Calculation of the H_∞ optimized design of a single-mass dynamic vibration absorber attached to a damped primary system

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Abstract

There are three criteria typically used in the design of dynamic vibration absorbers (DVAs): H_∞ optimization, H_2 optimization, and stability maximization. Recently, interest has shifted to the optimization of multi-mass DVAs, but in fact, in even the most basic single-mass DVA, the effect of primary system damping on the optimal solution is still not fully understood with respect to the H_∞ criterion. The author has recently reported an exact H_∞-optimal solution for a series-type double-mass DVA attached to a damped primary system. This article presents the application of this H_∞ optimization method developed for a double-mass DVA to the optimization of a single-mass DVA. In the H_∞ optimization of the mobility transfer function, a highly accurate numerical solution was successfully obtained by solving a single sixth-order algebraic equation. In the case of the optimization of the compliance and acceleration transfer functions, it is shown that a highly accurate numerical solution can be obtained by solving ternary systems of simultaneous algebraic equations. It should be noted that the equations presented in this paper can be factorized into simpler equations when there is no damping in the primary system. It is also demonstrated herein that the factorized expressions yield the previously published H_∞-optimal solutions.

Keywords : Dynamic vibration absorber, Damped primary system, H_∞ optimization, Formula manipulation, Exact algebraic equation, Compliance transfer function, Mobility and acceleration functions

1. Introduction

The dynamic vibration absorber (DVA) was devised by Watts in 1883 as a vibration control device (Watts, 1883) and was first patented by Frahm in 1909 and 1911 (Frahm, 1911). The design criteria for DVA optimization can be obtained with one of three approaches: H_∞ optimization, H_2 optimization, or stability maximization (Asami et al., 2002). Of these, H_∞ optimization was the first to be proposed and remains the most widely adopted approach in the design of DVAs. This approach involves minimizing the height of the resonance points of the primary vibratory system. Ormondroyd and Den Hartog (1928) proposed an approximate method for H_∞ optimization called the fixed-point method. Using this method, a solution that optimizes the tuning ratio for a single-mass DVA in the case of an undamped primary system was derived by Hahnkamm (1932), and the damping ratio of the DVA was later optimized by Brock (1946); details of this solution can be found in a book by Den Hartog (1956). The approximate optimal solution they derived was a solution to the minimization of the compliance transfer function (absolute displacement response) for a primary system subjected to force excitation. Approximate optimal solutions for the mobility and acceleration transfer functions (absolute velocity and acceleration responses, respectively) have also been found, as outlined in a specialized book (Korenev, 1993).

A method for deriving an exact solution for the H_∞ optimization of the DVA was proposed by Nishihara and colleagues around the turn of the century (Nishihara and Matsuhisa, 1997a; Nishihara and Asami, 2002). At this time, the optimal solution for the compliance transfer function was also presented, and it was confirmed that the accuracy of the approximate solution obtained by the fixed-point method was very high. After that, exact solutions for the mobility and acceleration transfer functions were also derived using Nishihara's method (Asami and Nishihara, 2003). The exact solutions are all special-case solutions where there is no damping in the primary vibratory system.

Before this significant advancement, many H_∞-optimal solutions considering the influence of damping in the primary system were proposed. However, all of them were numerical solutions (Ikeda and Ioi, 1978; Randall et al., 1981; Thompson, 1981; Soom and Lee, 1983; Sekiguchi and Asami, 1984) or perturbation-approximate solutions (Asami et al., 2002), and to date, there have been no algebraic solutions reported in the literature. In contrast, for the H_2 optimization criterion (Crandall and Mark, 1963) and the stability maximization criterion (Nishihara and Matsuhisa, 1997b), exact algebraic solutions can be obtained even if there is damping in the primary system (Asami et al., 2002; Nishihara and Matsuhisa, 1997b).

Recently, research interest has shifted to the optimization of multi-mass DVAs for the purpose of improving performance and robustness (Iwanami and Seto, 1984; Kamiya et al., 1996; Yasuda and Pan, 2003; Pan and Yasuda, 2005). The discovery of the exact H_∞-optimal solution (Asami, 2019) for a series-type double-mass DVA attached to a damped primary system has led to the hope of discovering the H_∞-optimal solution for a single-mass DVA. The solution found at that time was for the mobility transfer function, which was obtained by solving a quartic algebraic equation derived using a Jacobian matrix (Gradshteyn and Ryzhik, 2000) to search for the multiple root condition of a set of simultaneous equations. In the present study, when this method was applied to the H_∞ optimization of a single-mass DVA, a single sixth-order algebraic equation was derived for the mobility transfer function. Although this equation cannot be solved algebraically, a numerical solution can be easily obtained (for example, with the command `NSolve` in *Mathematica*). This method of obtaining the optimal solution greatly outperforms conventional numerical solutions that do not use mathematical formulas, in terms of both speed and accuracy. This article presents this sixth-order algebraic equation, which is shown to be a relatively simple equation.

Furthermore, for the compliance and accelerance transfer functions, ternary systems of simultaneous higher-order algebraic equations were derived. As with the sixth-order equation, these cannot be solved algebraically but can be solved numerically with the abovementioned command `NSolve`. This command outputs many sets of numerical solutions, but there is only one set with positive real roots. The simultaneous equations are also simple equations and are introduced here. Finally, the H_∞-optimal solutions for these three types of transfer functions and the steady-state response of the optimized primary system are discussed.

2. Analytical model and definition of dimensionless parameters

Figure 1(b) shows the analytical model of the two-degree-of-freedom (2-DOF) viscously damped system considered in this study. The m_1–k_1 system is the primary system P, and the m_2–k_2 system is the DVA. A harmonic excitation force acts on the primary system, and the DVA is designed to minimize the steady-state vibration of the primary system caused by this force. Figure 1(a) shows a single-degree-of-freedom (SDOF) system without a DVA. In this study, the calculation was conducted using the following dimensionless parameters:

$$\mu = m_2/m_1, \quad \nu = \omega_2/\omega_1, \quad \lambda = \omega/\omega_1, \quad \zeta_1 = c_1/(2m_1\omega_1), \quad \zeta_2 = c_2/(2m_2\omega_2), \tag{1}$$

where ω_1 and ω_2 are undamped natural angular frequencies defined as

$$\omega_1 = \sqrt{k_1/m_1}, \quad \omega_2 = \sqrt{k_2/m_2}. \tag{2}$$

Additionally, m_1, k_1, and c_1 are the mass, spring constant, and damping coefficient of the primary system, respectively; m_2, k_2, and c_2 are the same respective parameters of the DVA; and ω is the angular excitation frequency. These dimensionless parameters are the mass ratio μ, the tuning ratio ν, the excitation frequency ratio λ, the primary system damping ratio ζ_1, and the DVA damping ratio ζ_2. The excitation frequency ratio λ was varied from zero to infinity. For a given value of μ and ζ_1, the optimization problem involves finding the optimal values of ν and ζ_2 that give the desired spring constant k_2 and damping coefficient c_2 for the DVA.
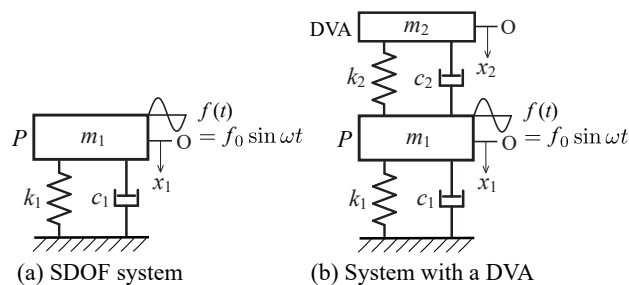

(a) SDOF system (b) System with a DVA

Fig. 1   Analytical model of viscously damped systems

3. Three evaluation criteria in H_∞ optimization and exact solutions for an undamped primary system

In this study, the H_∞ optimization of a DVA was performed with the following three transfer functions: compliance, mobility, and acceleration transfer functions. The transfer functions are presented here, along with the exact solution for the case of an undamped primary system.

