# Proof Sketch of Semmes' Theorem

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**Theorem 1** (Semmes [1]). *Suppose that $f$ is $\mathbf{\Sigma}^0_3$-measurable. If $f$ is not decomposable into $\mathbf{\Sigma}^0_2$-measurable functions on $\mathbf{\Pi}^0_2$ domains (denoted by $f \notin \mathbf{dec}_{2,2}$), then $f \notin \mathbf{\Sigma}_{2,3}$.*

We fix a $\mathbf{\Sigma}^0_3$ description of $f^{-1}[N_s]$ as $Q^s = \bigcup_i Q^s_i$. Given $X$, we define $X'_s$ as the set of all *irreducible points outside* $N_s$, that is,

$$X'_s = X \setminus \bigcup\{N_u : f|_{N_u \setminus Q^s} \in \mathbf{dec}_{2,2}\}.$$

Moreover, given $Y$, we consider the following $Y^\star_{s,i}$:

$$Y^\star_{s,i} = \mathrm{cl}_Y Q^s_i.$$

We iterate this derivation procedure. Let $X^0_{s,i} = X$, $X^{\alpha+1}_{s,i} = ((X^\alpha_{s,i})'_s)^\star_{s,i}$, and $X^\alpha_{s,i} = \bigcap_{\beta<\alpha} X^\beta_{s,i}$ if $\alpha$ is a limit ordinal. Note that there is a countable ordinal $\gamma(s,i)$ such that $X^{\gamma(s,i)+1}_{s,i} = X^{\gamma(s,i)}_{s,i}$. Clearly, $\gamma = \sup_{s,i} \gamma(s,i)$ is a countable ordinal. Then, define the $(s,i)$-*kernel* to be $K_{s,i}X = X^\gamma$.

We say that a point $x \in X$ is *generic* if for every $(s,i)$, either $x \in Q^s$ or $x \in Y \setminus \mathrm{cl}_Y Q^s_i$, where $Y = X^\alpha_{s,i}$ for some $\alpha$. We divide $X$ into three pieces. Define $C = \bigcup_{s,i} K_{s,i}X$, $B$ to be the set of all generic points, and $A$ to be the set of all other points.

**Lemma 2.** *Suppose that $x \notin C$. Then, $x \in A$ if and only if there are $(s_0,i_0),(s_1,i_1) \in \omega^{<\omega} \times \omega$ and $\alpha_0, \alpha_i < \gamma$ such that $N_{s_0} \cap N_{s_1} = \emptyset$ and $x \in X^{\alpha_n}_{s_n,i_n} \setminus (X^{\alpha_n}_{s_n,i_n})'_{s_n,i_n}$ for every $n < 2$.*

*Proof.* The condition $x \notin C$ means that for every $(s,i)$, there is $\alpha$ such that $x \in X^\alpha_{s,i} \setminus X^{\alpha+1}_{s,i}$. If $x \in Q^{s_0}$, then $x \in \mathrm{cl}_Y Q^{s_0}_{i_0}$ for any $i_0$ whenever $x \in Y$. Moreover, non-genericity implies that there exists $(s_1,i_1)$ such that $x \notin Q^{s_1}_{i_1}$ and $x \in \mathrm{cl}_Y Q^{s_1}_{i_1}$ with $Y = X^{\alpha_1}_{s_1,i_1}$. Hence, $x \in X^{\alpha_n}_{s_n,i_n} \setminus (X^{\alpha_n}_{s_n,i_n})'_{s_n,i_n}$ for every $n < 2$. By zero-dimensionality, we may choose $s_0$ such that $x \in Q^{s_0}$ and $N_{s_0} \cap N_{s_1} = \emptyset$. The converse direction is easily verified by choosing $s_n$ with $x \notin Q^{s_n}$, since $N_{s_0} \cap N_{s_1}$ implies that $x \notin Q^{s_n} = f^{-1}[N_{s_n}]$ for some $n < 2$. □

