Duality Theory and Categorical Universal Logic

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Outline

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Continuous vs. Discontinuous Worlds

Presumably, Zeno is the first person who elucidated a dichotomy b/w continuity and discreteness of space.

- By Zeno paradoxes, space is neither conti. nor discrete?
- In my view, space is continuous and discrete, or we must account for both aspects of the concept of space.
  - Different conceptions of space make sense in different contexts. We see different worlds in different environments.

It is, I believe, what (categorical) duality is about: duality puts equal emphasis on both sides. Categorical logic is a way to think of space in terms of logic. I follow its spirit in formulating a universal concept of space (and logic).

Wittgenstein's Conception of Space

Wittgenstein's point-free idea:

- "What makes it apparent that space is not a collection of points, but the realization of a law?" (from his *Philosophical Remarks*, p.216). "Geometrical" vs. "arithmetical" space.

As he says in *Philosophical Grammar*, Wittgenstein's intensional view on space is a consequence of his persistent disagreement with the set-theoretical extensional view of math:

- "Mathematics is ridden through and through with the pernicious idioms of set theory. One example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all." (p.211) Law??

M., "Wittgenstein's Conception of Space and the Modernist Transformation of Geometry via Duality", *Proc. of 36th Witt. Symposium*, Witt. Society, 2013.

Categorical Duality

Mathematically we have a categorical duality b/w set-theoretical, point-set and algebraic, point-free conceptions of space. Diverse categorical dualities:

	Ontic	Epistemic	Duality
Logic	Semantics	Syntax	Stone
Alg. Geometry	Variety	Ring	Grothendieck
Topology	Points	Opens (Pred.)	Isbell, Papert
Computer Sci.	System	Behaviour	Coalg|Modal
Quantum Phys.	State	Observable	OQM

Dualities in diverse fields have “something" in common. I think duality arises between the ontic and the epistemic. N/B: Gro. style sheaf-th. duality works even for non-com. alg.

Origin‘s’ of Duality

Similar ideas by a toposopher and a (genuine) philosopher:

- Lawvere: “duality b/w the conceptual and the formal", discussed in his hyperdoctrine paper “Adjointness in Foundations".
  - space-algebra duality, semantics-syntax duality, and state-observable duality would fall into this picture.
- Granger: “duality b/w objects and operations."

Modernisation, I think, is the shift from “objects" to “operations" (cf. cats., QM, ...). Where is the origin? May be no single origin.

- One origin could be Cassirer’s “Substance and Function". Whitehead’s process philosophy was later than Cassirer’s.

Even Gödel thought of the shift from “substance" to “function."

Gödel's idea of shift from "right" to "left"

Gödel's "The modern development of the foundations of mathematics in the light of philosophy" (in his *Collected Works*):

> *the development of philosophy since the Renaissance has by and large gone from right to left ... Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense ...*

In the physical context, Gödel's "right" means something like reality or substance, and "left" like observation or appearance. In other contexts, Gödel says metaphysics is "right", and formal logic is "left." Presumably, Newton is "right", and Leibniz is "left."

Two Aspects Integrated Together

Which of the Newtonian point-based concept of space and the Leibnizian relational point-free concept of space is better? (Cat.) duality would say they are equiv.; duality is not dualism.

- A universal concept of space must represent both (point-based and point-free) aspects of the concept of space within the one concept.
  - There are different ways to implement this idea.

One way: Chu space. It is $(S, A, e : S \times A \to \Omega)$, which makes no distinction b/w the point-based view $(S, \mathrm{Cont}(S), \mathrm{ev}_S)$ and the point-free view $(\mathrm{Spec}(A), A, \mathrm{ev}_A)$.

- This leads us to useful duality th.: M., From Operational Chu Duality to Coalgebraic Quantum Symmetry, *Springer LNCS* 8089, 2013.

This is not the direction we take here. We rely on categ. logic.