3.1. Compliance transfer function

The compliance transfer function represents the ratio of the absolute displacement response $x_1(t)$ to the external force $f(t)$ applied to the system. The H_∞ norm, which represents the maximum value of the function, is expressed in the following dimensionless form:

$$h_{\text{max}} = \left| \frac{x_1}{f/k_1} \right|_{\text{max}}.$$

(3)

Minimizing h_{max} is the goal with H_∞ optimization for the compliance transfer function, and the minimum value of h_{max} is denoted as h_{min}.

3.2. Mobility transfer function

The mobility transfer function represents the ratio of the absolute velocity response $\dot{x}_1(t)$ to the external force $f(t)$ applied to the system. The H_∞ norm in this case is expressed in the following dimensionless form:

$$h_{\text{max2}} = \left| \frac{\dot{x}_1}{\omega_1 f/k_1} \right|_{\text{max}} = \left| \frac{\lambda x_1}{f/k_1} \right|_{\text{max}}.$$

(4)

Minimizing h_{max2} is the goal of H_∞ optimization with the mobility transfer function, and the minimum value of h_{max2} is again denoted as h_{min}.

3.3. Accelerance transfer function

The accelerance transfer function represents the ratio of the absolute acceleration response $\ddot{x}_1(t)$ to the external force $f(t)$ applied to the system. The H_∞ norm in this case is expressed in the following dimensionless form:

$$h_{\text{max3}} = \left| \frac{\ddot{x}_1}{\omega_1^2 f/k_1} \right|_{\text{max}} = \left| \frac{\lambda^2 x_1}{f/k_1} \right|_{\text{max}}.$$

(5)

Minimizing h_{max3} is the goal of H_∞ optimization with the accelerance transfer function, and the minimum value of h_{max3} is also denoted as h_{min}.

3.4. Exact H_∞ solutions for an undamped primary system

For the case where there is no damping in the primary system, the exact H_∞-optimal solutions of the DVA have already been obtained for all of the above transfer functions (Asami and Nishihara, 2003). Because these solutions represent the starting point for our research, they are summarized here.

First, the H_∞-optimal solution for the compliance transfer function is

$$\left. \begin{aligned} \nu_{\text{opt}} &= \frac{2}{1+\mu} \sqrt{\frac{2}{3} \frac{16 + 23\mu + 9\mu^2 + 2(2+\mu)\sqrt{4+3\mu}}{64 + 80\mu + 27\mu^2}}, \quad \zeta_{\text{2opt}} = \frac{1}{4} \sqrt{\frac{8 + 9\mu - 4\sqrt{4+3\mu}}{1+\mu}} \\ h_{\text{min}} &= \frac{1}{3\mu} \sqrt{\frac{(8+9\mu)^2(16+9\mu) - 128(4+3\mu)^{3/2}}{3(32+27\mu)}}. \end{aligned} \right\}$$

(6)

Next, for the mobility transfer function, the solution is

$$\left. \begin{aligned} \nu_{\text{opt}} &= \frac{1}{1+\mu} \sqrt{-(1+\mu) + \sqrt{2(1+\mu)(2+\mu)}}, \quad \zeta_{\text{2opt}} = \sqrt{\frac{1}{2} - \frac{1}{\sqrt{2(1+\mu)(2+\mu)}}} \\ h_{\text{min}} &= \sqrt{\frac{1}{\mu} \left[\frac{2+\mu}{2(1+\mu)} + \sqrt{\frac{2+\mu}{2(1+\mu)}} \right]}. \end{aligned} \right\}$$

(7)

Finally, for the accelerance transfer function,

$$\left. \begin{aligned} \nu_{\text{opt}} &= \frac{2}{8+5\mu} \sqrt{\frac{2}{3}(16 + 7\mu + c_0)}, \quad \zeta_{\text{2opt}} = \frac{1}{8+5\mu} \sqrt{\frac{64 + 88\mu + 33\mu^2}{2} - (4+3\mu)c_0} \\ h_{\text{min}} &= \frac{8}{3\mu} \sqrt{\frac{16 + 25\mu - 2c_0}{3(32+27\mu)}}; \quad \text{where } c_0 = \sqrt{64 - 16\mu - 26\mu^2}. \end{aligned} \right\}$$
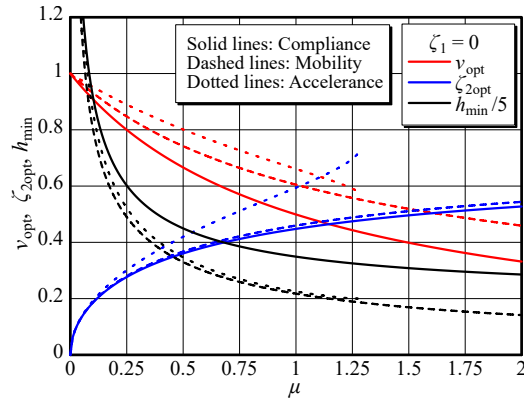
(8)

Fig. 2　H_∞-optimal solutions for a DVA attached to an undamped primary system ($\zeta_1 = 0$). The results demonstrate that there is no great difference between the three optimal solutions. For the compliance and acceleration transfer functions, h_{min} is never less than 1. For the acceleration transfer function, it becomes exactly 1 at $\mu = 1.291$, and no optimal solution exist for any greater value of μ.

These optimal solutions are plotted in Fig. 2. Note that because h_{min} is larger than the other two plotted parameters (v and ζ_2), it is scaled down by a factor of 5 to bring all three into the same range. In the case of the compliance and acceleration transfer functions, h_{min} is never less than 1. For the acceleration transfer function, it becomes exactly 1 at $\mu = 1.291$, and no optimal solution exist for any greater value of μ, whereas for the compliance transfer function, it asymptotically approaches 1 much more slowly. At $\mu = 1.291$, the parameter c_0 in Eq. (8) becomes an imaginary number. The frequency responses of the primary system for each optimized transfer function are shown in Figs. 4, 6, and 8 below.

3.5. Difference from the approximate solution of a dynamic vibration absorber

As mentioned in the introduction, for the H_∞ optimization criterion, an approximate solution based on the fixed-point method has conventionally been used for a long time. The solution is expressed as follows (Den Hartog, 1956):

$$v_{opt} = \frac{1}{1 + \mu}, \quad \zeta_{2opt} = \sqrt{\frac{3\mu}{8(1 + \mu)}}. \tag{9}$$

These solutions almost agree with the solutions for the compliance transfer function, plotted as solid lines in Fig. 2. For example, the exact solution shown in Eq. (6) calculated for a mass ratio of $\mu = 0.1$ yields $v_{opt} = 0.909058$ and $\zeta_{2opt} = 0.185470$, whereas in the approximate solution shown in Eq. (9), the calculated results are $v_{opt} = 0.909091$ and $\zeta_{2opt} = 0.184637$ (Asami, 2017). Thus, the approximate solution returns a value in the immediate vicinity of the exact solution; however this fixed-point method cannot be applied when there is damping in the primary system.

Traditionally, Eq. (9) has been used for the optimal design of the DVA for any purpose, but in the future it would be preferable to select a solution from Eqs. (6)–(8) depending on the purpose of the DVA installation. For example, the optimization of the mobility transfer function (Eq. (7)) should be used to reduce noise generated by mechanical vibrations (Asami et al., 2018), whereas the optimization of the acceleration transfer function (Eq. (8)) would be better applied to improve the ride quality of railroad vehicles and automobiles (Ohno, 1997; Takei and Ishiguro, 1995).

