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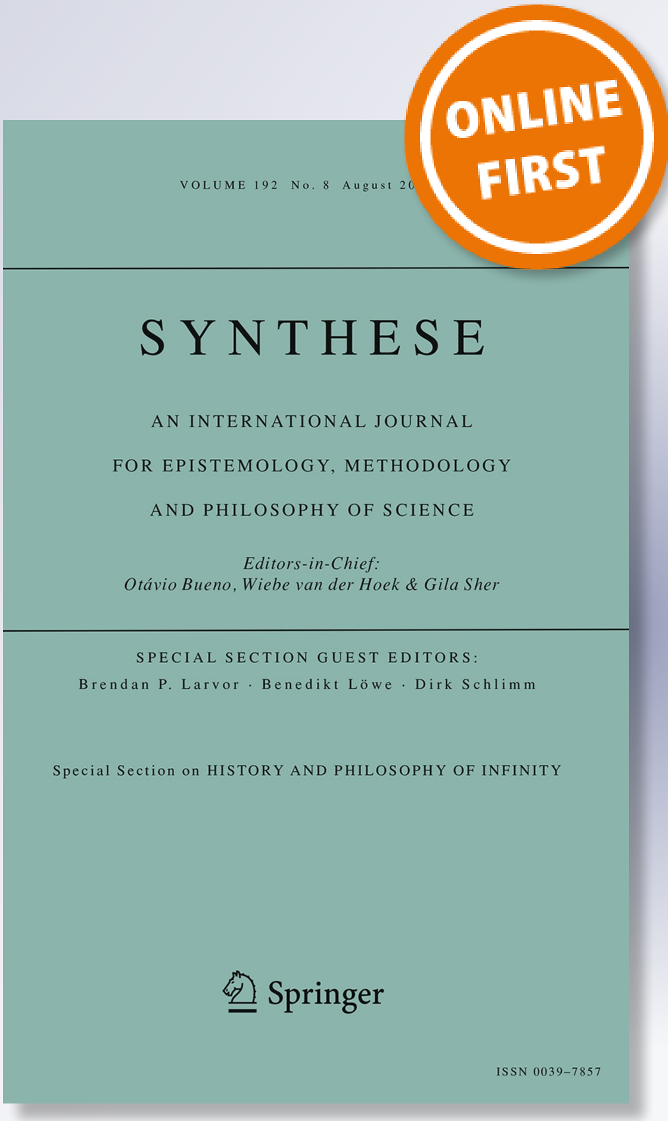

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S.I. : THE LOGIC AND PHILOSOPHY OF A.N. PRIOR

Prior's tonk, notions of logic, and levels of inconsistency: vindicating the pluralistic unity of science in the light of categorical logical positivism

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Abstract There are still on-going debates on what exactly is wrong with Prior's pathological "tonk." In this article I argue, on the basis of categorical inferentialism, that (i) two notions of inconsistency ought to be distinguished in an appropriate account of tonk; (ii) logic with tonk is inconsistent as the theory of propositions, and it is due to the fallacy of equivocation; (iii) in contrast to this diagnosis of the Prior's tonk problem, nothing is actually wrong with tonk if logic is viewed as the theory of proofs rather than propositions, and tonk perfectly makes sense in terms of the identity of proofs. Indeed, there is fully complete semantics of proofs for tonk, which allows us to link the Prior's old philosophical idea with contemporary issues at the interface of categorical logic, computer science, and quantum physics, and thereby to expose commonalities between the laws of Reason and the laws of Nature, which are what logic and physics are respectively about. I conclude the article by articulating the ideas of categorical logical positivism and pluralistic unified science as its goal, including the unification of realist and antirealist conceptions of meaning by virtue of the categorical logical basis of metaphysics.

Keywords Inferentialism · Theory of meaning · Identity of logic · Categorical logical positivism · Pluralistic unified science · Unity of realism and antirealism

1 Prior's tonk and categorical inferentialism

The inferentialist asserts that the meaning of a logical constant is conferred by the rôles it plays in the practice of inference or the inferential rules that govern its use (see, e.g.,

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Brandom (2000); Wittgenstein's thesis that meaning is use tends to be referred to in this context; notwithstanding, any formalist conception of rule would be misconceived in Wittgenstein's later philosophy, which was even against scientism in general, let alone symbolic logic qua formal science). Yet at the same time, does any pair of rules of inference determine a meaningful logical constant? Arthur Prior's answer was negative: to the end of showing that, he introduced a weird connective called "tonk" in his seminal paper Prior (1960), which may be defined, for example, by the following rules of natural deduction:

$$\frac{\xi \vdash \psi}{\xi \vdash \varphi \text{ tonk } \psi}\ (\text{tonk-l-intro.}) \qquad \frac{\xi \vdash \varphi}{\xi \vdash \varphi \text{ tonk } \psi}\ (\text{tonk-r-intro.})$$

$$\frac{\xi \vdash \varphi \text{ tonk } \psi}{\xi \vdash \psi}\ (\text{tonk-l-elim.}) \qquad \frac{\xi \vdash \varphi \text{ tonk } \psi}{\xi \vdash \varphi}\ (\text{tonk-r-elim.})$$

