Stochastic stability of dynamic user equilibrium in unidirectional networks: Weakly acyclic game approach

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ISTTT23@EPFL, 24th July, 2019
Stability of an equilibrium state

• Important condition for the state to be realized [Beckmann et al., 1956]

Traditional analysis method is Lyapunov approach [Smith, 1984]

1. Consider day-to-day dynamics describing users’ route choice behavior; e.g., the Smith dynamic

2. Investigate the existence of the Lyapunov function
   - Based on the monotonicity of route travel time functions
   - e.g., DUE with fixed depart: single-BN-per-route network [Smith and Ghali, 1990; Mounce, 2006]

• If exists, asymptotic stability of equilibria is guaranteed
Limitation of the Lyapunov approach

- Monotonicity is not a general property of DUE problems; e.g., multiple-BN-per-route networks [Kuwahara, 1990; Mounce and Smith, 2007]

We need to develop a different approach to examine stability property without requiring the monotonicity

Example of multiple-BN-per-route network

[Diagram showing a network with bottlenecks marked by '×']
Our approach

- Combines concepts developed in DTA and game theory
  1. DUE game dealing with atomic vehicles
  2. Ordering property of vehicles in unidirectional networks
  3. Weakly acyclic games
  4. Perturbed better/best dynamics and analysis of their stationary distributions

Research Objective

Examine the stochastic stability of DUE with fixed departure time in unidirectional networks
Table of contents

- Problem settings
  - DUE game and the concept of stochastic stability
  - Unidirectional network

- Analysis and results
  - Ordering property and weakly acyclic game
  - Stochastic stability analysis

- Numerical experiments
Components of a DUE game [cf., Rosenthal, 1973]

1. User = atomic vehicle
   - Departure time and OD are fixed
2. Strategy = route selected by the user
   - All users’ routes are called “route profile”
3. Disutility = the user’s travel time
   - Destination arrival time can be utilized

Notations

- Travel time is calculated by any dynamic loading model satisfying FIFO principle and Causality
  - Physical queue model is applicable;
  - e.g. Newell’s car-following model [Newell, 2002]
Nash equilibrium and Strictness

- Nash equilibrium

No user has an incentive to change the route individually in order to reduce the travel time

- Mathematical expression of equilibrium $r^*$ is

$$g_i(r_i^*; r_{-i}^*) \leq g_i(r_i; r_{-i}^*), \quad \forall r_i \in R_i, \forall i \in P.$$
Nash equilibrium and Strictness

- Nash equilibrium
  
  No user has an incentive to change the route individually in order to reduce the travel time

  - Mathematical expression of equilibrium $r^*$ is
    
    $$g_i(r_i^*; r_{-i}^*) < g_i(r_i; r_{-i}^*), \quad \forall r_i \in R_i, \forall i \in P.$$  
    
    **Strict inequality**

- If strict inequality holds: $r^*$ is called **strict** Nash equilibrium
  
  - Each user has **only one** shortest route in the state
Day-to-day dynamics

- Perturbed dynamics
  - Day-to-day dynamics that is perturbed by stochastic effect
    - e.g., misperception of utility; random route choice
  - This dynamics becomes Markov chain having the unique stationary distribution

- Consider two perturbed dynamics:
  1. Perturbed better response dynamics
  2. Perturbed best response dynamics
    - Logit response dynamics
Unperturbed dynamics

- Better response dynamics
  - One user is randomly selected and chooses the route so as to decrease the destination arrival time strictly

\[
D_i(r^\tau) := \left\{ r_i^* \mid r_i^* \in \mathcal{R}_i \text{ s.t. } g_i(r_i^*, r_{-i}^\tau) < g_i(r_i^\tau, r_{-i}^\tau) \right\}
\]

- Best response dynamics
  - Selected user chooses the route to minimize the arrival time

\[
B_i(r^\tau) := \left\{ r_i^* \mid r_i^* \in \mathcal{R}_i \text{ s.t. } g_i(r_i^*, r_{-i}^\tau) \leq \min_{r \in \mathcal{R}_i} g_i(r, r_{-i}^\tau) \right\}.
\]
If the **better response dynamics** is employed, users **can not** switch routes among multiple shortest routes.

If the **best response dynamics** is applied, users **can switch** routes among multiple shortest routes.

- Users can **deviate from their current shortest routes**.

- **Example:**

<table>
<thead>
<tr>
<th>Route</th>
<th>Destination arrival time</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9:00</td>
</tr>
<tr>
<td>B</td>
<td>8:30</td>
</tr>
<tr>
<td>C</td>
<td>8:30</td>
</tr>
</tbody>
</table>

- **Better response**

- **Best response**
Perturbed dynamics

- Perturbed better response dynamics
  - Users sometimes change the route randomly with probability $\varepsilon$

  Transition probability from route profile $r$ to $r'$

  \[
  P^\varepsilon_{r \rightarrow r'} = (1 - \varepsilon) \cdot P^0_{r \rightarrow r'} + \varepsilon \cdot \frac{1}{|P|} \cdot \frac{1}{|R_i|}
  \]
  
  Transition probability without perturbation
  Transition probability by random route choice