4. H_∞ optimization with the compliance transfer function

This section presents the H_∞ optimization of the compliance transfer function for a DVA attached to a damped primary system obtained by applying the Jacobian matrix (Gradshteyn and Ryzhik, 2000). For more details, please refer to Nishihara (2017) and Asami et al., (2018). The expressions in Eqs. (6)–(8) represent the H_∞-optimal solutions derived by Nishihara's method (Nishihara and Asami, 2002); however, at that time, the Sylvester matrix (Akritas, 1993) was used instead of the Jacobian matrix to search for the double root of simultaneous equations. As described below, when Nishihara's method is applied to the H_∞ optimization of a single-mass DVA, a binary system of simultaneous algebraic equations is first derived. Using the Sylvester matrix to find the double root of this system of equations imposes the constraint that the system of equations must be combined into a single equation. If there is no damping in the primary system, the Sylvester matrix can be applied because this integrated equation is at most a quartic equation; however, if damping is present in the primary system, the equation takes on a much higher order of at least eight. As a result, the Sylvester matrix is virtually inapplicable when searching for the double root. In previous work by the author and colleagues (Asami et al., 2018; Asami, 2019; Asami and Yamada, 2020), the H_∞ optimization of a double-mass DVA was

performed. The first equation to be obtained was a ternary system of simultaneous equations, and this system could not be integrated into a single equation even in the case of no damping in the primary system. Therefore, the H_∞-optimal solution for the series-type double-mass DVA was obtained using the Jacobian matrix instead of the Sylvester matrix to find the triple root of the equation.

The compliance transfer function for the primary system for the vibratory system shown in Fig. 1(b) is expressed as

$$\left. \begin{aligned} &\left| \frac{x_1}{f/k_1} \right| = \sqrt{\frac{\text{Num1}}{\text{Den1}}} \\ &\text{Num1} = (\lambda^2 - v^2)^2 + (2\zeta_2 \lambda v)^2 \\ &\text{Den1} = \{\lambda^4 - \lambda^2[1 + 4\zeta_1\zeta_2 v + (1 + \mu)v^2] + v^2\}^2 + 4\lambda^2\{\zeta_1(\lambda^2 - v^2) - \zeta_2[1 - \lambda^2(1 + \mu)]v\}^2. \end{aligned} \right\} \quad (10)$$

When the parameters of the vibratory system are varied, the heights of the two resonance points appearing in the system can be adjusted to be equal. The height of the resonance point is set to $h_{\max}$ and the following function f_n is defined:

$$\left. \begin{aligned} &f_n = \text{Den1} - \frac{\text{Num1}}{h_{\max}^2} = \lambda^8 + b_1\lambda^6 + b_2\lambda^4 + b_3\lambda^2 + b_4 \\ &b_1 = -2 + 4\zeta_1^2 + 8\zeta_1\zeta_2\mu v - 2(1 + \mu)[1 - 2\zeta_2^2(1 + \mu)]v^2 \\ &b_2 = r^2 + v^2[2(2 + \mu) + (1 + \mu)^2 v^2 - 8\zeta_1^2(1 - 2\zeta_2^2) - 8\zeta_2^2(1 + \mu)] \\ &b_3 = -2v^2[r^2(1 - 2\zeta_2^2) + (1 - 2\zeta_1^2 + \mu)v^2], \quad b_4 = r^2v^4, \end{aligned} \right\} \quad (11)$$

where

$$r = \sqrt{1 - \frac{1}{h_{\max}^2}} \quad (12)$$

is a parameter introduced to eliminate fractional expressions in Eq. (11). With this parameter transformation, the problem of minimizing $h_{\max}$ is replaced by the problem of minimizing r, and the minimum value of r is hereafter written as $r_{\min}$. As shown in Eq. (11), the function f_n is a quartic expression with respect to λ^2, and $f_n = 0$ yields the double root at the two resonance points. From that, the following two identities can be derived (Nishihara and Asami, 2002):

$$\left. \begin{aligned} &b_1 \sqrt{b_4} - b_3 = 0 \\ &(b_1/2)^2 - 2\sqrt{b_4} - b_2 = 0. \end{aligned} \right\} \quad (13)$$

Substituting Eq. (11) for the parameters b_1–b_4 in Eq. (13) and rearranging with respect to r yields the following simultaneous equations:

$$\left. \begin{aligned} &f_1 = a_0 r^2 + a_1 r + a_2 = 0, \quad f_2 = c_0 r^2 + c_1 r + c_2 = 0 \\ &a_0 = 1 - 2\zeta_2^2, \quad a_1 = -1 + 2\zeta_1^2 + 4\zeta_1\zeta_2\mu v - (1 + \mu)[1 - 2\zeta_2^2(1 + \mu)]v^2, \quad a_2 = (1 + \mu - 2\zeta_1^2)v^2 \\ &c_0 = 1, \quad c_1 = -2v^2, \quad c_2 = 4\zeta_2^2(1 + \mu)^3[1 - \zeta_2^2(1 + \mu)]v^4 + 8\zeta_1\zeta_2\mu(1 + \mu)[1 - 2\zeta_2^2(1 + \mu)]v^3 \\ &\quad + \{2 - 4\zeta_1^2[1 - \mu - 2\zeta_2^2(1 + \mu)(1 - 3\mu)] - 4\zeta_2(1 - \mu^2)\}v^2 + 8\zeta_1(1 - 2\zeta_1^2)\zeta_2\mu v - (1 - 2\zeta_1^2)^2. \end{aligned} \right\} \quad (14)$$

Equation (14) describes the conditions to make the heights of the two resonance points in a 2-DOF system equal.

Next, a condition to minimize the height of the resonance peaks (i.e., minimize $h_{\max}$ or r) is added to the conditions given in Eq. (14). This condition is satisfied by the total differential of r with respect to the two parameters v and ζ_2 to be optimized equaling zero, as given by

$$dr = \frac{\partial r}{\partial v}dv + \frac{\partial r}{\partial \zeta_2}d\zeta_2 = 0. \quad (15)$$

Because the parameter r is contained in the two functions f_1 and f_2, its total derivative dr can be rewritten as

$$\begin{bmatrix} dr \\ dr \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r}{\partial f_1} & 0 \\ 0 & \dfrac{\partial r}{\partial f_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial \zeta_2} \\ \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial \zeta_2} \end{bmatrix} \begin{bmatrix} dv \\ d\zeta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (16)$$

The second matrix in the center of Eq. (16) is called the Jacobian matrix. In Eq. (16), for v and ζ_2 to have non-trivial solutions, this Jacobian matrix must have rank deficiency (Nishihara, 2017). That is,

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial \zeta_2} \\ \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial \zeta_2} \end{vmatrix} = 0. \quad (17)$$