**Lemma 3.** *$A \in \mathbf{\Delta}^0_3$, $B \in \mathbf{\Delta}^0_3$, and $C \in \mathbf{\Sigma}^0_2$.*

*Proof.* Clearly, $C \in \mathbf{\Sigma}^0_2$. Hence, by the previous lemma, $A$ is the difference of two $\mathbf{\Sigma}^0_2$ sets. □

**Lemma 4.** *$f|_B$ is $\mathbf{\Sigma}^0_2$-measurable.*

*Proof.* Suppose that $x \in B$. It is not hard to see that for any $(s,i)$, $x \in Q^s = f^{-1}[N_s]$ if and only if there exists $\alpha < \gamma$ such that $x \in X^\alpha_{s,i} \setminus (X^\alpha_{s,i})^\star_{s,i}$. The latter condition is clearly $\mathbf{\Sigma}^0_2$. □

**Lemma 5.** *$f|_A \in \mathbf{dec}_{2,2}$.*

*Proof.* By Lemma 2, there are $(s_0,i_0),(s_1,i_1),\alpha_0,\alpha_1$ and a clopen neighborhood $N_u$ of $x$ such that $N_{s_0} \cap N_{s_1} = \emptyset$ and $f|_{Y \cap N_u \setminus Q^{s_n}} \in \mathbf{dec}_{2,2}$, where $Y = X^{\alpha_0}_{s_0,i_0} \cap X^{\alpha_1}_{s_1,i_1}$. Since $Q_{s_n}$ is $\mathbf{\Delta}^0_3$ and $(N_u \setminus Q^{s_0}) \cup (N_u \setminus Q^{s_1}) = N_u$, we also have $f|_{Y \cap N_u} \in \mathbf{dec}_{2,2}$. Hence, $f|_A$ is decomposable into $\mathbf{dec}_{2,2}$-functions on closed domains $(X^{\alpha_0}_{s_0,i_0} \cap X^{\alpha_1}_{s_1,i_1} \cap N_u)_{s_0,i_0,\alpha_0,s_1,i_1,\alpha_1,u}$. □

**Lemma 6.** *If $f \notin \mathbf{dec}_{2,2,}$, then $C$ is nonempty, that is, there is $(s,i)$ such that the $(s,i)$-kernel $K_{s,i}X$ is nonempty.* □

The following are key properties of the $(s,i)$-kernel $K_{s,i}X$:

1. $f$ is nowhere decomposable on $K_{s,i}X$ outside $Q^s = f^{-1}[N_s]$, i.e., $f|_{N_u \cap K_{s,i}X \setminus Q^s} \notin \mathbf{dec}_{2,2}$.

2. $Q_i^s$ is dense in $K_{s,i}X$.

**Lemma 7.** *For every $j$, $Q_j^t$ is not dense in $K_{s,i}X$ whenever $N_s \cap N_t = \emptyset$.*

*Proof.* Otherwise, $Q_i^s$ and $Q_j^t$ are dense $\mathbf{\Pi}_2^0$ sets, that is, intersections of sequences of dense open sets. By the Baire category theorem, $Q_i^s$ and $Q_j^t$ have an intersection. However, $Q_i^s \cap Q_j^t$ must be empty since $N_s \cap N_t = \emptyset$. □

Let $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_{a-1}$ be a chain of kernels such that $f$ is nowhere decomposable on $K_l$ outside $f^{-1}[V]$ for any $l < a$, where $V$ is a fixed clopen set.

**Lemma 8.** *There exist nonempty clopen sets $N_s$ and $N_u$ such that $V \cap N_s = \emptyset$ such that $f$ is nowhere decomposable on $K_l \cap N_u$ outside $f^{-1}[V \cup N_s]$ for any $l \leq a$, where $K_a$ is of the form $K_{s,i}Y$ for some $Y \subseteq K_{a-1}$.*