- Logit response dynamics (perturbed best response)
  - Users choose the route according to the travel time

  \[
  P^\beta_{r \rightarrow r'} = \frac{\exp(-\beta g_i(r'_i, r_{-i}))}{\sum_{r \in R_i} \exp(-\beta g_i(r, r_{-i}))}
  \]
  
  $\beta$: perturbation parameter
Stochastic stability

- Stability concept of perturbed dynamics [Young, 1993]
  - State $r$ is **stochastically stable** if the observation probability is positive when the perturbation approaches zero, i.e.,
    \[
    \lim_{\varepsilon \to 0} \mu_r^\varepsilon > 0
    \]
    $\mu_r^\varepsilon$ : stationary distribution with perturbation parameter $\varepsilon$
Stochastic stability

- Stability concept of perturbed dynamics [Young, 1993]
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\]

\( \mu^\epsilon : \) stationary distribution with perturbation parameter \( \epsilon \)

![Diagram showing probability distribution with equilibrium points and medium and high values of \( \epsilon \)]
Stochastic stability

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- Numerical experiments
Definition

The **earliest arrival time** of a vehicle at each node can be represented as a **function of a reference time**

- Reference time: departure time from an arbitrary origin
- The function is called **node potential function**: \( p_n(t) \)
- Earliest node arrival time of vehicle having a reference time can be determined unrelated to the OD
Definition

The **earliest arrival time** of a vehicle at each node can be represented as a function of a reference time

Typical example: **single origin** – multiple destination network

- Reference time: **departure time from the single origin**

Unidirectional network [Iryo and Smith, 2018; ISTTT22]
Definition

The earliest arrival time of a vehicle at each node can be represented as a function of a reference time.

Typical example: single origin – multiple destination network

- Reference time: departure time from the single origin

![Diagram of unidirectional network](attachment:network_diagram.png)
Definition

The earliest arrival time of a vehicle at each node can be represented as a function of a reference time.

Typical example: single origin – multiple destination network

- Reference time: departure time from the single origin

\[ t = 0 \]

\[
\begin{align*}
p_a(t) &= 1 \\
p_b(t) &= 4 \\
p_e(t) &= 5 \\
p_c(t) &= 2 \\
p_d(t) &= 5 \\
p_f(t) &= 6
\end{align*}
\]

Shortest route

Number:
Earliest arrival time
Table of contents

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Theorem: there exists monotonic relationship between the potential function and the reference time, i.e.,

\[ t < t' \implies p_n(t) \leq p_n(t'), \quad \forall n \in \mathbb{N} \]

- User departing at a reference time is not overtaken by the Users departing at later reference times

Diagram:

- \( p_a(t) = 1 \) \quad \( \rightarrow 2 \) \quad \( p_b(t) = 4 \) \quad \( \rightarrow 5 \) \quad \( p_e(t) = 5 \) \quad \( \rightarrow 6 \)
- \( p_c(t) = 2 \) \quad \( \rightarrow 3 \)
- \( p_d(t) = 5 \) \quad \( \rightarrow 6 \) \quad \( p_f(t) = 6 \) \quad \( \rightarrow 7 \)

Ordering property in unidirectional net.
• The shortest route of the user having earliest the reference time becomes the **ex-post shortest route** for that user.

➢ Nash equilibrium is achieved by assigning all vehicles **one by one** to their shortest routes in the order of reference time.

❖ Ordering property

There exists an **assignment order** of the users for achieving equilibrium in a unidirectional network.
Weakly acyclic games (WAG)

Definition

In WAG, from any state, there exists at least one better response path ending at a Nash equilibrium

- Better response path: sequence of better responses

Better response dynamics converges almost surely to a Nash equilibrium

<table>
<thead>
<tr>
<th>User 1</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(2,1)</td>
<td>(1,2)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>B</td>
<td>(-1,2)</td>
<td>(2,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>C</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

Number: utility
Theorem

A DUE game in a unidirectional network is a WAG

The proof can be made by proving the existence of better response path based on ordering property.

From any state, a Nash equilibrium is achieved by the users’ better responses in the assignment order obtained from the ordering property.
Table of contents

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  - DUE game and the concept of stochastic stability
  - Unidirectional network

- Analysis and results
  - Ordering property and weakly acyclic game
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Existence of stable equilibrium

- Proposition

If perturbed better response dynamics is employed, there exists a stochastically stable equilibrium in unidirectional network.

- The proof can be made straightforwardly by combining:
  - Young’s theorem
  - Ordering property of a DUE game in a unidirectional network
  - Properties of WAG
Existence of stable equilibrium

- Proposition

If perturbed better response dynamics is employed, there exists a stochastically stable equilibrium in unidirectional network.

