Tonk and Harmony from a Categorical Perspective*

Yoshihiro Maruyama
Oxford University Computing Laboratory
maruyama@cs.ox.ac.uk

1 Overview

In his seminal paper [2], William Lawvere proposed to understand logical constants as adjoint functors, giving rise to the discipline of categorical logic and a structural perspective on proof theory. From such a categorical point of view, logical constants have to be defined by a specific form of bi-directional inference rules determined by adjointness (in a logical context, adjunctions are equivalent to certain bi-directional rules). The idea of logical constants as adjoints may thus be regarded as a principle of harmony discussed in proof-theoretic semantics, and I shall call it the principle of categorical harmony.

In this paper, I aim at clarifying the scope and limit of the principle of categorical harmony, especially in relation to Prior's weird connective "tonk" [3] and Belnap's response [1] to it. Tonk and Belnap's harmony principle (which I consider consists of both conservativity and uniqueness conditions) have been discussed much; however, the categorical perspective hopefully sheds a new light on such old topics (as Lawvere did on several aspects of logic). In the final analysis, I shall conclude that, although tonk is an adjoint functor once added to a logical system, however, it cannot be defined as an adjoint functor in a logical system without tonk; hence the principle of categorical harmony excludes tonk. In terms of inference rules corresponding to adjunctions, the bi-directional rule for tonk that represent adjointness is derivable (in two senses) in a system with tonk, while we cannot define tonk by adding to a system without tonk any bi-directional

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rule representing an adjunction (so, the situation is similar to the case of multiplicative connectives in Girard's terms). In addition, I shall argue that, essentially due to Freyd's adjoint functor theorem, there must be a conflict between Belnap's conservativity condition and the principle of categorical harmony; on the other hand, his uniqueness condition is just a natural consequence of the principle of categorical harmony. The close relationships with the reflection principle that Sambin et al. [4] introduced for their Basic Logic shall briefly be pointed out as well.

2 Is tonk an adjoint functor?

Let us review the concept of adjoint functors in the simple case of preorders. A preorder $(L, \vdash_L)$ consists of a set P with a reflexive and transitive relation $\vdash_L$ on L. Especially, the deductive relations of most logical systems are preorders . It is well known that a preorder can be seen as a category in which the number of morphisms between fixed two objects in it are at most one. Then, a functor $F : L \to L'$ between preorders L and L' is just a monotone map . Now, a functor $F : L \to L'$ is called left adjoint to $G : L' \to L$ (or G is right adjoint to F) if and only if $F(\varphi) \vdash_{L'} \psi \Leftrightarrow \varphi \vdash_L G(\psi)$ for any $\varphi \in L$ and $\psi \in L'$. A left or right adjoint of a given functor does not necessarily exist.

In this formulation, it would already be clear that adjunctions are equivalent to (or defined by) bi-directional inference rules. Let us look at examples. Suppose L is intuitionistic logic. Define a diagonal functor $\Delta : L \to L \times L$ by $\Delta(\varphi) = (\varphi, \varphi)$. Then, the right adjoint of Δ is $\wedge : L \times L \to L$, and the left adjoint of Δ is $\vee : L \times L \to L$. According to the principle of categorical harmony, we can thus define $\wedge$ and $\vee$ as the right and left adjoints of Δ. Moreover, Lawvere has shown that even quantifiers and equality can be defined as adjoint functors, establishing strong foundations of his idea of logical constants as adjoints.

Let L be a logical system with a deductive relation $\vdash_L$ that is reflexive and transitive. And suppose L contains truth constants 0 and 1, for which $0 \vdash \varphi$ and $\varphi \vdash 1$ hold for any formula φ. The first observation is that, if L has tonk, then tonk has both left and right adjoints. Recall that the inferential role of tonk is given by rules $A \vdash A\mathrm{tonk}B$ and $A\mathrm{tonk}B \vdash B$ (see [3, 1]). I mean by "L has tonk" that L has a connective "tonk" with these rules. We can see tonk as a functor from $L \times L$ to L. Define functors $\Delta_0 : L \to L \times L$ and $\Delta_1 : L \to L \times L$ as follows: $\Delta_0(\varphi) = (0, 0)$ and $\Delta_1(\varphi) = (1, 1)$. We can then prove that Δ_0 is the left adjoint of tonk, and that Δ_1 is the right adjoint of tonk. In other words, tonk is the right adjoint of Δ_0 and the left adjoint of Δ_1; therefore, tonk is an adjoint functor in two senses, if L is already endowed with tonk.

At the same time, however, this does not mean that the principle of categorical harmony cannot exclude tonk, a pathological connective we should not have in a "logical" system. Indeed, this is a problem in the other way around: in order to define tonk in a logical system, the principle of categorical harmony forces us to add it by requiring the right or left adjoint of some functor, or equivalently by requiring a bi-directional rule that represents adjointness. Thus, when one wants to define tonk in a logical system L according to the principle of categorical harmony, the task is: (1) specify a functor $F : L \to L \times L$ that has a (right or left) adjoint; (2) prove that the rules for tonk are derivable in the system L extended with the bi-directional rules corresponding to the adjunction. As a matter of fact, however, this turns out to be impossible.

