(y, b) = 1\}$. Now we define $a = g(b)$, which is in A. Since we have $x \in f^{-1}(Y)$ iff $e'(f(x), b) = 1$, and since the Chu morphism condition tells us that $e'(f(x), b) = 1$ iff $e(x, a) = 1$, it finally follows that $f^{-1}(Y) = \{x \in S \mid e(x, a) = 1\}$.

We then define another functor $F : \mathbf{PtSp}_{\mathfrak{X}} \to \mathbf{ExtChu}_{\mathfrak{X}}$ as follows. A point-set $\mathfrak{X}$-space $(S, \mathcal{F})$ induces a Chu space $F(S, \mathcal{F}) := (S, \mathcal{F}, e_{(S,\mathcal{F})})$, and a $\mathfrak{X}$-preserving map $f : (S, \mathcal{F}) \to (S', \mathcal{F}')$ induces a Chu morphism $F(f) := (f : S \to S', f^{-1} : \mathcal{F}' \to \mathcal{F})$. $F(f)$ is indeed a Chu morphism, since $e_{(S',\mathcal{F}')}(f(x), X') = 1$ iff $f(x) \in X'$ iff $x \in f^{-1}(X')$ iff $e_{(S,\mathcal{C})}(x, f^{-1}(X')) = 1$. Here note also that $\mathrm{Col}(F(S, \mathcal{C}))$ satisfies the closure conditions $\mathfrak{X}$, and that $F(S, \mathcal{F})$ is extensional.

Now it is straightforward to see that F and G defined above give a categorical equivalence between $\mathbf{PtSp}_{\mathfrak{X}}$ and $\mathbf{ExtChu}_{\mathfrak{X}}$.

Proof of Proposition 8. Assume that (S, A, e) is T_1 and extensional. Obviously, $(A, S, \hat{e})$ is separated. We show it is atomistic. For each $x \in S$ we can choose $a_x \in A$ such that $e(x, a_x) = 1$ and $e(y, a_x) = 0$ for any $y \neq x$. Note that, since a_x is unique by extensionality (see Lemma 5), we do not need the axiom of choice to choose a_x for $x \in S$. Let $A' = \{a_x \mid x \in S\}$, and define $\eta : S \to A'$ by $\eta(x) = a_x$. The afore-mentioned property of a_x ensures that η is a bijection and any two elements of $\mathrm{Row}(A', S, \hat{e}')$ are incomparable. Assume $\hat{e}(a, x) = 1$. Then, $e(x, \eta(x)) = 1$, and $e(y, \eta(x)) = 0$ for $y \neq x$. By assumption, $\hat{e}(\eta(x), \text{-}) \leq \hat{e}(a, \text{-})$. Conversely, assume $\hat{e}(\eta(x), \text{-}) \leq \hat{e}(a, \text{-})$. Then, we have $1 = \hat{e}(\eta(x), x) \leq \hat{e}(a, x)$, and thus $\hat{e}(a, x) = 1$.

To show the converse, assume that $(A, S, \hat{e})$ is atomistic and separated. Extensionality is obvious. We show (S, A, e) is T_1. Fix $x \in S$. We claim that $e(x, \eta(x)) = 1$ and that $e(y, \eta(x)) = 0$ for any $y \neq x$. By assumption, we have: $e(x, \eta(x)) = 1$ iff $e(\text{-}, \eta(x)) \leq e(\text{-}, \eta(x))$, whence it follows that $e(x, \eta(x)) = 1$. Now, suppose for contradiction that there is $y \neq x$ such that $e(y, \eta(x)) = 1$. It then follows from assumption that

$$e(\text{-}, \eta(y)) \leq e(\text{-}, \eta(x)).$$

This is a contradiction for the following reason: $e(\text{-}, \eta(y))$ and $e(\text{-}, \eta(x))$ are different because η is bijective and $(A, S, \hat{e})$ is separated, and hence must be incomparable by assumption. We thus obtain $e(y, \eta(x)) = 0$ for any $y \neq x$.

Proof of Lemma 9. The proof of Proposition 8 tells us that (A, S, e) is T_1, and for each $x \in S$ we have $\eta(x) \in A$ such that $e(\eta(x), x) = 1$ and, for $y \neq x$, $e(\eta(x), y) = 0$. By this property of $\eta(x)$, $\{e(\text{-}, \eta(x)) \mid x \in S\}$ gives us the required set of atoms of $\mathrm{Row}(A, S, e)$.