- The proof can be made straightforwardly by combining:
  - Young’s theorem

Stochastically stable states of perturbed dynamics are contained in the absorbing states of the Markov chain of the unperturbed dynamics.
In a DUE game in unidirectional networks, which is WAG, absorbing state of better response = **Nash equilibrium**

Therefore, stochastically stable states are **contained in a set of Nash equilibrium states**

<table>
<thead>
<tr>
<th>Route profile</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Destination</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>User $i$</td>
<td>9:00</td>
<td>8:30</td>
<td>8:30</td>
</tr>
<tr>
<td>User $j$</td>
<td>9:00</td>
<td>8:30</td>
<td>9:00</td>
</tr>
<tr>
<td>User $k$</td>
<td>9:00</td>
<td>8:30</td>
<td>9:30</td>
</tr>
</tbody>
</table>

**Nash equilibrium**
Remark

If logit response dynamics is applied, the existence of stochastically stable equilibrium is "not" guaranteed in general.

There is a difference in the existence of stable equilibrium between "better" and "best!"
Perturbed best response dynamics

- Absorbing state of best response = **Strict Nash equilibrium**

- When a user has multiple shortest routes, the route profile can deviate from a non-strict Nash equ. by the best response.

- Therefore, the existence of a strict Nash equilibrium is required to claim the stochastic stability.

- However, a DUE game **might not have** strict Nash equilibrium.
Example of a network that the strictness is lost due to queues:

- One origin and one destination network with two routes
- A user can catch up to the queue using either route
- Not overtaken by the users departing later
Non-strictness of Nash equilibrium

- Example of a network that the strictness is lost due to queues
  - Destination arrival times of both routes become the same
  - Multiple shortest routes exist -> Strictness is lost
Non-strictness of Nash equilibrium

- Example of a network that the strictness is lost due to queues
  - Destination arrival times of both routes become the same
    → Multiple shortest routes exist –> Strictness is lost

- Dynamics with “strict improvement property” is desirable
  - Such as “better” response dynamics
Table of contents

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  - DUE game and the concept of stochastic stability
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Network and demand

- Two origins and Two destinations with four routes for each OD
- 50 users driving for each OD
  - Total: 200 users

Perturbed dynamics

1. Perturbed better response dynamics
2. Logit response dynamics
   - with three perturbation level
3. Iterated 200,000 times from the fixed initial state
   - Shortest distance route profile

Numerical experiments

We finally show the numerical experiments to demonstrate the convergence and stochastic stability of the dynamic user equilibrium in a unidirectional network. Specifically, we first investigate the convergence property of the better response dynamics. In this experiment, we compare the convergence speed of the better response with physical-queue model and point-queue model to consider how queue spillbacks affect the convergence speed.

Next, we investigate the stationary distributions of the perturbed dynamics with and without the additional criteria of DP to confirm the validity of the previous discussion.

5.1. Settings

We considered a unidirectional network with many-to-many OD, which is the modification of Nguyen Dupuis network (as shown in Fig. 4). Nodes \( \{o_1, o_2\} \) are origins, and nodes \( \{d_1, d_2\} \) are destinations. The physical condition of each link (e.g., free-flow travel time, capacity) are summarized in Table 1. Each link has a bottleneck section with a bottleneck capacity at the end of the link. The network has 4 routes for each destination, respectively; this translates to 4 actions available for each user. These routes are numbered as shown in the figure. The number of users departing from each origin is 100; the total number of users is 200. Each user from each origin departs with fixed time-headway (we specify later). We set the ratio of users for the destinations as one-to-one.

Traffic dynamics within the network is simulated using a mesoscopic LWR model proposed by Leclercq and Bécarie (2012). This simulator provides the event time when each user crosses the specific points of the network (e.g., node); then according to the event time, the trajectory of each user is calculated based on the dynamic loading model. We here employ the simplified car following model of Newell (2002).

Origins

\[ \begin{array}{c}
\text{Route 4 for both} \\
\text{Route 3 for both} \\
\text{Route 2 for both} \\
\text{Route 1 for } d_1
\end{array} \]

Dests.

\[ \begin{array}{c}
\text{Route 1 for } d_2
\end{array} \]
Stationary distributions about route choice patterns

- PBBD: shows Nash equilibria (5 Nash)
- Logit: does not seem to converge to equilibria
Investigate how users' route choices influence each other

- Divide the users into four groups in the order of reference time
  - Users in group with earlier number: having earlier reference time
- Aggregate the total number of route changes in each group for every time slot (4000 iterations)

Reference time becomes later
Perturbed better response dynamics

- # of route changes: decreases to zero from earlier groups
  - Ordering property + strict improvement property:
    - Users having earlier reference time can choose their ex-post shortest routes earlier
    - Once users choose their shortest routes, they do not change
# of route changes: **does not decrease to zero**

- Changes among multiple shortest routes in **group1** act like perturbation for the other groups
  - Users’ routes are **not fixed to their shortest routes**
- Route profile does not go toward a Nash equilibrium
Conclusion

- Analyze the stochastic stability of DUE
  - Prove that **DUE games in unidirectional networks are weakly acyclic games** with the ordering property
  - Prove **the existence of stochastically stable equilibrium of perturbed better response dynamics**
  - Found that **strict improvement property of the applied dynamics** is important for the existence of a stable equilibrium

- Future research direction
  - Analyze the cost pattern of stable equilibrium
    - Examine whether efficient equilibrium is selected or not
  - Design an incentive scheme to achieve efficient pattern
    - Converge to an efficient equilibrium
Thank you for listening!

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References


References