Suppose for contradiction that it is possible. Then we have a functor $F : L \to L \times L$ by (1) and its right or left adjoint is tonk, which must have both the right adjoint and the left adjoint by (2) and the argument above. Since category theory tells us a right (left) adjoint of a functor is unique (if it exists), F must be either Δ_0 or Δ_1 defined above. Assume F is Δ_0; then tonk is the right adjoint of F. The bi-directional rule corresponding to the adjunction in this case is actually equivalent to: $\varphi \vdash_L \psi_1 \mathrm{tonk} \psi_2$ for any formula φ, ψ_1, ψ_2. But this condition is not sufficient to make the rules for tonk derivable; thus the right adjoint of F cannot be tonk, a contradiction. The case that F is Δ_1 is similar, and the proof is done.

It has thus been shown that tonk cannot be defined as an adjoint functor (of some functor) in a logical system without tonk, even though tonk is an adjoint functor in a logical system that is already endowed with tonk. This is a subtle phenomenon, and we have to be careful of what exactly the question "Is tonk an adjoint functor" means.

3 Categorical harmony and other principles

The categorical approach to harmony poses several questions to Belnap's notion of hamorny. It's been well known since Lawvere that the implication $\psi \to$ (-) of intuitionistic logic is right adjoint to the conjunction (-) $\wedge\, \psi$, since it holds that $\varphi \wedge \psi \vdash \chi$ if and only if $\varphi \vdash \psi \to \chi$. Assume we have the logical system L with logical constants $\wedge$ and $\vee$ only, which are defined as the right and left adjoints of Δ. And suppose we want to add $\to$ to L.

Of course, this can naturally be done by requiring the right adjoint of $\wedge$. Now, Freyd's adjoint functor theorem tells us that any right adjoint functor preserves limits (e.g., products), and any left adjoint functor preserves colimits (e.g., coproducts). In the present case, this implies that $\wedge$ preserves $\vee$; in other words, $\wedge$ distributes over $\vee$. Thus, defining $\rightarrow$ according to categorical harmony is not conservative over the original system L, since the bi-directional rules for $\wedge$ and $\vee$ do not imply the distributivity. This non-conservativity is very natural form a category-theoretical point of view, and seems to be in harmony with the Quinean, holistic theory of meaning, even though it violates Belnap's conservativity condition. Anyway, we may at least say that the principle of categorical harmony, or Lawvere's idea of logical constants as adjoints is in conflict with Belnap's notion of harmony.

In the process of the proof above, we have encountered the fact that an adjoint of a functor is uniquely determined. It actually implies that Belnap's uniqueness condition automatically holds if we define a logical constant according to the principle of categorical harmony. Thus, the uniqueness condition is inherently present in the concept of categorical harmony. It should be mentioned that, as a matter of fact, exponentials in linear logic do not have the uniqueness property. At the same time, however, we could doubt that exponentials are proper logical constants.

A relevant problem for categorical harmony is that multiplicative connectives in Graird's sense cannot be defined as adjoint functors in an "intrinsic" manner. This involves a tension between Cartesian and monoidal structures in category theory. Categorically speaking, multiplicatives correspond to monoidal structures (e.g., monoidal product), while additive connectives correspond to Cartesian structures (e.g., categorical product). In general, monoidal structures can only be given from "outside" a category; the same category can have different monoidal structures. Since adjunctions are determined via their universal properties "inside" a category, monoidal structures on a category cannot be defined as adjoint functors unless we already have some monoidal structures on it. But, once we have a monoidal structure, we can define further monoidal structures on it. For example, let us consider the additive fragment of linear logic, denoted by ALL; then we cannot define the multiplicative conjunction as an adjoint functor of a functor derived from the additive structue. Once we have a multiplicative conjunction in ALL, however, we can define a multiplicative implication as the right adjoint of it; then, the multiplicative conjunction becomes an adjoint functor as well, since it is the left adjoint of the implication. This is a subtlety similar to the case of tonk. And as in the case of exponentials we could doubt that multiplicative connectives are truly logical constants;

linear logic is regarded by Girard not as a logic on its own right but as a perspective to have a better understanding of classical and intuitionistic logics. The last two points are thus not necessarily disadvantages of the categorical approach to harmony.

At the same time, however, the worries above can also be resolved by extending the concept of categorical harmony. Actually, the reflection principle of Sambin et al. is implicitly doing this. In the case of additives, their definitional equivalences for logical constants are exactly the bi-directional rules induced by the corresponding adjunctions. In addition, they have definitional equivalences for multiplicatives, and thus give a proper extension of the scope of the principle of categorical harmony. They do not mention anything categorical, probably being unaware of the connection with Lawvere's idea of logical constants as adjoints. The relationships between the reflection principle and the principle of categorical harmony would thus deserve further investigation.

4 Conclusions

I proposed and examined the categorical approach to harmony, building upon Lawvere's idea of logical constants as adjoints. The main focus of analysis was on Prior's "tonk" in the light of the principle of categorical harmony. Simply speaking, it finally turned out that tonk is an adjoint functor in a logical system with tonk, and that tonk is not an adjoint functor in a logical system without tonk. Apart from the validity of the concept of categorical harmony, these observations on tonk seem to be worth knowing on their own rights. I also attempted to clarify the relationships of categorical harmony with Belnap's notion of harmony and with the reflection principle by Sambin et al. There is a sharp conflict on conservativity between categorical harmony and Belnap's. The reflection principle is, in a certain sense, an extension of Lawvere's idea of logical constants as adjoints to the case of multiplicative connectives.

References

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