Equation (17) can be used to find the multiple root of the set of simultaneous equations given in Eq. (14). From this equation, the third expression required for the H_∞-optimal solution search can be obtained as

$$
\left.\begin{aligned}
f_3 &= d_0 r^3 + d_1 r^2 + d_2 r + d_3 = 0 \\
d_0 &= \zeta_2 \nu \\
d_1 &= -\zeta_2(1+\mu)^2[1 + 4\zeta_2^2(1+\mu) - 4\zeta_2^4(1+\mu)^2]\nu^3 - \zeta_1\mu[1 + 6\zeta_2^2(1+\mu) - 12\zeta_2^4(1+\mu)^2]\nu^2 \\
&\quad - \zeta_2\{1 - 2\zeta_2^2(1-\mu^2) - 2\zeta_1^2[1-\mu-2\zeta_2^2(1+\mu)(1-3\mu)]\}\nu - 2\zeta_1(1-2\zeta_1^2)\zeta_2^2\mu \\
d_2 &= [\zeta_2(1+\mu)^4\nu^3 + \zeta_1\mu(1+\mu)^2\nu^2 + \zeta_2(1+\mu-4\zeta_1^2)\mu(1+\mu)\nu + \zeta_1\mu(2+\mu-4\zeta_1^2)]\nu^2 \\
d_3 &= (1+\mu-2\zeta_1^2)\{-\zeta_2(1+\mu)^3[1-2\zeta_2^2(1+\mu)]\nu^3 - \zeta_1\mu(1+\mu)[1-6\zeta_2^2(1+\mu)]\nu^2 \\
&\quad + \zeta_2(1+\mu)[1-\mu-2\zeta_1^2(1-3\mu)]\nu - \zeta_1(1-2\zeta_1^2)\mu\}\nu^2 .
\end{aligned}\right\} \tag{18}
$$

Equations (14) and (18) build a ternary system of simultaneous equations containing three unknowns: ν, ζ_2, and r. This system of equations can be solved using the command NSolve in the formula manipulation software *Mathematica* with the desired values of the remaining parameters μ and ζ_1 input into the equations. In this way, a large number of solution sets are obtained (14 sets of real roots and 56 sets of complex roots for a given set of μ and ζ_1 values), among which only one is the positive real root set.

In the special case where the primary system damping ζ_1 is zero, the expressions in Eqs. (14) and (18) can be further factorized, and the algebraic solution shown in Eq. (6) can be derived from the simplified equation (see the Appendix for this derivation). As a practical matter, it is recommended that the Newton–Raphson method (the FindRoot command in *Mathematica*) is used to solve Eqs. (14) and (18) with initial values computed from Eq. (6), rather than solving them directly with the NSolve command.

Figure 3 shows the optimal tuning and damping ratios ν_{opt} and $\zeta_{2\mathrm{opt}}$ for the DVA and the minimized resonance amplitude $h_{\min}$ for the primary system, obtained as the solution to the simultaneous equations given in Eqs. (14) and (18), plotted against the primary system damping ratio ζ_1 for various mass ratios μ. In this figure, all curves are plotted in the range of $\zeta_1 = 0$ to $1/\sqrt{2} = 0.7071$. As is well known, the compliance transfer function for the viscously damped SDOF system shown in Fig. 1(a) starts from 1 at zero frequency and decays monotonically with increasing frequency beyond a damping ratio ζ_1 of 0.7071. Therefore, optimal values of the DVA parameters exist only up to $\zeta_1 = 0.7071$. As shown in Fig. 3, the optimal tuning ratio decreases monotonically to zero as ζ_1 approaches 0.7071, whereas the optimal damping ratio increases monotonically. The resonance amplitude $h_{\min}$ decreases with both the mass of the DVA and the primary system damping and is equal to 1 at $\zeta_1 = 0.7071$ regardless of the mass ratio. The gray curve in Fig. 3(c) representing the case of $\mu = 0$ shows the resonance amplitude ratio for a SDOF system without a DVA. The large difference between this curve and the $\mu = 0.02$ curve indicates that even a small DVA is able to significantly reduce the resonance amplitude.

Figure 4 shows the compliance transfer function for a primary system with an optimally tuned and damped DVA for representative mass ratios ranging from $\mu = 0.05$ to 0.2. As shown in this figure, a greater primary system damping yields a curve with a lower resonance height that is closer to the response curve of the SDOF system ($\zeta_1 = 0.7071$) shown in gray. Furthermore, this figure provides some insight into why the H_∞-optimal solution for the DVA that minimizes the compliance transfer function exists only up to $\zeta_1 = 0.7071$.

5. H_∞ optimization with the mobility transfer function

A ternary system of simultaneous algebraic equations was obtained for the mobility transfer function in the same way as before. Here, some of the expressions making up the simultaneous equations can be solved algebraically, eventually leading to the problem of solving the following single sixth-order algebraic equation with respect to ν^2:

$$
\left.\begin{aligned}
f_n &= a_0\nu^{12} + a_1\nu^{10} + a_2\nu^8 + a_3\nu^6 + a_4\nu^4 + a_5\nu^2 + a_6 = 0 \\
\zeta_{2\mathrm{opt}} &= \frac{-\zeta_1\mu\nu_{\mathrm{opt}}^3 + \sqrt{\zeta_1^2\mu^2\nu_{\mathrm{opt}}^6 + (1-\nu_{\mathrm{opt}}^2)[1-(1+\mu)\nu_{\mathrm{opt}}^2][1-2\nu_{\mathrm{opt}}^2+(1+\mu)^2\nu_{\mathrm{opt}}^4]}}{1-2\nu_{\mathrm{opt}}^2+(1+\mu)^2\nu_{\mathrm{opt}}^4} \\
h_{\min} &= \frac{1}{2}\sqrt{\frac{1-\nu_{\mathrm{opt}}^2}{[\zeta_1(1+\mu)+\zeta_{2\mathrm{opt}}(1+\nu_{\mathrm{opt}}+\mu\nu_{\mathrm{opt}})][\zeta_1(1-\mu)-\zeta_{2\mathrm{opt}}(1-\nu_{\mathrm{opt}}-\mu\nu_{\mathrm{opt}})]}} \\
a_0 &= (1+\mu)^6, \quad a_1 = (1+\mu)^4[(1+\mu)(3-\mu)-\zeta_1^2(9-\mu)] \\
a_2 &= -(1+\mu)[2(1+\mu)^2(3+3\mu+2\mu^2)-\zeta_1^2(1+\mu)(21+18\mu+13\mu^2)+8\zeta_1^4\mu^2] \\
a_3 &= -2[(1+\mu)^2(5+2\mu+\mu^2)+\zeta_1^2(1+\mu)(5+4\mu-3\mu^2)+4\zeta_1^4\mu^2] \\
a_4 &= (3+\mu)(7+9\mu+4\mu^2)-2\zeta_1^2(3+2\mu+\mu^2), \quad a_5 = -(3+\mu)^2+3\zeta_1^2(1-\mu), \quad a_6 = \zeta_1^2 .
\end{aligned}\right\} \tag{19}
$$

Because the first expression in Eq. (19) is a sixth-order function of v^2, solving this equation yields six roots. The equation has several positive real roots, one of which is the optimal tuning ratio v_{opt}. Substituting this solution into the second expression yields the optimal damping ratio ζ_{2opt}. At this point, if the optimal tuning ratio v_{opt} is selected incorrectly, the optimal damping ratio ζ_{2opt} becomes a negative real number or a complex number. Finally, with these optimal solutions substituted into the third expression, the value of minimized resonance amplitude h_{min} can be calculated.