It is then immediate to see that adding tonk makes the consequence relation $\vdash$ trivial in the sense that $\varphi \vdash \psi$ is derivable for any formulae φ, ψ. In some non-standard systems, tonk actually does not cause triviality, as shown in Cook (2005); yet, they are just tailor-made ad hoc for tonk so as to avoid inconsistency, and there is no well-established system which does not trivialise in the presence of tonk.

Prior's tonk has urged philosophical logicians to pose a sort of demarcation problem in logic: Is there any criterion to demarcate between genuine logical constants, like conjunction, and pseudo-logical constants, like tonk? Especially, when do inferential rules yield a meaningful logical constant? When are they really "meaning-conferring"? Michael Dummett coined the term "harmony" for such a criterion, and the harmony principle requires paired inferential rules (e.g., introduction and elimination rules) to be in "harmony" with each other [see, e.g., Dummett (1991)]. Well-known examples of harmony include Prawitz's inversion principle, and Belnap's conservativeness and uniqueness conditions in his paper Belnap (1962) addressing the Prior's tonk problem. Today, there are so many different concepts of harmony proposed by different philosophers of logic [particularly relevant to this work is Maruyama (forthcoming)]. Each concept of harmony gives us its own diagnosis on what is wrong with tonk or its defining rules.

In this article, I present a diagnosis of the Prior's tonk problem from the standpoint of categorical inferentialism based on categorical semantics, according to which meaning is conferred by the structure of inference rather than inference *per se*, which basically boils down to the structure of arrows in categories interlaced by different adjunctions to represent logical constants and inferential rules intrinsic to them. Then, the meaning of a logical constant is not given by its defining rules of inference, which are syntax-dependent, but rather given by the adjunction structure of the inferential rules, which is presentation-independent. Categorical inferentialism may thus be called "structural inferentialism" in more general terms. Combined with structural realism in the philosophy of science [see, e.g., Ladyman (2014)], it yields "inferential structural realism", which arguably sheds new light on the scientific realism debate, e.g., by telling us what is really preserved in theory change is the structure of inference. Such broader issues, however, shall be addressed elsewhere.

2 Notions of logic and inconsistency

What is logic at all? Some think logic is, or may be identified with, a collection of valid propositions or "theorems" in the sense of symbolic logic rather than pure mathematics. This conception of logic, however, is misguided because different, or non-equivalent, logical systems can have the same set of theorems: e.g., classical logic and Graham Priest's paraconsistent logic LP have the same set of theorems, as Priest (2002) emphasises the fact himself.

More careful logicians consider logic a consequence relation rather than a mere set of propositions; this would be a possible, and to some extent received, view in the philosophy of logic, apart from the seemingly problematic fact that different theories can have isomorphic consequence relations, just as Pour-El-Kripke's theorem tells us (first-order) Peano arithmetic and ZF set theory are, to be surprise, recursively isomorphic with regard to their consequence relations (it is not obvious if "logic" and "theory" are intrinsically different; some logics can be theories over others). In this conception, logic is about what proposition(s) entails what proposition (and not about why one proposition entails another); as such, logic is the theory of (equivalences of) propositions.

Modern developments of proof theory, type theory, and categorical logic have suggested yet another notion of logic as represented in Girard's dictum "A logic without cut-elimination is like a car without engine" in his paper Girard (1995). In more philosophical terms due to Martin-Löf and Prawitz, the identity of proofs is what matters in the concept of logic, and it is determined by cut-elimination or normalisation as Girard says (hence "logic without the identity of proofs" $\sim$ "logic without cut-elimination" $\sim$ "car without engine"). In this conception, logic is the theory of (equalities of) proofs rather than propositions.

This modernist conception of logic as the theory of proofs strikingly deviates from the traditional conception of logic. Old-style logicians would think that natural deduction NJ and sequent calculus LJ for intuitionistic logic represent the same logic; however, their induced identities of proofs are essentially different as is well known in proof theory [see, e.g., Zucker (1974)]. And also classical logic suffers from an intrinsic difficulty in its identity of proofs; simply saying, all proofs get identified in classical logic [see Girard et al. (1989)]. There is still no generally agreed remedy for this paradox of classical proofs.

