Proof Sketch of Semmes' Theorem

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Theorem 1 (Semmes [1]). *Suppose that f is $\mathbf{\Sigma}^0_3$-measurable. If f is not decomposable into $\mathbf{\Sigma}^0_2$-measurable functions on $\mathbf{\Pi}^0_2$ domains (denoted by $f \notin \mathbf{dec}_{2,2}$), then $f \notin \mathbf{\Sigma}_{2,3}$.*

We fix a $\mathbf{\Sigma}^0_3$ description of $f^{-1}[N_s]$ as $Q^s = \bigcup_i Q^s_i$. Given X, we define X'_s as the set of all *irreducible points outside* N_s, that is,

$$X'_s = X \setminus \bigcup\{N_u : f|_{N_u \setminus Q^s} \in \mathbf{dec}_{2,2}\}.$$

Moreover, given Y, we consider the following $Y^\star_{s,i}$:

$$Y^\star_{s,i} = \mathrm{cl}_Y Q^s_i.$$

We iterate this derivation procedure. Let $X^0_{s,i} = X$, $X^{\alpha+1}_{s,i} = ((X^\alpha_{s,i})'_s)^\star_{s,i}$, and $X^\alpha_{s,i} = \bigcap_{\beta<\alpha} X^\beta_{s,i}$ if α is a limit ordinal. Note that there is a countable ordinal $\gamma(s,i)$ such that $X^{\gamma(s,i)+1}_{s,i} = X^{\gamma(s,i)}_{s,i}$. Clearly, $\gamma = \sup_{s,i} \gamma(s,i)$ is a countable ordinal. Then, define the (s,i)-*kernel* to be $K_{s,i}X = X^\gamma$.

We say that a point $x \in X$ is *generic* if for every (s,i), either $x \in Q^s$ or $x \in Y \setminus \mathrm{cl}_Y Q^s_i$, where $Y = X^\alpha_{s,i}$ for some α. We divide X into three pieces. Define $C = \bigcup_{s,i} K_{s,i}X$, B to be the set of all generic points, and A to be the set of all other points.

Lemma 2. *Suppose that $x \notin C$. Then, $x \in A$ if and only if there are $(s_0,i_0),(s_1,i_1) \in \omega^{<\omega} \times \omega$ and $\alpha_0, \alpha_i < \gamma$ such that $N_{s_0} \cap N_{s_1} = \emptyset$ and $x \in X^{\alpha_n}_{s_n,i_n} \setminus (X^{\alpha_n}_{s_n,i_n})'_{s_n,i_n}$ for every $n < 2$.*

Proof. The condition $x \notin C$ means that for every (s,i), there is α such that $x \in X^\alpha_{s,i} \setminus X^{\alpha+1}_{s,i}$. If $x \in Q^{s_0}$, then $x \in \mathrm{cl}_Y Q^{s_0}_{i_0}$ for any i_0 whenever $x \in Y$. Moreover, non-genericity implies that there exists (s_1,i_1) such that $x \notin Q^{s_1}_{i_1}$ and $x \in \mathrm{cl}_Y Q^{s_1}_{i_1}$ with $Y = X^{\alpha_1}_{s_1,i_1}$. Hence, $x \in X^{\alpha_n}_{s_n,i_n} \setminus (X^{\alpha_n}_{s_n,i_n})'_{s_n,i_n}$ for every $n < 2$. By zero-dimensionality, we may choose s_0 such that $x \in Q^{s_0}$ and $N_{s_0} \cap N_{s_1} = \emptyset$. The converse direction is easily verified by choosing s_n with $x \notin Q^{s_n}$, since $N_{s_0} \cap N_{s_1}$ implies that $x \notin Q^{s_n} = f^{-1}[N_{s_n}]$ for some $n < 2$. □

Lemma 3. *$A \in \mathbf{\Delta}^0_3$, $B \in \mathbf{\Delta}^0_3$, and $C \in \mathbf{\Sigma}^0_2$.*

Proof. Clearly, $C \in \mathbf{\Sigma}^0_2$. Hence, by the previous lemma, A is the difference of two $\mathbf{\Sigma}^0_2$ sets. □

Lemma 4. *$f|_B$ is $\mathbf{\Sigma}^0_2$-measurable.*

Proof. Suppose that $x \in B$. It is not hard to see that for any (s,i), $x \in Q^s = f^{-1}[N_s]$ if and only if there exists $\alpha < \gamma$ such that $x \in X^\alpha_{s,i} \setminus (X^\alpha_{s,i})^\star_{s,i}$. The latter condition is clearly $\mathbf{\Sigma}^0_2$. □

Lemma 5. *$f|_A \in \mathbf{dec}_{2,2}$.*

Proof. By Lemma 2, there are $(s_0,i_0),(s_1,i_1),\alpha_0,\alpha_1$ and a clopen neighborhood N_u of x such that $N_{s_0} \cap N_{s_1} = \emptyset$ and $f|_{Y \cap N_u \setminus Q^{s_n}} \in \mathbf{dec}_{2,2}$, where $Y = X^{\alpha_0}_{s_0,i_0} \cap X^{\alpha_1}_{s_1,i_1}$. Since Q_{s_n} is $\mathbf{\Delta}^0_3$ and $(N_u \setminus Q^{s_0}) \cup (N_u \setminus Q^{s_1}) = N_u$, we also have $f|_{Y \cap N_u} \in \mathbf{dec}_{2,2}$. Hence, $f|_A$ is decomposable into $\mathbf{dec}_{2,2}$-functions on closed domains $(X^{\alpha_0}_{s_0,i_0} \cap X^{\alpha_1}_{s_1,i_1} \cap N_u)_{s_0,i_0,\alpha_0,s_1,i_1,\alpha_1,u}$. □

Lemma 6. *If $f \notin \mathbf{dec}_{2,2,}$, then C is nonempty, that is, there is (s,i) such that the (s,i)-kernel $K_{s,i}X$ is nonempty.* □

The following are key properties of the (s, i)-kernel $K_{s,i}X$:

1. f is nowhere decomposable on $K_{s,i}X$ outside $Q^s = f^{-1}[N_s]$, i.e., $f|_{N_u \cap K_{s,i}X \setminus Q^s} \notin \mathbf{dec}_{2,2}$.