Proof of Lemma 10. Corollary 9 tells us that if $(L, \mathrm{Spm}(L), e_L)$ is atomistic then L is atomistic. Assume that a point-free $\mathfrak{X}$-space L is atomistic. Item 1 in the definition of an atomistic Chu space is obvious. Item 2 follows from the fact that for any $a \in L$ and $x \in \mathrm{Spm}(L)$, $x \leq a$ iff, for any $y \in \mathrm{Spm}(L)$, $y \leq x$ implies $y \leq a$ iff $e_L(x, \text{-}) \leq e_L(a, \text{-})$; this intuitively says that all the elements of the algebra L are classified by the maximal filters or ideals of it.

Proof of Lemma 11. Assume that $(f, g) : (L, \mathrm{Spm}(L), e_L) \to (L', \mathrm{Spm}(L'), e_{L'})$ is a Chu morphism. Then we have $e_L(f(x), b) = e_{L'}(x, g(b))$, and so $b \leq f(x)$ iff $g(b) \leq x$; this equivalence shall freely be exploited in the following. We first show that f is monotone. Suppose $x \leq y$ in L. In order to show $f(x) \leq f(y)$, by the atomisticity of L', it suffices to prove that for any $a \in \mathrm{Spm}(L')$, if $a \leq f(x)$ then $a \leq f(y)$. If $a \leq f(x)$, then we have $g(a) \leq x \leq y$, and hence $a \leq f(y)$.

We next show that f is a homomorphism. Suppose that L has a meet operation $\bigwedge$ under a condition φ. We must prove that $f(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} f(x_i)$ for $\{x_i \mid i \in I\} \subset L$ satisfying φ. By atomisticity, it is enough to show that for any $a \in \mathrm{Spm}(L)$, $a \leq f(\bigwedge_{i \in I} x_i)$ iff $a \leq \bigwedge_{i \in I} f(x_i)$. If $a \leq f(\bigwedge_{i \in I} x_i)$, then $g(a) \leq \bigwedge_{i \in I} x_i \leq x_i$, whence $a \leq f(x_i)$, and so $a \leq \bigwedge_{i \in I} f(x_i)$. The converse follows simply by reversing this argument.

Suppose that L has a join operation $\bigvee$ under a condition φ. We must prove that $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$ for $\{x_i \mid i \in I\} \subset L$ satisfying φ. By atomisticity, it is enough to show that for any $a \in \mathrm{Spm}(L)$, $a \leq f(\bigvee_{i \in I} x_i)$ iff $a \leq \bigvee_{i \in I} f(x_i)$. If $a \leq f(\bigvee_{i \in I} x_i)$, then $g(b) \leq \bigvee_{i \in I} x_i$. Since $g(b)$ is an atom, there is $i \in I$ such that $g(b) \leq x_i$. We then have $b \leq f(x_i)$ and hence $b \leq \bigvee_{i \in I} f_(x_i)$.

Suppose that L has a boolean negation $\neg$ under a condition φ. We must prove that $f(\neg x) = \neg f(x)$ for $\{x\}$ satisfying φ. By atomisticity, it is enough to show that for any $a \in \mathrm{Spm}(L)$, $a \leq f(\neg x)$ iff $a \leq \neg f(x)$. Assume $a \leq f(\neg x)$. Since a is an atom in a Boolean algebra, we have either $a \leq \neg f(x)$ or $a \leq f(x)$. If $a \leq f(x)$, then $a \leq f(x) \wedge f(\neg x) = f(x \wedge \neg x) = f(0) = 0$ (note that by our previous convention boolean negation $\neg$ is only defined in the presence of $\vee$ and $\wedge$; we have already shown $f(x \wedge y) = f(x) \wedge f(y)$ and $f(0) = 0$ in the above arguments), which contradicts that a is an atom. Thus we have $a \leq \neg f(x)$. The converse follows by a similar argument.