The modernist shift in the conception of logic is arguably in parallel with the shift in the conception of space in algebraic geometry from varieties (zeros of polynomials) to schemes (spectrums of rings, or rather representable functors from the category of rings, and their glueings), in which two systems of polynomials whose zeros are the same can sometimes be considered, as in Grothendieck's EGA, to represent different spaces because they have different algebraic or "proof-theoretic" structures. Having in mind the fact that theories correspond to rings, and Stone spectra (spaces of models or prime filters) to Zariski spectra (spaces of zeros/generic points or prime ideals), the two shifts are indeed mathematically intertwined and in precise parallel to each other. In a nutshell, the notions of space and logic have been algebraised or relationalised in modernisation, having finally been unified in the concept of topos, which is space and logic at once. The two shifts in logic and geometry instantiate the conceptual shift from "substance" to "function" in Cassirer's terms or processes in Whitehead's terms; modernisation is functionalisation in the light of Cassirer's philosophy, which is now considered a key to transgress the analytic-continental divide in philosophy [see Friedman (2000)], whilst lying at the root of structural realism in the contemporary philosophy of science, together with Leibniz's philosophy, from which the early Cassirer started. Physics (the concepts of physical theory, spacetime, and matter) has arguably been functionalised or informationalised in the course of the quantum revolution in parallel with the modernisation of logic and mathematics.

The modernist conception of logic lurks behind the Curry–Howard correspondence, the BHK interpretation, the semantics of computation, and the like (intensional differences between extensionally equivalent programs do matter in computer science). The Curry–Howard correspondence tells us that a type theory is a theory of proofs (propositions $\sim$ types; proofs $\sim$ terms or programs; identity of proofs $\sim$ equality of terms). According to the Curry–Howard–Lambek correspondence, the equality of arrows in a category represents the identity of proofs in its internal logic, and this view plays a fundamental rôle in categorical logic, lying at the heart of categorical proof theory in particular.

In a nutshell, a category is a theory of proofs. Indeed, a category may be defined as a proof-theoretic consequence relation given by the following data: (i) the concept of formulae φ (aka. objects); the concept of hypothetical proofs (aka. arrows) from φ to ψ; there must be an identity proof id_φ from φ to φ; we write $\varphi \vdash_p \psi$ when there is a proof p from φ to ψ (ii) sequential composition $\circ$ of proofs; composing $\varphi \vdash_p \psi$ and $\psi \vdash_q \xi$, we obtain $\varphi \vdash_{q \circ p} \xi$; in a monodical category or substructural logic, we also have parallel composition $\otimes : \varphi \otimes \varphi' \vdash_{p \otimes p'} \psi \otimes \psi'$ where $\varphi \vdash_p \psi$ and $\varphi' \vdash_{p'} \psi'$; (iii) the concept of proof reduction such that the identity proofs may be canceled (i.e., $id_\varphi \circ p = p$ and $q \circ id_\varphi = q$ modulo reducibility), and proofs may locally be reduced in any order (i.e., $(p \circ q) \circ r = p \circ (q \circ r)$ modulo reducibility); reduction must of course respect composition (i.e., if $p = q$ and $r = s$ modulo reducibility, $p \circ q = r \circ s$ modulo reducibility). This is the concept of a proof-theoretic consequence relation (resp. with parallel composition) or category (resp. monoidal category).

There are different conceptions of inconsistency in logic. To the end of explicating what is wrong with tonk, and what is not, I introduce two notions of inconsistency corresponding to the aforementioned two conceptions of logic:

- Inconsistency qua theory of propositions;
- inconsistency qua theory of proofs.

The definitions are as follows. A logical system qua theory of propositions is called inconsistent if and only if its consequence relation $\vdash$ is trivial, i.e., for any formulae φ, ψ, it holds that $\varphi \vdash \psi$ and $\psi \vdash \varphi$, i.e., all propositions are equivalent with respect to the consequence relation. On the other hand, a logical system qua theory of proofs is inconsistent if and only if its identity of proofs is trivial, i.e., for any formulae φ, ψ and any proofs π, π' from φ to ψ, it holds that $\pi = \pi'$. This means all proofs (of each type) are equivalent.

The former concept of inconsistency, therefore, is concerned with the trivialisation of identity of propositions, whilst the latter with that of identity of proofs. In the former, a logical system is apprehended as a consequence relation, which may be expressed either proof-theoretically or model-theoretically. In the latter, however, we suppose the logical system is given in an essentially proof-theoretic manner, together with a concept of proof reduction, which is required to define the identity of proofs within the system.