Outline

From Space to Logic

Frm is the alg. cat. of frames, a point-free abstraction of **Top**.

- In this context, to represent both aspects of the concept of space means to take into account both **Top** and **Frm**.
- We have a dual adjunction b/w **Top** and **Frm**, induced by any Ω (in both **Frm** and **Top**) as a dualising object.
- We consider the functor $\mathrm{Hom}(\text{-}, \Omega) : \mathbf{Top}^{\mathrm{op}} \rightarrow \mathbf{Frm}$ per se to be space, which has geometric logical structures (e.g., quantifier $\exists$ as adjoint) in terms of Lawvere hyperdoctrines, i.e., “logical substances" of toposes or logoses.
- In general we take universally-algebraised hyperdoctrines $P : \mathbf{C}^{\mathrm{op}} \rightarrow \mathbf{Alg}$ to be space, which can include quantum sp.

This is the place where logic and space meet, and at the same time, where traditional and modern quantum logics meet.

Abramsky-Coecke vs. Birkhoff-von Neumann

Does Traditional Quantum Logic (TQL) make no sense after modern CQM? If asked, I could argue for TQL, against CQM.

- CQM (at present) is mostly limited by finite-dimensionality; TQL is not. Some work, yet it cannot account, e.g., for unbounded op.
- CQM (at present) cannot account for quantum symmetries (esp., anti-unitaries); TQL can.
- CQM is logically too strong; AND and OR collapse; e.g., $\neg(A \otimes B) \leftrightarrow \neg B \otimes \neg A$. TQL is not.
  - This is caused by compact structures (SMC is okay).
- Curry-Howard-Lambek interpretation of $\dagger$ in CQM: a proof from A to B implies a proof from B to A. (Cf. tonk)

But I do not argue for any of the two, against the other. Rather, I want to reconcile them; in my view, they are not really in conflict.

Quantum Systems vs. Propositions

CQM is logic of what? The obvious point:

- CQM is the logic of quantum systems: objects are systems; arrows are processes.
  - Then it is not very strange that a process from a system A to a system B implies another process from B to A.
- In CQM, objects are not propositions but systems.
- CQM is about how systems compose, and not about how propositions compose.
- CQM's $\otimes$ and others are not logical connectives, but system constructors, or in logical terms, type constructors.

CQM is logic of quantum systems; "logic" in type th.'s sense.
TQL is logic of quantum propositions. Different logics.

Discrepancy b/w Logic and Types

CQM and TQL are different, i.e., logic (type th.) of quantum systems and logic of quantum propositions are different.

- The Curry-Howard iso. asserts that logic of propositions is iso. to logic of types.
- It is not valid in the quantum sense above.
- The concept of hyperdoctrine primarily separates logic and type structures: i.e., **C** and **Alg** in $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$.
    - Aczel's "logic-enriched type theory" is along a similar line, separating logic and type structures.

In the intuitionistic case, logic of **C** (CCC) is in harmony with that of **Alg** (**HA**). But it is not always the case, as in the quant.

Quantum Proof Theory

TQL has been criticized in the light of proof theory, sometimes referred to as “non-logic."

- The Faggian-Sambin (1998) sequent calculus for quantum logic enjoys nice proof-theoretic properties, most notably the cut elimination.
  - Orthocomplement is defined rather than assumed according to Girard’s conception of negation.
  - Context formulae are restricted to treat non-distributivity. This leads us to restricting Frobenius Reciprocity.
- According to the standard idea of logic over type theory ($\Gamma | \Phi \vdash \Psi$), we can expand it with linear type theory (Ambler’s) in order to represent both logic of quantum propositions and logic of quantum systems.

Completeness wrt. quantum hyperdoctrines is then obtained. No completeness has been known for quantified quantum logic.