If the primary damping ζ_1 is zero, the first expression of Eq. (19) can be factorized as follows:

$$f_n = v^2(v^2 - 1)[(1 + \mu)^3 v^4 + 2(1 + \mu)^2 v^2 - (3 + \mu)]^2 = 0. \tag{20}$$

The quadratic equation for v^2 in the last factor of this equation contains the optimal solution v_{opt}. The solution agrees with the first expression in Eq. (7).
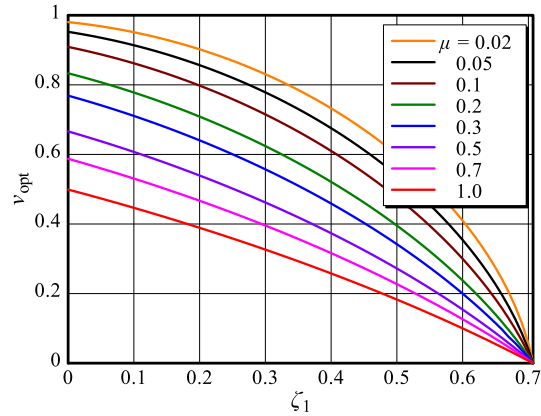
(a) Optimal tuning ratio v_{opt}
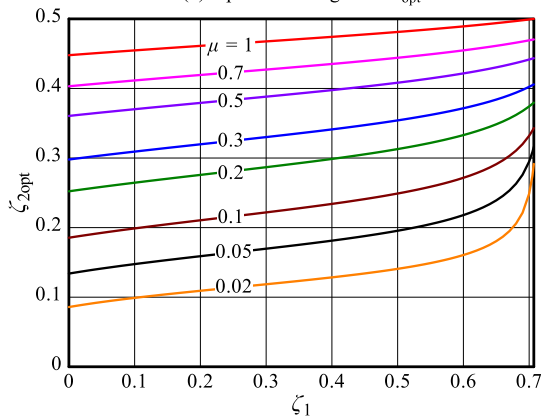
(b) Optimal damping ratio ζ_{2opt}
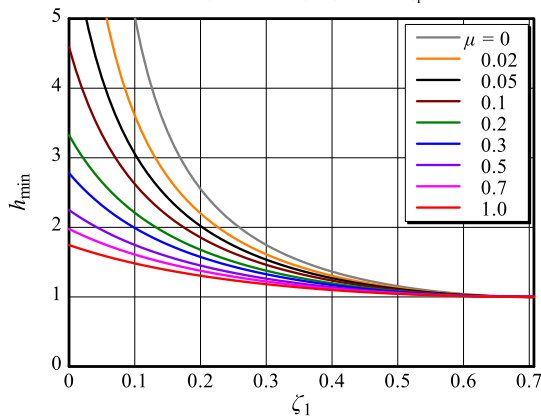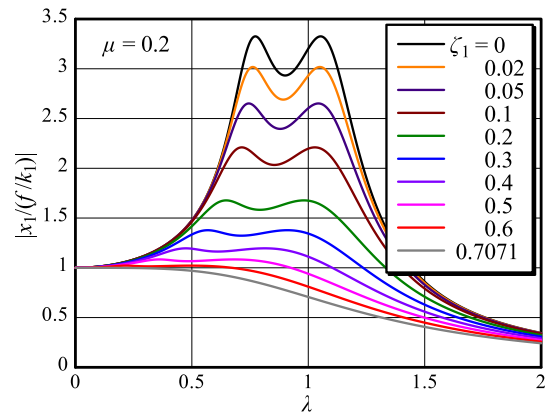
(c) Minimized resonance amplitude h_{min}

Fig. 3　Optimization of the compliance transfer function for a damped primary system by the H_∞ criterion. All optimal values for the DVA parameters exist only up to $\zeta_1 = 1/\sqrt{2} = 0.7071$. At the value of $\zeta_1 = 0.7071$, h_{min} takes a minimum value of 1.
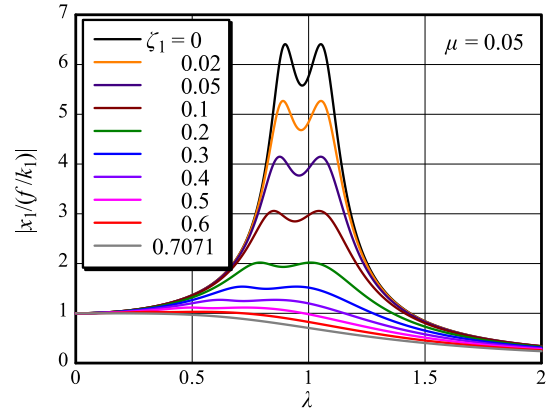
(a) Mass ratio $\mu = 0.05$
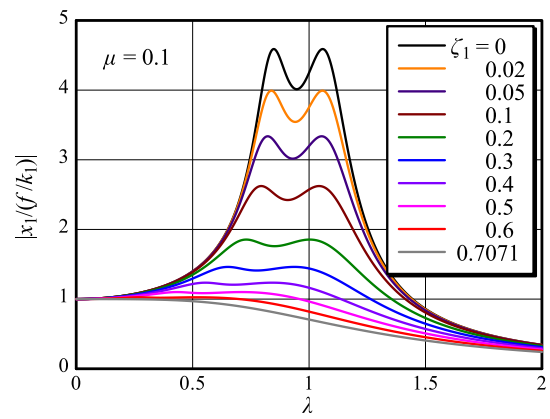
(b) Mass ratio $\mu = 0.1$

(c) Mass ratio $\mu = 0.2$

Fig. 4　Optimized compliance transfer functions for a damped primary system based on the H_∞ criterion. A greater primary system damping yields a curve with a lower resonance height that is closer to the response curve of the SDOF system ($\zeta_1 = 0.7071$) shown in gray.

The optimal solutions and minimized resonance amplitude calculated by Eq. (19) are shown in Fig. 5. A comparison of this figure with the optimal solutions for the compliance transfer function shown in Fig. 3 reveals that the optimal values vary slowly and almost linearly with the primary damping ζ_1. The optimal solutions shown in Fig. 5(a) and (b) exist for $\zeta_1 \to \infty$. The mobility transfer function starts from zero at $\lambda = 0$ and converges to zero as λ approaches ∞. Therefore, the value of $h_{\min}$ shown in Fig. 5(c) may be less than 1. It should be noted that the parameter r defined in Eq. (12) is imaginary when the resonance amplitude $h_{\max}$ is less than 1, demonstrating that the parameter r can produce valid solutions even if it takes on an imaginary value. Instead of Eq. (12), the variable transformation $r_4 = \sqrt{4 - 1/h_{\max}^2}$ or $r_{10} = \sqrt{10 - 1/h_{\max}^2}$ can be used to derive the same expression as the third expression in Eq. (19).

Figure 6 shows examples of optimal mobility transfer functions optimized at some representative mass ratios μ. In the force excitation system shown in Fig. 1, the vibratory system is mounted on a fixed foundation, and the vibration decreases to zero as the primary damping ζ_1 is increased.
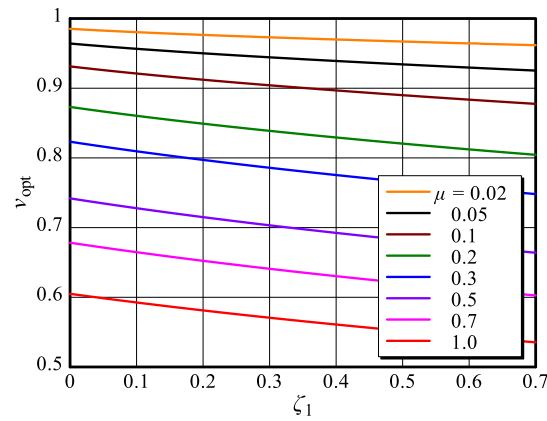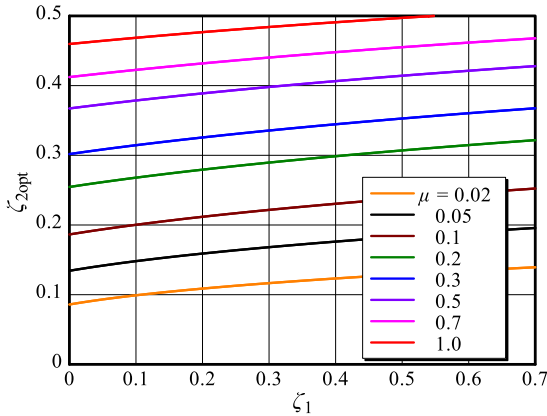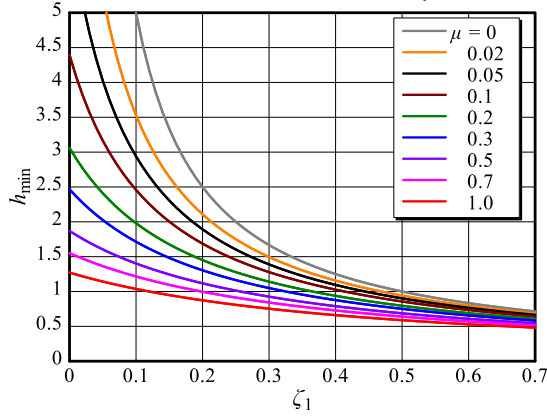