The two concepts of inconsistency are incomparable. To see this, let us think of classical logic LK in the sequent calculus formulation, which is obviously consistent as a theory of propositions. And yet it is well known that the identity of proofs is trivial in classical logic; this phenomenon is known by what is called Lafont's critical pair in proof theory or Joyal's lemma in categorical logic [see Girard et al. (1989) and Lambek and Scott (1986) respectively]. This means that inconsistency qua theory of proofs does not imply inconsistency qua theory of propositions. Likewise, it does not hold that inconsistency qua theory of proofs implies inconsistency qua theory of propositions. This is actually what Prior's tonk tells us. As we shall see later, certain logic with tonk is inconsistent as a theory of propositions, and yet consistent as a theory of proofs. Thus, the two concepts of inconsistency are incomparable, indeed.

There is yet another, third kind of inconsistency combining the two concepts of inconsistency. Let us formulate it in terms of proof-theoretic consequence relation $\vdash_p$ explained above. A logical system qua proof-theoretic consequence relation is inconsistent if and only if for any φ, ψ there are proofs p, q such that $\varphi \vdash_p \psi$, and $\psi \vdash_q \varphi$, and the sequential combinations of proofs p and q (i.e., both $q \circ p$ and $p \circ q$) are equal to identity proofs (i.e., id_φ and id_ψ, respectively). Here, p and q are mutually inverse; φ and ψ are isomorphic (not just equivalent); and so the structure of formulae collapses completely. Note that logical equivalence $\leftrightarrow$ on formulae is different from proof-theoretic isomorphism witnessed by equalities of proofs. Whereas the third concept of inconsistency, by definition, implies the first one, the second and third ones are incomparable. This may be seen by noticing that classical logic is inconsistent as a theory of proofs, yet consistent as a proof-theoretic consequence relation, and that a category having one object only can allow for more than one morphisms. The highest degree of inconsistency can be seen in intuitionistic higher-order logic HOL (i.e., the internal logic of toposes) expanded with fixpoint operators, which is known to be inconsistent in the third sense and in any other sense (this tells us topos theory is inconsistent with domain theory, the semantics of computation, which requires fixpoint operators to interpret recursive processes). Now, the following table wraps up the discussion:

	$\vdash$ inconsis.	Proof id. inconsis.	$\vdash_p$ inconsis.
Prior's tonk logic	Yes	No	No
Classical logic	No	Yes	No
HOL with fix.	Yes	Yes	Yes

In the following I shall argue that logic with Prior's tonk is inconsistent as a theory of propositions with equivocation lying at the heart of the inconsistency of tonk, yet at the same time, it is totally consistent as a theory of proofs with coherent semantics of proofs in terms of both type theory and category theory.

3 What is wrong with Prior's tonk

In this section I elucidate the inconsistent nature of Prior's tonk in the light of categorical inferentialism, in particular Lawvere's idea of logical constants as adjoint functors [see Lawvere (1969), in which he sheds light on "duality between the formal and the conceptual" as well as the categorical conception of logic as so-called hyperdoctrines]. This section mostly builds upon Maruyama (forthcoming).

Is tonk an adjoint functor, namely a logical constant in Lawvere's conception? This is actually a subtle question; we have to be careful of what exactly the question means. Let us think of a logical system L with the consequence relation $\vdash_L$ validating the identity and cut rules. We suppose the language of L contains truth constants 0 and 1, for which $0 \vdash \varphi$ and $\varphi \vdash 1$ for any formula φ. In this setting, tonk comes with the following rules: $\varphi \vdash \varphi$ tonk ψ; $\psi \vdash \varphi$ tonk ψ; φ tonk $\psi \vdash \psi$; φ tonk $\psi \vdash \varphi$. And regard tonk as a functor from $L \times L$ to L. Let us define functors $\Delta_0 : L \to L \times L$ and $\Delta_1 : L \to L \times L$ by $\Delta_0(\varphi) = (0, 0)$ and $\Delta_1(\varphi) = (1, 1)$. It is immediate to see that Δ_0 is the left adjoint of tonk, and that Δ_1 is the right adjoint of tonk. Put another way, tonk is the right adjoint of Δ_0, and at the same time, the left adjoint of Δ_1.

In a nutshell, if logic concerned comes equipped with tonk, then tonk is an adjoint functor (in the two senses) within that logic. Does tonk, therefore, count as a logical constant? Notwithstanding the fact that tonk is an adjoint functor in a system with tonk, categorical inferentialism allows us to preclude tonk from counting as a logical constant, on the ground that tonk is not an adjoint functor in a system without tonk, to be precise, tonk cannot be defined or introduced as an adjoint functor, to be even more precise, there is no single adjunction characterising tonk [for more details, see Maruyama (forthcoming)].