It remains to show the maximality of f, i.e., $f^{-1}(\uparrow b) = \uparrow g(b)$, which is proven as follows: $x \in f^{-1}(\uparrow b)$ iff $f(x) \in \uparrow b$ iff $b \leq f(x)$ iff $g(b) \leq x$ iff $x \in \uparrow g(b)$.

Proof of Proposition 12. We define a functor $F : \mathbf{AtmsPfSp}_{\mathfrak{X}} \to \mathbf{AtmsSepChu}_{\mathfrak{X}}$ as follows. Given a point-free $\mathfrak{X}$-space L, we let $F(L) = (L, \mathrm{Spm}(L), e_L)$. Given a maximal homomorphism $f : L \to L'$, we define $F(f) = (f, \tilde{f})$. It is easy to show that $F(L)$ is a Chu space in $\mathbf{AtmsSepChu}_{\mathfrak{X}}$; note that the atomisticity of L implies both the separatedness and the atomisticity of $F(L)$.

We show that $F(f)$ is a Chu morphism, i.e., $e_{L'}(f(x), b) = e_L(x, \tilde{f}(b))$. This is equivalent to: $b \leq f(x)$ iff $\tilde{f}(b) \leq x$. If $b \leq f(x)$ then $\uparrow b \supset \uparrow f(x)$, and so $f^{-1}(\uparrow b) \supset f^{-1}(\uparrow f(x))$. Since $\uparrow \tilde{f}(b) = f^{-1}(\uparrow b)$ by the definition of $\tilde{f}$, and since $f^{-1}(\uparrow f(x)) \supset \uparrow x$, we have $\uparrow \tilde{f}(b) \supset \uparrow x$, whence it follows that $\tilde{f}(b) \leq x$. Conversely, if $\tilde{f}(b) \leq x$ then $\uparrow \tilde{f}(b) \supset \uparrow x$, and so $f^{-1}(\uparrow b) \supset \uparrow x$. We then have $x \in f^{-1}(\uparrow b)$, and hence $f(x) \in \uparrow b$, which means $b \leq f(x)$.

We define a functor $G : \mathbf{AtmsSepChu}_{\mathfrak{X}} \to \mathbf{AtmsPfSp}_{\mathfrak{X}}$ as follows. Given a Chu space (A, S, e) in $\mathbf{AtmsSepChu}_{\mathfrak{X}}$, we define $G(A, S, e) = \mathrm{Row}(A, S, e)$ where $\mathrm{Row}(A, S, e)$ is ordered pointwise. Then, $G(A, S, e)$ is an atomistic point-free $\mathfrak{X}$-space. Given a Chu morphism $(f, g) : (A, S, e) \to (A', S', e')$, we define $G(f, g) = f$ by identifying $\mathrm{Row}(A, S, e)$ with A (i.e., $e(a, \text{-})$ with a); this identification is allowed because (A, S, e) is separated. We must show that $G(f, g)$ is a maximal homomorphism. Once we prove (A, S, e) and (A', S', e') are isomorphic to $(A, \mathrm{Spm}(A), e_A)$ and $(A', \mathrm{Spm}(A'), e_{A'})$ respectively, Lemma 11 tells us $G(f, g)$ is indeed a maximal homomorphism.

Let us show that (A, S, e) is isomorphic to $(A, \mathrm{Spm}(A), e_A)$. Since (A, S, e) is atomistic, we have a bijection $\eta : S \to A'$ for some $A' \subset A$, and A' is in turn the set of atoms in A when A is identified with $\mathrm{Row}(A, S, e)$; this is a consequence of Corollary 9. We thus have a canonical bijection $\varepsilon : S \to \mathrm{Spm}(A)$. Since $e(a, x) = 1$ iff $e(\eta(x), \text{-}) \leq e(a, \text{-})$ iff $\varepsilon(x) \leq a$ iff $e_A(a, \varepsilon(x)) = 1$, it follows that (A, S, e) is isomorphic to $(A, \mathrm{Spm}(A), e_A)$.

It is straightforward to see that F and G give us a categorical equivalence between $\mathbf{AtmsPfSp}_{\mathfrak{X}}$ and $\mathbf{AtmsSepChu}_{\mathfrak{X}}$.

Proof of Theorem 13. The first part is a corollary of Proposition 8. The second part follows immediately from Proposition 6, Proposition 12, and the first part.