Outline

1. What is Space?
   - Space as Ontic Structure
   - Space as Epistemic Structure
2. What is Logic (of Quantum Mechanics)?
   - One Conception: How Propositions Compose
   - Another Conception: How Systems Compose
3. What is a Universal Concept of Logic and Space?
   - How to Unify Topos- and **Hilb**-like structures?
   - Lawvere Hyperdoctrine Universally-Algebraised

Categorical Universal Logic

A topos **E** gives rise to

- $\mathrm{Sub}_{\mathbf{E}}(\text{-}) : \mathbf{E}^{\mathrm{op}} \to$ **HeytAlg** (and the fibration $\int \mathrm{Sub}_{\mathbf{E}} \to \mathbf{E}$)
- Lawvere-Pitts hyperdoctrines abstract such structures, providing semantics for both intuitionistic FOL and HOL.
  - Hyperdoctrines are of the form $P : \mathbf{C}^{\mathrm{op}} \to$ **HeytAlg** (Pitts' tripos version rather than Lawvere's original one).
  - Topoi are categories whose subobject functors form higher-order hyperdoctrines, or equivalently, fibrations.
  - Cat. of topoi embeds into cat. of hypdoc. via HJP const.

A $\dagger$-kernel cat **H** by Heunen-Jacobs yields

- $\mathrm{KSub}_{\mathbf{H}}(\text{-}) : \mathbf{H}^{\mathrm{op}} \to$ **OrthModLat** (subtleties on arrows).

Both are of the form

- $P : \mathbf{C}^{\mathrm{op}} \to$ **Alg** for a variety **Alg** of ordered algebras.

Categorical Universal Logic $=$ theory of universally algebraised hyperdoctrines; a universal concept of logic *and* space.

Universally algebraised hyperdoctrines

An **Alg**-hyperdoctrine is a functor (or “fibred universal algebra")

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$$

where **C** has at least products, and may be monoidal.

- It has $\forall$ iff for any proj. π in **C**, $P(\pi)$ has a right adjoint $\forall_\pi$.
- It has $\exists$ iff for any proj. π in **C**, $P(\pi)$ has a left adjoint $\exists_\pi$.

Why do algebras have $\leq$? B/c “quantifiers as adjoints" means adjointness wrt. logical entailment relations $\vdash$, which is $\leq$.

- Different translations (Gödel, Girard, etc.) can be treated as transformations b/w different types of **Alg**-hypdoc.

CUL allows us to compare different categorical logics on the one setting. They have mostly been studied separately.

More Logical Structures

An **Alg**-hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$ gives rise to a fibred category $\int P$ via the Grothendieck construction.

- $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Alg}$ has comprehenion $\{\text{-}\}$ iff the truth functor $\top : \mathbf{C} \to \int P$ has a right adjoint $\{\text{-}\} : \int P \to \mathbf{C}$.
  - Each fibre $P(C)$ is supposed to have the truth const. $\top_{P(C)}$.
- It has $=$ iff for any diagonal δ, $P(\delta)$ has a left adjoint.
  - Most such ideas come from Lawvere.

Higher-order logic can be treated as follows.

- A **Alg**-hyperdoctrine P is higher order iff the base cat. $\mathbf{C}$ is an MCC, and it has an object classifier (or truth value object), i.e., $\exists \Omega \in \mathbf{C} \;\; P \simeq \mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega)$.

Topoi are cats. whose $\mathrm{Sub}(\text{-})$ form higher-order hyperdoctrines.