Nonetheless, tonk can be defined via double adjunctions, i.e., as a functor that is a right adjoint of Δ_0 and a left adjoint of Δ_1 at the same time, and so tonk is a "bi-adjoint" functor. Bi-adjointness may be considered a sort of equivocation: tonk mixes up the two different logical constants, which are the right adjoint of Δ_0 and the left adjoint of Δ_1, namely, the binary truth constant (mapping any $(\varphi, \psi) \in L$ to 1) and the binary falsity constant (mapping any (φ, ψ) to 0), respectively. Prior's tonk thus mixes up the two different logical constants; this is the reason why tonk causes inconsistency qua theory of propositions. The problem of tonk, therefore, is a problem of equivocation.

Building upon this insight into tonk, we can draw an even bigger picture: as in Schroeder-Heister (2012), let us define Genuine Paradox as a nullary connective R (R means "Russell" with the Russell paradox in mind) such that $\vdash R$ if and only if $\vdash \neg R$. We can rephrase "$\vdash \neg R$" as "$R \vdash$". By regarding R as a unary constant functor, R then turns out to be right and left adjoint to itself, i.e., R is a self-adjoint functor. Let us also define disconjunction as a functor that is right and left adjoint to Δ, i.e., it is a "uniformly bi-adjoint" functor. Whereas tonk is right adjoint to one functor, and left adjoint to another functor which is different from the former, disconjunction is right and left adjoint to the same functor. Uniform bi-adjointness is a stronger condition than bi-adjointness, and self-adjointness is obviously the strongest one.

We thus have three degrees of paradoxicality of logical constants. The paradoxicality of tonk and disconjunction is caused by equivocation, thus being able to be resolved by "disambiguation", i.e., by giving different names to two adjoint functors involved. Nonetheless, the case of Genuine Paradox is different: it cannot be resolved by disambiguation, since a functor that is right (resp. left) adjoint to itself is, at the same time, left (resp. right) adjoint to itself. In a nutshell, self-adjointness can be expressed by a single adjunction alone, unlike bi-adjointness requiring two adjunctions. This is exactly the reason why we call it Genuine Paradox. The following table wraps up the discussion:

	Right adjoint to	Left adjoint to
Paradox	Itself	Itself
Disconjunction	Diagonal	Diagonal
Tonk	Truth diagonal	Falsity diagonal

Note that self-adjunction may be seen as a sort of categorical self-reference. A broader perspective on the conception of logical constants as adjoint functors is given in Maruyama (forthcoming), which pursues the idea of categorical harmony.

4 What is not wrong with Prior's tonk

Now, I shall explicate the perfectly consistent nature of Prior's tonk in the light of categorical inferentialism. In a nutshell, if you conceive of logic as the theory of proofs rather than propositions, or as concerned with semantics of proofs rather than provability, then you can make sense of tonk in a totally consistent, proof-theoretically and category-theoretically meaningful manner.

Let us define a natural deduction system NP for tonk (rules shall be formulated via $\vdash$ to explicate assumptions involved; this is standard in theoretical computer science). NP has implication, (binary) tonk, and what is called nullary tonk (or null object in terms of categories), which allows the system to admit proof normalisation (the *raison d'etre* of natural deduction). The intro. and elim. rules for tonk have been given above. The rules for nullary tonk are:

$$\frac{\xi \vdash \varphi}{\xi \vdash 0}\ (\text{0-intro.}) \qquad \frac{\xi \vdash 0}{\xi \vdash \psi}\ (\text{0-elim.})$$

It would be obvious why this is called nullary tonk. Since tonk plays the rôles of conjunction and disjunction at the same time, we do not have to equip NP with conjunction or disjunction; they are just redundant in the presence of tonk. NP is thus equivalent to NJ expanded with binary and nullary tonk's.

It is crucial to have nullary tonk 0 in making the system admit proof normalisation, even though adding 0 does not look like a big deal at first sight. Proof reductions for tonk are defined as follows. Firstly,

$$\cfrac{\cfrac{\xi \vdash \varphi}{\xi \vdash \varphi \text{ tonk } \psi}\ \text{(tonk-r-intro.)}}{\xi \vdash \varphi}\ \text{(tonk-r-elim.)}$$

is reduced to $\xi \vdash \varphi$. Categorically, this reduction means that injecting into the first component and then projecting onto the same component boil down to an identity arrow (used below). The case of combining tonk-r-intro. and tonk-r-elim. is reduced in a similar manner. Secondly,

$$\cfrac{\cfrac{\xi \vdash \psi}{\xi \vdash \varphi \text{ tonk } \psi}\ \text{(tonk-l-intro.)}}{\xi \vdash \varphi}\ \text{(tonk-r-elim.)}$$

is reduced to

$$\cfrac{\cfrac{\xi \vdash \psi}{\xi \vdash 0}\ \text{(0-intro.)}}{\xi \vdash \varphi}\ \text{(0-elim.)}$$

Categorically, this reduction means that injecting into the first component and then projecting onto the second component boil down to the canonical zero morphism (used below). The case of combining tonk-l-elim. and tonk-r-intro. is reduced in a similar manner. Proof reductions for the other logical constants are defined as usual. In this way, NP admits proof normalisation.