Proof of Lemma 14. It is sufficient to prove that $\mathbf{B}(f)(\mathrm{cl}(Z)) \subset \mathrm{cl}(\mathbf{B}(f)(Z))$ for a subset Z of $(\{0\} + (0,1] \times X)^{\mathcal{C}(X)}$. Let $C \in \mathcal{C}(Y)$. We then have:

$$\begin{aligned}
\mathbf{B}(f)(\mathrm{cl}(Z))(C) &= (id_{\{0\}} + id_{(0,1]} \times f) \circ \mathrm{cl}(Z) \circ f^{-1}(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times f) \circ \mathrm{cl}(Z \circ f^{-1}(C)) \\
&\subset \mathrm{cl}((id_{\{0\}} + id_{(0,1]} \times f) \circ Z \circ f^{-1}(C)) \\
&= \mathrm{cl}(\mathbf{B}(f)(Z)(C)) \\
&= \mathrm{cl}(\mathbf{B}(f)(Z))(C).
\end{aligned}$$

The second equality and the fifth equality hold because those closure operators are defined for the product spaces $(\{0\} + (0,1] \times X)^{\mathcal{C}(X)}$ and $(\{0\} + (0,1] \times Y)^{\mathcal{C}(Y)}$ respectively. The third inclusion follows from the assumption that f is closure-preserving. We have thus shown: $\prod_{C \in \mathcal{C}(Y)} \mathbf{B}(f)(\mathrm{cl}(Z))(C) \subset$

$\prod_{C\in\mathcal{C}(Y)} \mathrm{cl}(\mathbf{B}(f)(Z))(C)$. Since $\mathrm{cl}(\mathbf{B}(f)(Z))$ is a closed subset of the product space, we actually have $\mathrm{cl}(\mathbf{B}(f)(Z)) = \prod_{C\in\mathcal{C}(Y)} \mathrm{cl}(\mathbf{B}(f)(Z))(C)$. These, together with the following fact that $\mathbf{B}(f)(\mathrm{cl}(Z)) \subset \prod_{C\in\mathcal{C}(Y)} \mathbf{B}(f)(\mathrm{cl}(Z))(C)$, imply that $\mathbf{B}(f)(\mathrm{cl}(Z)) \subset \mathrm{cl}(\mathbf{B}(f)(Z))$.

Proof of Lemma 15. Since (i) is trivial, we prove (ii) in the following. For $C \in \mathcal{C}(Z)$ and $h \in (\{0\} + (0,1] \times X)^{\mathcal{C}(X)}$, the following holds:

$$
\begin{aligned}
(\mathbf{B}(g) \circ \mathbf{B}(f)(h))(C) &= (id_{\{0\}} + id_{(0,1]} \times g) \circ \mathbf{B}(f)(h) \circ g^{-1}(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times g) \circ (id_{\{0\}} + id_{(0,1]} \times f) \circ h \circ f^{-1}(g^{-1}(C)) \\
&= (id_{\{0\}} + id_{(0,1]} \times g \circ f) \circ h \circ (g \circ f)^{-1}(C) \\
&= \mathbf{B}(g \circ f)(h)(C)
\end{aligned}
$$

This completes the proof.

Proof of Lemma 20. For $C \in \mathcal{C}(Y)$, we have:

$$
\begin{aligned}
(F^{\mathcal{C}(Y)}(\tilde{f}) \circ t_X^{f^{-1}} \circ \alpha(x))(C) &= F^{\mathcal{C}(Y)}(\tilde{f})(\alpha(x) \circ f^{-1})(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times \tilde{f}) \circ \alpha(x) \circ f^{-1}(C) \\
&= (id_{\{0\}} + id_{(0,1]} \times f) \circ \alpha(x) \circ f^{-1}(C) \\
&= \mathbf{B}(f)(\alpha(x))(C) \\
&= (\beta \circ f(x))(C) \\
&= (\beta \circ \tilde{f}(x))(C).
\end{aligned}
$$

This completes the proof.