(a) Optimal tuning ratio v_{opt}

(b) Optimal damping ratio $\zeta_{2\mathrm{opt}}$

(c) Minimized resonance amplitude $h_{\min}$

Fig. 5 Optimization of the mobility transfer function for a damped primary system by the H_∞ criterion. The mobility transfer function starts from zero at $\lambda = 0$ and converges to zero as λ approaches ∞. Therefore, the value of $h_{\min}$ may be less than 1.
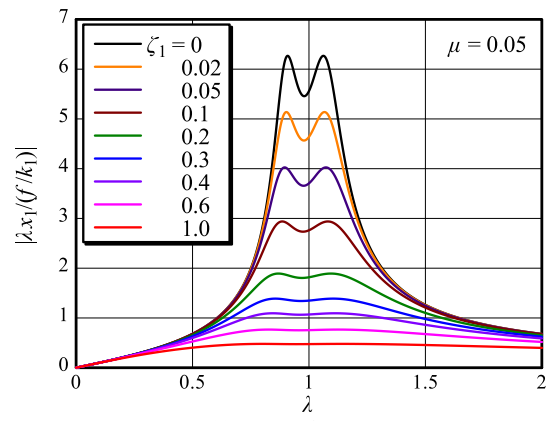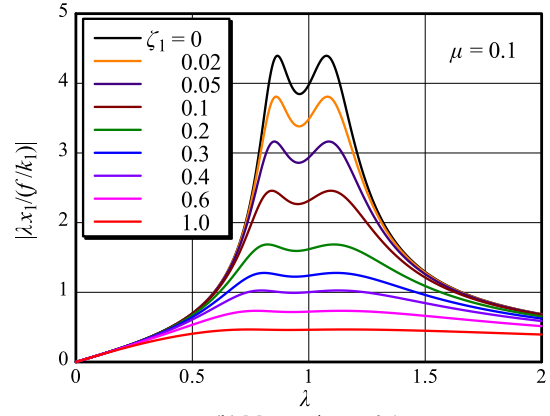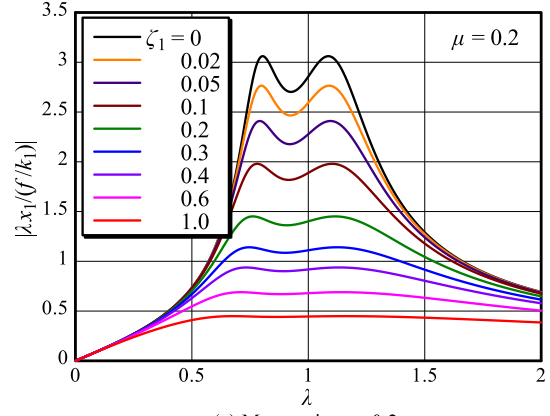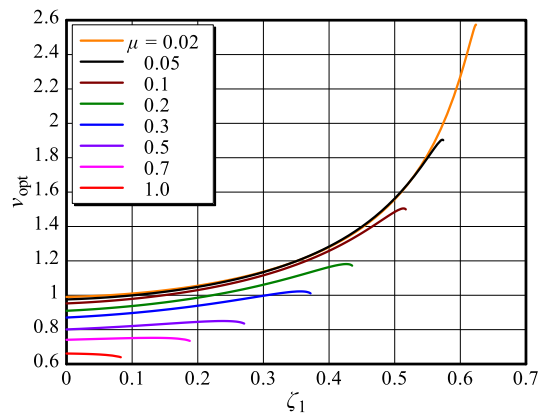
(a) Mass ratio $\mu = 0.05$

(b) Mass ratio $\mu = 0.1$

(c) Mass ratio $\mu = 0.2$

Fig. 6 Optimized mobility transfer functions for a damped primary system based on the H_∞ criterion. In the force excitation system shown in Fig. 1, the vibratory system is mounted on a fixed foundation, so the vibration decreases to zero as ζ_1 is increased.
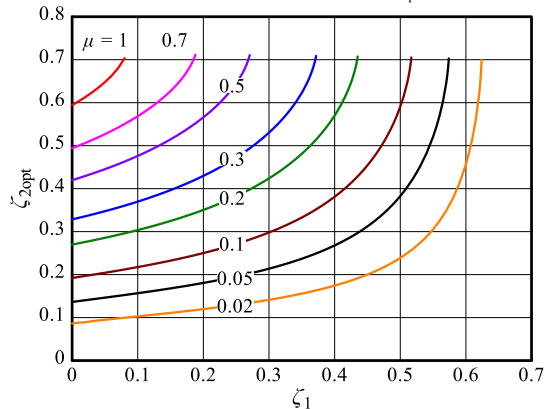
6. H_∞ optimization with the accelerance transfer function

A ternary system of simultaneous algebraic equations was also obtained for the accelerance transfer function. First, the condition that the heights of the two resonance points are equal yields
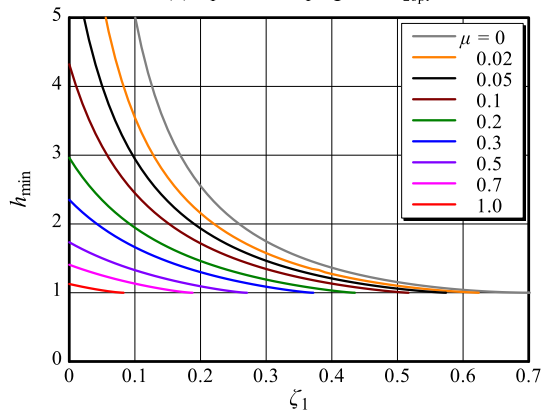
$$
\left.
\begin{aligned}
f_1 &= a_0 r^2 + a_1 r + a_2 = 0 \\
f_2 &= c_0 r^4 + c_1 r^3 + c_2 r^2 + c_3 r + c_4 = 0 \\
a_0 &= (1 - 2\zeta_2^2)^2, \quad a_1 = -1 + 2\zeta_2^2 - (1 + \mu - 2\zeta_1^2)v^2, \quad a_2 = 1 - 2\zeta_1^2 - 4\zeta_1\zeta_2\mu v + (1 - 4\zeta_2^2)\mu v^2 - 2\zeta_2^2\mu^2 v^2 \\
c_0 &= -4\zeta_2^2(1 - \zeta_2^2)v^4, \quad c_1 = 2v^2, \quad c_3 = 0, \quad c_4 = [1 - 2\zeta_1^2 - 4\zeta_1\zeta_2\mu v + (1 - 4\zeta_2^2)\mu v^2 - 2\zeta_2^2\mu^2 v^2]^2 \\
c_2 &= -1 - 2[1 + \mu - 2\zeta_1^2(1 - 2\zeta_2^2) - 2\zeta_2^2(1 + 2\mu)]v^2 - 8\zeta_1\zeta_2(1 - 2\zeta_2^2)\mu v^3 - \mu[\mu + 4\zeta_2^2(3 + \mu) - 8\zeta_2^4(2 + \mu)]v^4.
\end{aligned}
\right\} \quad (21)
$$