*Proof.* Let $C_0$ and $C_1$ be a pair of pairwise disjoint clopen sets. If $K_l$ contains a dense subset of reducible points outside $f^{-1}[V \cup C_i]$ for every $i < 2$, i.e., $f|_{N_u \cap K_l \setminus f^{-1}[V \cap C_i]} \in \mathbf{dec}_{2,2}$, then by combining these two decomposable functions, we can see that $f$ is somewhere decomposable on $K_l$ outside $f^{-1}[V]$. By the similar argument, if $f$ is nowhere decomposable on $K_l$ outside $f^{-1}[V]$, given a collection $(C_i)_{i<m}$ of pairwise disjoint clopen sets, there can be at most one $i < m$ such that $K_l$ contains a dense subset of reducible points outside $f^{-1}[V \cup C_i]$. Hence, if $m$ is sufficiently big, there is $i < m$ such that $f$ is nowhere decomposable on $K_l \cap N_u$ outside $f^{-1}[V \cup C_i]$ for any $l \leq a$.

By Lemma 6, we have an arbitrarily long chain $K_{a-1} \supseteq K_a^0 := K_{s_0,i_0}K_{a-1} \supseteq K_a^1 := K_{s_1,i_1}K_a^0 \supseteq \dots$ of kernels, where we apply Lemma 6 to $f|_{K_a^n \setminus f^{-1}[V \cup U_n]}$ to obtain $K_a^{n+1}$, where $U_n = \bigcup_{j \leq n} N_{s_j}$. Then, there is $n$ such that $K_a^n$ satisfies the desired condition. □

*Proof Sketch of Theorem 1.* Now, the $\mathbf{\Sigma}_3^0$ set $f^{-1}[N_s]$ is described as $\bigcup_i \bigcap_j \bigcup_k Q_{i,j,k}^s$. Suppose that a $\mathbf{\Sigma}_3^0$ set $S = \bigcup_a \bigcap_b \bigcup_c S_{a,b,c}$ is given. We construct a continuous function $\psi$ and an open set $V$ such that $y \in S$ iff $\psi(y) \in f^{-1}[V]$ for every $y \in \omega^\omega$. Thus, this ensure that $f^{-1}[V]$ is $\mathbf{\Sigma}_3^0$-complete for some open set $V$, which implies $f \notin \mathbf{\Sigma}_{2,3}$.

Suppose that $y \in \omega^\omega$ is given. The $a$-strategy predicts that $y \in \bigcap_b \bigcup_c S_{a,b,c}$ is true. The $a$-strategy acts on a some closed *restraint* $R \subseteq \omega^\omega$ such that $f|_R \notin \mathbf{dec}_{2,2}$ (which is constructed by higher-priority strategies $a' < a$). When the $a$-strategy first acts, say stage $s$,

1. the $a$-strategy defines $(s_a, i_a)$ such that $K_a = K_{s_a,i_a}Y$ for some $Y \subseteq R$ as in Lemma 8, and enumerate $s_a$ into our open set $V_{s+1}$.

2. Then, ensure $K_a \cap [\psi(y \upharpoonright s)] \cap \bigcup_{d \leq a} Q_d^{s[l]} = \emptyset$ for every $l \in I_s$ by Lemma 7, where $V_s$ is written as $V_s = \bigcup_{l \in I_s} N_{s[l]}$.

The $a$-strategy tries to keep $\psi(y \upharpoonright s) \in K_a$.

The $(a,b)$-substrategy waits for a stage $s$ such that $N_{y \upharpoonright s} \subseteq \bigcap_{d \leq b} \bigcup_{c \leq s} S_{a,d,c}$ is true. If such a stage occurs, we may find $\psi(y \upharpoonright s) \in K_a$ extending $\psi(y \upharpoonright s-1)$ such that $N_{\psi(y \upharpoonright s)} \subseteq \bigcap_{l \leq j} \bigcup_k Q_{i_a,j,k}^{s_a}$, since $Q_{i_a}^{s_a}$ is dense in $K_a$. Then, the $a$-strategy makes a restraint $R = K_a \cap \psi(y \upharpoonright s)$, and injures all lower-priority strategies. Then, the $(a,b)$-substrategy halts, and the $(a,b+1)$-strategy starts to act. □

# References

[1] Semmes, B., A game for the Borel functions. *Ph.D. thesis, Universiteit van Amsterdam.* 2009.