Duality as Semantics

Duality induced by a Janusian (aka. schizophrenic) object Ω:

- $\mathrm{Hom}_{\mathbf{D}}(\text{-}, \Omega) \dashv \mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.
  - Such duality theories have been developed by Johnstone, Porst-Tholen, M., ... **C** and **D** have faithful functors into **Set**.
- **D** $:=$ **Alg** as in many cases. $\mathrm{Hom}_{\mathbf{C}}(\text{-}, \Omega) : \mathbf{C}^{\mathrm{op}} \to$ **Alg** give **Alg**-relativized hyperdoctrines under certain conditions.
- The right adjoint functor $\mathrm{Hom}_{\mathbf{Top}}(\text{-}, \mathbf{2}) : \mathbf{Top}^{\mathrm{op}} \to$ **Frm** is a geometric hyperdoctrine.
- Adj. functor $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to$ **Alg** is an **Alg**-hyprdoc.
  - In the Heyting case, these give rise to sheaf topoi via the tripos-topos construction by Hyland-Johnstone-Pitts.
  - In the quantum case, these yield quantum-valued sets by Takeuti and Ozawa. (substr. case, quantale-valued sets)

Duality hypdoc. always have obj. classifiers as dualising obj. Ω.

Remarks on Duality Theory

Johnstone's theory of "general concrete dualities" in his *Stone Spaces* seems flawed. Context: the two structures of a dualising object Ω in **C** and **D** must *commute* to get duality.

- Indeed, he explicitly says: "we choose not to involve ourselves in giving a precise meaning to the word 'commute' in the last sentence" (p. 254). But his dual adjunction thm. actually depends on this.

Porst-Tholen's 1991 theory of dualities is rigorous. My theory in:

- M., Categorical Duality Theory: With Applications to Domains, Convexity, and the Distribution Monad, *Proc. of CSL'13*.

The concept of "commute" is precisely formulated as what is called the harmony condition b/w alg. and topo. structures.

Convex Geometric Logic

Let $\mathcal{D}$: **Set** $\to$ **Set** be the distribution monad. $\mathcal{D}(X)$ is the set of probability distributions on X with finite supports.

- Jacobs showed a dual adjunct. b/w **Alg**$(\mathcal{D})$ and **PFrm**. **Alg**$(\mathcal{D})$ is the cat. of sets with abst. convex combinations.
  - Jacobs left open the equivalence induced by this adj. My CSL paper above characterises it as dual equiv. b/w idempotent **D**-algs. and **AlgLat**.
- Another dual adj. b/w **ConvSp** and **ContLat**, which restricts to dual equiv. b/w **SobConvSp** and **AlgLat**.
  - M., Fundamental Results for Pointfree Convex Geometry, *Ann. Pure Appl. Logic*, 2010. The rel. in my CSL paper.

Both adj. give hyperdoctrines with quantifier $\forall$. Convex geometric logic only allows $\forall$, as geometric logic only admits $\exists$. Such phenomena are uniformly treated in the QPL'13 paper.

Conclusions

- CQM is logic (type theory) of quantum systems. TQL is logic of quantum propositions.
  - They are different logics of QM, and, in a sense, would exhibit a counterexample to the Curry-Howard iso.
- We can combine CQM and TQL via universal-algebraised Lawvere hyperdoctrines. Completeness thm. holds wrt. Faggian-Sambin's sequent calculus over linear type th.
  - This is not just for quantum logic, but indeed works for a wide variety of substructural logics and their extensions: classical, intuitionistic, linear, fuzzy, (convex) geometric ...
  - M., "Full Lambek Hyperdoctrine", Springer LNCS 8071. Includes categ. accounts of Gödel and Girard translations.
- Duality yields hyperdoctrine models of logic / set theory, clarifying links with forcing-like models. This is not just for intuitionistic, but for quantum, substrct., ... Hence universal.

Conclusions of Conlusions

My provisional answer to the three big questions:

- A universal concept of space must represent the two aspects of the concept of space, and encompass both topos-like and **Hilb**-like structures.
- A universal concept of logic must represent both logic (of propositions) and type theory (of systems) while of course encompassing a wide variety of logical systems in practice.
- A universal concept of logic *and* space is universally algebraised hyperdoctrines, which, in particular, can represent both logic of quantum systems and logic of quantum propositions.

To think of what space is to think of what logic is. Further topics: completeness lifting, translation as LT topology, etc.