2. Q^s_i is dense in $K_{s,i}X$.

Lemma 7. *For every j, Q^t_j is not dense in $K_{s,i}X$ whenever $N_s \cap N_t = \emptyset$.*

Proof. Otherwise, Q^s_i and Q^t_j are dense $\mathbf{\Pi}^0_2$ sets, that is, intersections of sequences of dense open sets. By the Baire category theorem, Q^s_i and Q^t_j have an intersection. However, $Q^s_i \cap Q^t_j$ must be empty since $N_s \cap N_t = \emptyset$. □

Let $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_{a-1}$ be a chain of kernels such that f is nowhere decomposable on K_l outside $f^{-1}[V]$ for any $l < a$, where V is a fixed clopen set.

Lemma 8. *There exist nonempty clopen sets N_s and N_u such that $V \cap N_s = \emptyset$ such that f is nowhere decomposable on $K_l \cap N_u$ outside $f^{-1}[V \cup N_s]$ for any $l \leq a$, where K_a is of the form $K_{s,i}Y$ for some $Y \subseteq K_{a-1}$.*

Proof. Let C_0 and C_1 be a pair of pairwise disjoint clopen sets. If K_l contains a dense subset of reducible points outside $f^{-1}[V \cup C_i]$ for every $i < 2$, i.e., $f|_{N_u \cap K_l \setminus f^{-1}[V \cap C_i]} \in \mathbf{dec}_{2,2}$, then by combining these two decomposable functions, we can see that f is somewhere decomposable on K_l outside $f^{-1}[V]$. By the similar argument, if f is nowhere decomposable on K_l outside $f^{-1}[V]$, given a collection $(C_i)_{i<m}$ of pairwise disjoint clopen sets, there can be at most one $i < m$ such that K_l contains a dense subset of reducible points outside $f^{-1}[V \cup C_i]$. Hence, if m is sufficiently big, there is $i < m$ such that f is nowhere decomposable on $K_l \cap N_u$ outside $f^{-1}[V \cup C_i]$ for any $l \leq a$.

By Lemma 6, we have an arbitrarily long chain $K_{a-1} \supseteq K^0_a := K_{s_0,i_0}K_{a-1} \supseteq K^1_a := K_{s_1,i_1}K^0_a \supseteq \ldots$ of kernels, where we apply Lemma 6 to $f|_{K^n_a \setminus f^{-1}[V \cup U_n]}$ to obtain K^{n+1}_a, where $U_n = \bigcup_{j \leq n} N_{s_j}$. Then, there is n such that K^n_a satisfies the desired condition. □

Proof Sketch of Theorem 1. Now, the $\mathbf{\Sigma}^0_3$ set $f^{-1}[N_s]$ is described as $\bigcup_i \bigcap_j \bigcup_k Q^s_{i,j,k}$. Suppose that a $\mathbf{\Sigma}^0_3$ set $S = \bigcup_a \bigcap_b \bigcup_c S_{a,b,c}$ is given. We construct a continuous function ψ and an open set V such that $y \in S$ iff $\psi(y) \in f^{-1}[V]$ for every $y \in \omega^\omega$. Thus, this ensure that $f^{-1}[V]$ is $\mathbf{\Sigma}^0_3$-complete for some open set V, which implies $f \notin \mathbf{\Sigma}_{2,3}$.

Suppose that $y \in \omega^\omega$ is given. The a-strategy predicts that $y \in \bigcap_b \bigcup_c S_{a,b,c}$ is true. The a-strategy acts on a some closed *restraint* $R \subseteq \omega^\omega$ such that $f|_R \notin \mathbf{dec}_{2,2}$ (which is constructed by higher-priority strategies $a' < a$). When the a-strategy first acts, say stage s,

1. the a-strategy defines (s_a, i_a) such that $K_a = K_{s_a,i_a}Y$ for some $Y \subseteq R$ as in Lemma 8, and enumerate s_a into our open set V_{s+1}.

2. Then, ensure $K_a \cap [\psi(y \upharpoonright s)] \cap \bigcup_{d \leq a} Q^{s[l]}_d = \emptyset$ for every $l \in I_s$ by Lemma 7, where V_s is written as $V_s = \bigcup_{l \in I_s} N_{s[l]}$.

The a-strategy tries to keep $\psi(y \upharpoonright s) \in K_a$.

The (a, b)-substrategy waits for a stage s such that $N_{y \upharpoonright s} \subseteq \bigcap_{d \leq b} \bigcup_{c \leq s} S_{a,d,c}$ is true. If such a stage occurs, we may find $\psi(y \upharpoonright s) \in K_a$ extending $\psi(y \upharpoonright s-1)$ such that $N_{\psi(y \upharpoonright s)} \subseteq \bigcap_{l \leq j} \bigcup_k Q^{s_a}_{i_a,j,k}$, since $Q^{s_a}_{i_a}$ is dense in K_a. Then, the a-strategy makes a restraint $R = K_a \cap \psi(y \upharpoonright s)$, and injures all lower-priority strategies. Then, the (a, b)-substrategy halts, and the $(a, b+1)$-strategy starts to act. □

References

[1] Semmes, B., A game for the Borel functions. *Ph.D. thesis, Universiteit van Amsterdam.* 2009.