What is the type theory that corresponds to the natural deduction system NP in the Curry–Howard correspondence's sense? It is the λP-calculus defined in the following. The λP-calculus, denoted λP, is the simply typed λ-calculus $\lambda_{\rightarrow}$ expanded with the following type constructors:

- Null object 0, which is a nullary type constructor. It comes equipped with two kinds of terms $!_1 : 0 \rightarrow \sigma$ and $!_2 : \sigma \rightarrow 0$.
- Biproduct $\oplus$, a binary type constructor. Biproduct $\otimes$ comes equipped with four kinds of terms, that is, projections $p_1 : \sigma_1 \oplus \sigma_2 \rightarrow \sigma_1$ and $p_2 : \sigma_1 \oplus \sigma_2 \rightarrow \sigma_2$, injections (aka. insertions) $i_1 : \sigma_1 \rightarrow \sigma_1 \oplus \sigma_2$ and $i_2 : \sigma_2 \rightarrow \sigma_1 \oplus \sigma_2$, product pairing $\langle t, s\rangle : \tau \rightarrow \sigma_1 \oplus \sigma_2$ where $t : \tau \rightarrow \sigma_1$ and $s : \tau \rightarrow \sigma_2$, and coproduct pairing $[t, s] : \sigma_1 \oplus \sigma_2 \rightarrow \tau$ where $t : \sigma_1 \rightarrow \tau$ and $s : \sigma_2 \rightarrow \tau$.

The corresponding equations, which represent the reduction rules in natural deduction, are as follows. Note that 1_σ is a shorthand for $\lambda x.x$ where $x : \sigma$, and $t \circ s$ is a shorthand for $\lambda x.t(sx)$ where $x : \sigma_1$, $s : \sigma_1 \rightarrow \sigma_2$, and $t : \sigma_2 \rightarrow \sigma_3$.

- For any $t : 0 \rightarrow \sigma$, $!_1 = t$. For any $t : \sigma \rightarrow 0$, $!_2 = t$.
- For product pairing and projections, $p_1 \circ \langle t, s\rangle = t$, $p_2 \circ \langle t, s\rangle = s$, and $t = \langle p_1 \circ t, p_2 \circ t\rangle$. For coproduct pairing and injections, $[t, s] \circ i_1 = t$, $[t, s] \circ i_2 = s$, and $[i_1, i_2] = 1_{\sigma\oplus\tau}$. For $k = 1, 2$, $p_k \circ i_k = 1_{\sigma_k}$. For $k \neq l$, $p_l \circ i_k = 0_{\sigma_k \rightarrow \sigma_l}$ where $0_{\sigma_k \rightarrow \sigma_l}$ denotes $!_1 \circ !_2$ for $!_1 : 0 \rightarrow \sigma_l$ and $!_2 : \sigma_k \rightarrow 0$.

Note that $p_k \circ i_k = 1_{\sigma_k}$ represents the first reduction rule in NP presented above, and $p_l \circ i_k = 0_{\sigma_k \to \sigma_l}$ represents the second reduction rule in NP. This is crucial in the Curry–Howard isomorphism between NP and λP.

Prior categories are defined as categories with a null object, biproduct, and exponential; note that biproduct $\oplus$ presupposes a null object 0. Exponential is the categorical concept of implication or function type $\to$. Biproduct is the categorical concept of tonk or type-theoretic biprpduct $\oplus$ defined above. The above presentation of λP is actually almost the same as the concept of categories with a null object, biproduct, and exponential. In light of categorical logic, a type theory is a way of presenting a concept of categories.

There is a Curry–Howard–Lambek correspondence between the natural deduction system NP, the type theory λP, and the Prior categories $PCat$: $NP \simeq \lambda P \simeq PCat$. The last two give fully complete semantics of proofs for NP, which of course admits proof normalisation. Full completeness is a relatively new concept in proof theory with its origin in the concept of full abstraction in the semantics of programming languages, intuitively meaning that the space of proofs and the space of semantic denotations of proofs are isomorphic, i.e., in exact one-to-one correspondence respecting the identity of proofs (technically, the term "full" originally comes from the fullness of a semantic functor mapping a proof to an arrow in a category concerned; besides, we require faithfulness as well). Thus, full completeness may be seen as completeness on proofs rather than propositions.