(a) Optimal tuning ratio v_{opt}

(b) Optimal damping ratio ζ_{2opt}
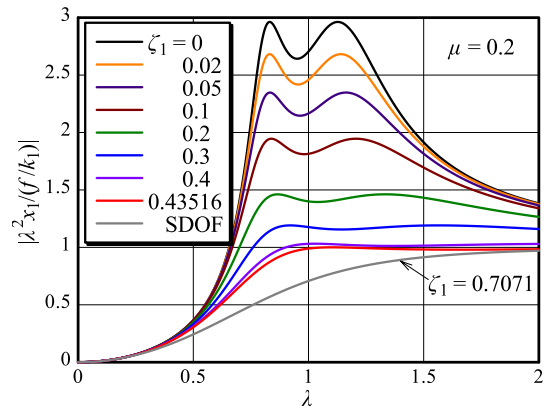
(c) Minimized resonance amplitude h_{min}

Fig. 7 Optimization of the accelerance transfer function for a damped primary system by the H_∞ criterion. The region in which the optimal solution exists narrows as μ increases. This is because the minimized amplitude h_{min} reaches its minimum value of 1 at a small value of ζ_1.
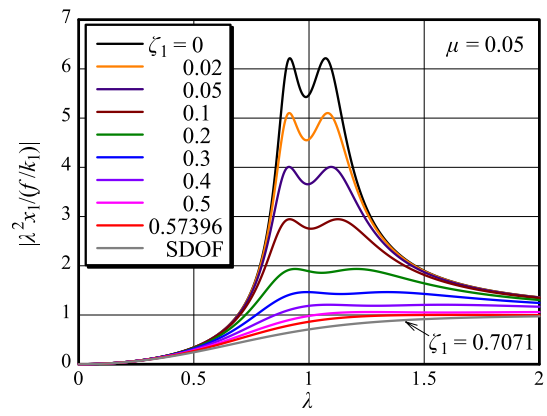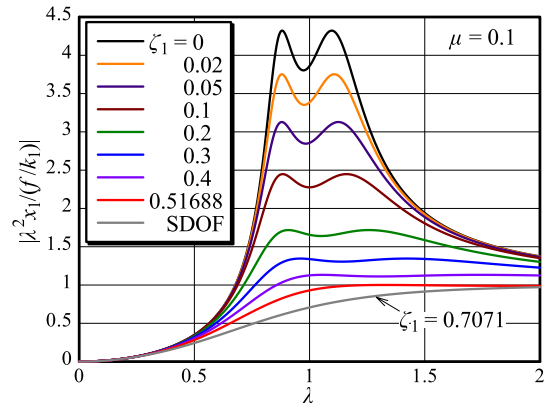
(a) Mass ratio $\mu = 0.05$

(b) Mass ratio $\mu = 0.1$

(c) Mass ratio $\mu = 0.2$

Fig. 8 Optimized accelerance transfer functions for a damped primary system. The red curve represents the shape of the transfer function in the limit where an optimal solution is present, and the gray curve shown by the arrow $\zeta_1 = 0.7071$ corresponds to the SDOF system.

Then, the condition for minimizing the heights of the two equally adjusted resonance points is fulfilled by

$$
\left.\begin{aligned}
&f_3 = d_0 r^5 + d_1 r^4 + d_2 r^3 + d_3 r^2 + d_4 r + d_5 = 0 \\
&d_0 = \zeta_2 \nu^5, \quad d_1 = -\zeta_2 \nu^3\{1 - 4\zeta_2^4 + (1 + \mu - 2\zeta_1^2)\nu^2 + 2\zeta_2^2[2 - (1 + \mu - 2\zeta_1^2)\nu^2]\} \\
&d_2 = \nu[\zeta_1 \mu \nu^3 + \zeta_2 - \zeta_2 \mu \nu^2 + 2\zeta_2 \mu(2 + \mu)\nu^4] \\
&d_3 = -\zeta_2[1 + \mu - 2\zeta_1^2(1 - 2\zeta_2^2) - 2\zeta_2^2(1 + 2\mu)]\nu - \zeta_1(1 + 6\zeta_2^2 - 12\zeta_2^4)\mu\nu^2 \\
&\quad - \zeta_2[1 + \mu - 2\zeta_1^2(2 + 3\mu) + 4\zeta_1^4 - 4\zeta_2^2\mu(3 + \mu) + 8\zeta_2^4\mu(2 + \mu)]\nu^3 \\
&\quad - \zeta_1(1 + \mu - 2\zeta_1^2)(1 - 6\zeta_2^2)\mu\nu^4 - \zeta_2(1 + \mu - 2\zeta_1^2)[3 + \mu - 4\zeta_2^2(2 + \mu)]\mu\nu^5 \\
&d_4 = \mu\nu^2\{\zeta_1[2 + \mu + \mu(2 + \mu)\nu^2] - 4\zeta_1^3 + \zeta_2\nu[2 + 2\mu + \mu^2 + \mu(2 + \mu)^2\nu^2] - 4\zeta_1^2\zeta_2(1 + \mu)\nu\} \\
&d_5 = -2\zeta_1(1 - 2\zeta_2^2)\zeta_2^2\mu + \mu\{\zeta_2 - \zeta_2^3(2 + \mu) - 2\zeta_1^2\zeta_2[1 - 2\zeta_2^2(2 + 3\mu)]\}\nu \\
&\quad - \zeta_1\mu[1 + \mu - 2\zeta_1^2(2 + \mu) + 4\zeta_1^4 + 6\zeta_2^2\mu - 12\zeta_2^4\mu(2 + \mu)]\nu^2 \\
&\quad - \zeta_2\mu[2 + 2\mu + \mu^2 - 2\zeta_1^2(4 + 6\mu + 3\mu^2) + 4\zeta_1^4(2 + 3\mu) + 4\zeta_2^2\mu(2 + \mu) - 4\zeta_2^4\mu(2 + \mu)^2]\nu^3 \\
&\quad - \zeta_1\mu^2(1 + \mu - 2\zeta_1^2)[1 - 6\zeta_2^2(2 + \mu)]\nu^4 - \zeta_2\mu^2(2 + \mu)(1 + \mu - 2\zeta_1^2)[1 - 2\zeta_2^2(2 + \mu)]\nu^5.
\end{aligned}\right\} \quad (22)
$$

The set of simultaneous equations given by Eqs. (21) and (22) can also be factorized and solved algebraically for a primary damping ζ_1 of zero. The solution is confirmed to be consistent with Eq. (8). If the primary damping is not zero, it can be solved with the command `NSolve` in *Mathematica*; the results are shown in Fig. 7.

As is evident from Fig. 7(a), in the accelerance transfer function, the optimal tuning ratio ν_{opt} increases with increasing damping ratio ζ_1. In addition, unlike for the other two transfer functions, the region in which the optimal solution exists for the accelerance transfer function narrows as the mass ratio μ increases. This is because the minimized resonance amplitude h_{min} reaches its minimum value of 1 at a small value of ζ_1, as shown in Fig. 7(c).

Further details can be understood from the optimized accelerance transfer functions shown in Fig. 8. In each plot in Fig. 8, the red curve represents the shape of the transfer function in the limit where an optimal solution is present, and the gray curve is the response curve in the case of $\zeta_1 = 0.7071$, which corresponds to the SDOF system shown in Fig. 1(a).