Proof of Lemma 21. We prove only (ii), since (i) is easier to show. By definition we have:

$$
\begin{aligned}
\mathbf{BF}(g \circ f) &= ((g \circ f)^{-1}, \widetilde{g \circ f}) \\
\mathbf{BF}(g) \circ \mathbf{BF}(f) &= (f^{-1} \circ g^{-1}, \tilde{g} \circ \mathbf{F}(g^{-1})(\tilde{f})).
\end{aligned}
$$

Since $\mathbf{F}(g^{-1})(\tilde{f})(x) = f(x)$ for $x \in X$, it follows that $\widetilde{g \circ f}$ and $\tilde{g} \circ \mathbf{F}(g^{-1})(\tilde{f})$ have the same underlying function. Thus it only remains to show that their types are also the same. The type of $\widetilde{g \circ f}$ is:

$$(X, t_X^{(g\circ f)^{-1}} \circ \alpha) \to (Z, \gamma).$$

The type of $\tilde{g} \circ \mathbf{F}(g^{-1})(\tilde{f})$ is

$$(X, t_X^{g^{-1}} \circ t_X^{f^{-1}} \circ \alpha) \to (Y, t_Y^{g^{-1}}) \to (Z, \gamma).$$

These, together with the fact that $t_X^{g^{-1}} \circ t_X^{f^{-1}} = t_X^{(g\circ f)^{-1}}$, complete the proof.

Proof of Proposition 22. If f and g are different morphisms in **Coalg**(**B**), then **BF**(f) and **BF**(g) are different as well, since $\tilde{f}, \tilde{g}$ have different underlying functions; hence the faithfulness of **BF**, which tells us that **Coalg**(**B**) can be embedded into $\int \mathbf{F}$.

BF is not full for the following reason. In $\mathbf{BF}(f : (X, \alpha) \to (Y, \beta))$, transformations from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$ are always inverse image maps f^{-1}, whilst, in morphisms of $\int \mathbf{F}$, transformations from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$ may be arbitrary functions from $\mathcal{C}(Y)$ to $\mathcal{C}(X)$.

Proof of Lemma 24. A semi-untary map of Hilbert spaces preserves the closure operator (-)$^{\perp\perp}$ on a Hilbert space, since it is linear and preserves limits. Hence, U preserves (-)$^{\perp\perp}$ on the projective space. Moreover, we can show that

$$P_S U = U P_{U^{-1}(S)},$$

and this also implies that

$$\frac{\langle \varphi | P_{U^{-1}(S)} \varphi \rangle}{\langle \varphi | \varphi \rangle} = \frac{\langle U\varphi | P_{U^{-1}(S)} U\varphi \rangle}{\langle U\varphi | U\varphi \rangle}.$$

It thus follows that **SC**(U) is indeed a morphism of **B**-coalgebras.

Proof of Theorem 25. We show that the functor **SC** is full; faithfulness is obvious. Let us consider an injective **B**-coalgebra morphism

$$f : (\mathrm{P}(H), \alpha_H) \to (\mathrm{P}(H'), \alpha_{H'}).$$

It is enough to verify that f actually arises from a morphism in **QSym** via the functor **SC**. Let us consider quantum Chu spaces $(\mathrm{P}(H), \mathrm{L}(H), e_H)$ and $(\mathrm{P}(H'), \mathrm{L}(H'), e_{H'})$. Since f is a **B**-coalgebra morphism,

$$(f, f^{-1}) : (\mathrm{P}(H), \mathrm{L}(H), e_H) \to (\mathrm{P}(H'), \mathrm{L}(H'), e_{H'})$$

is a Chu morphism: i.e., the **B**-coalgebra morphism condition

$$\mathbf{B}(f) \circ \alpha_H(\varphi)(S) = (\alpha_{H'} \circ f(\varphi))(S)$$

where $S \in \mathcal{C}(\mathrm{P}(H'))$ gives us

$$\frac{\langle \varphi | P_{f^{-1}(S)} \varphi \rangle}{\langle \varphi | \varphi \rangle} = \frac{\langle f(\varphi) | P_S f(\varphi) \rangle}{\langle f(\varphi) | f(\varphi) \rangle},$$

which in turn verifies the Chu morphism condition

$$e_{H'}(f(\varphi), S) = e_H(\varphi, f^{-1}(S)).$$

It then follows from Theorem 3.12 in Abramsky [2] that f arises from a unique semi-unitary map $U : H \to H'$ identified up to a phase. We have thus shown the fullness of **SC**.