Let us sketch how to prove the Curry–Howard–Lambek isomorphism: $NP \simeq \lambda P \simeq PCat$. First think of NP and $PCat$. How can we interpret proofs in NP as arrows in a given Prior category? The intro. rules for tonk are interpreted as injections on biproduct, and the elim. rules as projections on biproduct. In such a way, every rule of inference is associated with an arrow in the category, and inductively, every proof gets associated with an arrow as its denotation (for a general account of such a semantic process, see Abramsky and Tzevelekos (2011)). Since the proof reduction procedures for tonk precisely correspond to the defining equations of biproduct (esp. $p_k \circ i_k = 1_{\sigma_k}$ and $p_l \circ i_k = 0_{\sigma_k \to \sigma_l}$) as explained just after the proof reduction procedures above, and since similar correspondences exist for the other connectives, we have got the soundness of the semantics of NP by $PCat$. A completeness proof can be given by constructing from NP a syntactic category. The syntactic category of NP is the category the objects of which are the formulae and the arrows of which are the equivalence classes of proofs under the identity of proofs. The syntactic category forms a $PCat$. If some identity between proofs does not hold in NP, then it does not hold in the syntactic category. Thus, we have got the completeness theorem, and this yields the isomorphism between NP and $PCat$. In a similar way, we can show the isomorphism between λP and $PCat$ (soundness by the correspondence between term equations and arrow equations; completeness by the syntactic category construction), thus establishing the trinity correspondence.

Although $\lambda_{\to}$ is used as a base system in the above formulation, we may instead use linear λ-calculus $L\lambda$ (see, e.g., Abramsky and Tzevelekos (2011)), and equip $L\lambda$ with binary and nullary tonks as type constructors, just in the same way as above. Let us denote the resulting calculus by $L\lambda P$, and also its linear logical counterpart by LLP. Then, monoidal Prior categories $MPCat$ are defined as symmetric monoidal closed categories with biproduct, and we lead to the following Curry–Howard–Lambek isomorphism by arguing as in the above proof: $LLP \simeq L\lambda P \simeq MPCat$. This is actually better than the above one in that it allows for natural models appearing in ordinary mathematics, and enables us to readily prove the consistency of the calculus (on the ground of the existence of those models rather than involved proof-theoretic computation). For example, the category of Hilbert spaces and bounded linear operators (category of quantum systems), and also the category of sets and relations (a toy model of quantum theory), give categorical models of $L\lambda P$ and LLP.

More generally, fundamental categorical structures used in categorical quantum mechanics [see, e.g., Abramsky and Coecke (2008)] are rich enough to form models of $L\lambda P$ and LLP. Underpinning this is what I call the Abramsky-Coecke correspondence "propositions as systems, proofs as processes", where systems and processes are primarily intended to be physical. Particularly relevant to the present discussion is the insight that the categorical structure of quantum systems and quantum processes yields a coherent model of Prior's tonk, from which the proof-theoretic or type-theoretic consistency of tonk follows immediately. If Prior had known about categorical quantum mechanics, he might have been happy to accept tonk as a meaningful logical constant in the substructural logic (or linear type theory) of quantum physics.

5 Categorical logical positivism and pluralistic unified science: the unity of realism and antirealism in the logical basis of metaphysics

In this way, the Prior's old philosophical idea turns out to be tightly intertwined with contemporary developments at the interface of logic (the laws of reason), computer science (the laws of information), and physics (the laws of nature), in which Prior's tonk plays a ubiquitous rôle whilst appearing in the category-theoretic or type-theoretic guise of biproduct, which is even used in formal modelling of protocols in quantum computation [see, e.g., Abramsky and Coecke (2008)]. The Curry–Howard–Lambek correspondences presented in this article are, to my knowledge, new results themselves, but the relevant technical machinery has been more or less known in the categorical proof theory community. Nonetheless, the compelling tight connection between Prior's tonk and biproduct has never been articulated before. And the identity of proofs in logic with tonk has never been explicated, either. In a nutshell, the consistent nature of Prior's tonk has never been elucidated so far.

As seen in the Curry–Howard–Lambek correspondences for Prior's tonk and their close link with quantum physics, category theory allows us to make conceptual bridges between different sciences, including pure mathematics, symbolic logic, computer science, and physics (and even more), thus establishing the unity of ideas scattered in different fields of science. Even some theoretical biologists rely on categorical methods as well. Category theory today is actually applied beyond mathematical and natural sciences, for instance, in linguistics, economics, and philosophy as in the work of the Oxford school of quantum foundations, thus unveiling and articulating novel analogies and disanalogies across diverse sciences in a mathematically precise and rigorous fashion. We are presumably heading towards a new kind of unified science. The idea of categorical unified science, nevertheless, is in sharp contrast to the logical positivist's old-fashioned idea of unified science, which was monistic and reductionistic under the foundationalist conception of epistemology.