Both of these curves have a maximum value of 1, but the red curve begins to decrease after it peaks, and then increases again and approaches 1, whereas the gray curve monotonically increases towards 1. That is, in the compliance transfer function shown in Fig. 4, the optimal value approaches the curve of $\zeta_1 = 0.7071$, corresponding to the SDOF system, but the optimal value in the accelerance transfer function shown in Fig. 8 does not approach this curve.

7. Conclusion

Despite the fact that research on the H_∞ optimization of a single-mass viscously damped DVA has been going on for more than 90 years, many questions remained unanswered. The optimal solution for the DVA when viscous damping is present in the primary vibratory system had previously been obtained only in the form of an incomplete numerical or approximate solutions; the author believes that this paper puts an end to this long search. The H_∞ optimization of the viscously damped DVA for three representative transfer functions (compliance, mobility, and accelerance) for the primary system was carried out, and the following conclusions were obtained.

(1) For the mobility transfer function, the optimal tuning condition for the DVA can be obtained by solving a single sixth-order algebraic equation, and the optimal solutions were numerically obtained; these solutions are not inferior to the exact solution. Solving the equation with the *Mathematica* command `NSolve` yields six solutions, but there is only one positive real root for the optimal damping ratio. Therefore, it is easy to identify the optimal solution among the six solutions.

(2) For the compliance and accelerance transfer functions, the derivation of the optimal solution reduces to the problem of solving a ternary system of simultaneous algebraic equations. This system of equations can also be solved with the command `NSolve`, and out of the approximately 70 sets of solutions obtained with this approach, there is only one optimal solution. Here again, there exists only one set of positive real roots, simplifying the selection of the correct solution.

(3) The above single sixth-order equation or sets of simultaneous equations can be further simplified by factorization when the primary damping is zero, and the algebraic solutions of these equations yield the same exact solutions as previously reported.

Acknowledgment

Appendix: Derivation of the undamped solution for the compliance transfer function from the corresponding solution with damping

From the ternary system of simultaneous equations obtained in this study, the exact solutions shown in Sec. 3.4 (Asami and Nishihara, 2003) can be derived for the special case in which no damping is present in the primary system. In this appendix, one method for the derivation process is explained using the compliance transfer function as an example.

If the primary system damping ζ_1 is set to zero in Eqs. (14) and (18), the simultaneous equations become

$$\left.\begin{aligned}
&f_1 = a_0 r^2 + a_1 r + a_2 = 0, \quad f_2 = c_0 r^2 + c_1 r + c_2 = 0, \quad f_3 = d_0 r^3 + d_1 r^2 + d_2 r + d_3 = 0 \\
&a_0 = 1 - 2\zeta_2^2, \quad a_1 = -1 - (1+\mu)[1 - 2\zeta_2^2(1+\mu)]v^2, \quad a_2 = (1+\mu)v^2 \\
&c_0 = 1, \quad c_1 = -2v^2, \quad c_2 = 4\zeta_2^2(1+\mu)^3[1 - \zeta_2^2(1+\mu)]v^4 + 2[1 - 2\zeta_2(1-\mu^2)]v^2 - 1 \\
&d_0 = 1, \quad d_1 = -(1+\mu)^2[1 + 4\zeta_2^2(1+\mu) - 4\zeta_2^4(1+\mu)^2]v^2 - 1 + 2\zeta_2^2(1-\mu^2) \\
&d_2 = (1+\mu)^4 v^4 + \mu(1+\mu)^2 v^2, \quad d_3 = -(1+\mu)^4[1 - 2\zeta_2^2(1+\mu)]v^4 + (1+\mu)^2(1-\mu)v^2.
\end{aligned}\right\} \quad (23)$$

First, $f_1 = 0$ is solved for ζ_2, which gives

$$\zeta_2^2 = \frac{-(1-r)[r - (1+\mu)v^2]}{2r[r - (1+\mu)^2 v^2]}. \quad (24)$$

Substituting Eq. (24) into the formulas in Eq. (23) and removing unnecessary factors after factorization yields

$$\left.\begin{aligned}
&f_2 = c_0 r^5 + c_1 r^4 + c_2 r^3 + c_3 r^2 + c_4 r + c_5 = 0, \quad f_3 = d_0 r^5 + d_1 r^4 + d_2 r^3 + d_3 r^2 + d_4 r + d_5 = 0 \\
&c_0 = (1+\mu)^6 v^8, \quad c_1 = -(1+\mu)^4 v^6[4 - (1+\mu)^2 v^2], \quad c_2 = 2(1+\mu)^2 v^4[3 + \mu - 2(1+\mu)^2 v^2] \\
&c_3 = -2(1+\mu)v^2[2 - (1+\mu)(3+\mu)v^2], \quad c_4 = 1 - 2(2 + 2\mu + \mu^2)v^2, \quad c_5 = 1 \\
&d_0 = (1+\mu)^8 v^8, \quad d_1 = -(1+\mu)^6 v^6[4 - (1+\mu)^2 v^2], \quad d_2 = 2(1+\mu)^4 v^4[3 - 2(1+\mu)^2 v^2] \\
&d_3 = -2(1+\mu)^2 v^2[2 - 3(1+\mu)^2 v^2], \quad d_4 = (1+\mu)[1 - \mu - (1+\mu)(4 - \mu^2)v^4], \quad d_5 = 1.
\end{aligned}\right\} \quad (25)$$

In this way, two quintic equations with respect to r are obtained.

Next, the order of r is incrementally reduced by performing four arithmetic operations on these expressions. There are two approaches to selecting these operations: one is to eliminate the highest-order terms of the equation and the other is to remove the lowest-order terms. Finally, two linear equations with respect to r are obtained. With the equation obtained by the former operation expressed as $f_7 = 0$ and that obtained by the latter operation expressed as $g_7 = 0$, they are given by

$$\left.\begin{aligned}
&f_7 = c_0 r + c_1 = 0, \quad g_7 = d_0 r + d_1 = 0 \\
&c_0 = 32 - 8(16 + 23\mu + 9\mu^2)v^2 + (1+\mu)^2(96 + 119\mu + 40\mu^2)v^4 \\
&c_1 = -(1+\mu)^2 v^2[16 - 2(32 + 45\mu + 17\mu^2)v^2 + 3(1+\mu)^2(16 + 19\mu + 6\mu^2)v^4] \\
&d_0 = 16 - 2(32 + 45\mu + 17\mu^2)v^2 + 3(1+\mu)^2(16 + 19\mu + 6\mu^2)v^4, \quad d_1 = \mu(1+\mu)^2 v^4[4 - 3(1+\mu)^2 v^4].
\end{aligned}\right\} \quad (26)$$

Eliminating r from these equations yields the following single equation:

$$4(1+v+\mu v)(1-v-\mu v)[1-(3+3\mu+\mu^2)v^2][64-16(16+23\mu+9\mu^2)v^2+3(1+\mu)^2(64+80\mu+27\mu^2)v^4] = 0. \quad (27)$$

Setting the final factor in Eq. (27) to zero yields an equation, and solving this equation for v gives the first optimal solution v_{opt} of Eq. (6). Substituting this solution into v in Eq. (26) yields the minimum value r_{min} of r as follows:

$$r_{\mathrm{min}} = \frac{8[(4+3\mu)^{3/2} - \mu]}{64 + 80\mu + 27\mu^2}. \quad (28)$$

Additionally, r_{min} can be converted to h_{min} using the relationship given in Eq. (12).

Finally, the optimal damping ratio $\zeta_{2\mathrm{opt}}$ is obtained from Eq. (24).

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