Categorical unified science is pluralistic unified science, emancipating logical positivism from the foundationalist doctrine of reductive physicalism. Indeed, it does not aim at grounding all sciences on one and the same absolute global foundation (cf. set theory primarily aiming at this sort of foundation of mathematics, apart from the recent multiverse view of set theory), nor revising existing sciences by reducing them into a single science of the most fundamental sort. Pluralism and the idea of relative and local foundations are arguably inherent in category theory. Think, for instance, of Grothendieck's relative point of view. Base change is a fundamental idea of category theory. There is no single category that gives the ultimate ontological foundation of everything (hence no universe in category theory, unlike set theory). There are just different categories to give local relative foundations of different fields of science (as such category theory intrinsically supports the multiverse view).

Still they are categories, so that we can apprehend what structure is shared by them, and what is not. What is here aimed at is not giving ontological or epistemological foundations, but rather giving a structural account of how each discipline's local ontology and epistemology can be interlinked with others' in the whole web of human knowledge. For example, quantum mechanics and substructural logic share quite a lot; among other things, the lack of contraction and weakening in logic corresponds to the so-called no-cloning and no-deleting properties of quantum states, in particular quantum information or qubits. This is, so to say, unity from below, originating from and sticking to the actual practice of science. In such a way, category theory yields conceptual understanding across different sciences (in the present case, logic and physics). There are numerous cases of local unity already achieved by category theory. Was there any such fruit in the logical positivist's unified science movement? Unified science must not merely be a philosophical idea; it must be practiced.

Thus I hereby declare the birth of categorical logical positivism: its theory of meaning is given by categorical inferentialism, which keeps a hint of verificationist flavour, if in Dummett's style rather than the logical positivist's, and its ultimate goal is pluralistic unified science or the antifoundational naturalist unity of science including humanities, which is, I believe, of the utmost importance in overcoming the fragmentation of science after modernisation, and in refurbishing the lost scientific image of the world as an integrated whole, in this nihilistic age of the *Destruktion* of *cogito sum* (*á la* Heidegger), the deconstruction of logocentrism (*á la* Derrida), the abandonment of the Cartesian goal of a first philosophy (*á la* Quine), or the end of grand narrative (*á la* Lyotard). Both analytic and continental philosophers in the postmodern era are quite generally disposed to nihilism and skepticism on meaning and reality as seen in these cases and in Kripkenstein's scepticism, Putnam's model-theoretic argument, Davidson's linguistic philosophy asserting "there is no such thing as a language", Derrida's indeterminacy of meaning, Foucault's archaeology showing "man is an invention of recent date", and so fourth. Postmodernisation is, so to say, *Daseinisation*, or a return to the contextuality of Being-in-the-World surrounded by a myriad of contingencies and uncertainties. If so, the unity of science may already be anachronistic. Such postmodern nihilism in favour of disunity notwithstanding, categorical logical positivism still seeks unity as the meaning of the entire human endeavour and as the meaning of our very existence; as such, it is akin to the existential enterprise of the Camusian absurdist struggling at the very edge of the discrepancy between the human being who yearns for the unity of the meaning of the world (i.e., the universe and beings therein) and the plurality and diversity of the world which repudiate the human longing for the unity of meaning at the end of the day. The neo-positivist must live this absurdism.

Categorical logical positivism allows for a new direction in the theory of meaning by virtue of the efficacy of categorical unification. The two major theories of meaning are the truth-conditional, denotationalist theory of meaning as advocated by Davidson, which is founded on model-theoretic semantics, and the justification-conditional, inferentialist theory of meaning as advocated by Dummett, which is logically underpinned by what is now called proof-theoretic semantics. According to Dummett's constitution thesis [see, e.g., Miller (2014)], which is one of the tenets of his *The Logical Basis of Metaphysics*, the content of metaphysical realism (resp. antirealism) consists in the former (resp. latter) kind of semantic realism (resp. antirealism) allowing verification-transcendent truth-conditions (resp. finitary verification-conditions only). A fundamental insight of categorical semantics, however, is that model-theoretic and proof-theoretic semantics, and thus the denotationalist and inferentialist theories of meaning, may be regarded as two instances of the one categorical notion of interpretation (technically, interpretation in set-based categories amounts to model-theoretic semantics, and that in syntactic categories to proof-theoretic semantics). Categorical inferentialism, therefore, does encompass both realist and antirealist conceptions of meaning, thus elucidating the common structural underpinning of them. And hence no realism versus antirealism debate in the categorical logical basis of metaphysics. The unity of realism and antirealism is thus vindicated in the light of categorical inferentialism as the theory of meaning in the pluralistic unified science of categorical logical positivism.

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