Sets and Categories as Foundations of Mathematical Practice

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Abstract

Kreisel distinguishes between foundations and organization of mathematics. Broadly speaking, however, both could be seen as foundational studies on mathematics; the former is concerned with epistemological and/or ontological basis of mathematics in an ideal sense, while the latter is involved in conceptual, methodological machinery for organizing actual mathematics (as both problem solving and theory building) in an effective manner, which shall be called foundations of mathematical practice. The issue of sets versus categories have been discussed mainly in the context of the former, and they are usually supposed to be in conflict as a foundational enterprise. I thus aim to shed light on the issue from the second perspective, and clarify how sets and categories have worked as foundations of mathematical practice through the modernization of mathematics, with an emphasis on the vital role of sets in category theory, and on the way sets and categories interact in mathematical practice. They are not really in conflict when regarded as foundations of mathematical practice.

1 What are Foundations?

The dictionary tells us that foundations of X are the basis on which X stands or can stand. It makes much difference whether X "actually stands" or "can stand", and foundations of mathematics (set theory, category or topos theory, and others as formal theories) usually mean the latter. For the very reason, however, discussions on foundations of mathematics these days tend to be ignored by pure mathematicians (of course, there are some exceptions like Voevodsky); they or their mathematics do not stand on any formal foundations, even though it is possible in principle. Still, such pure mathematicians often use set theory and category theory as conceptual, non-formal-logical machinery for developing mathematics, but not as foundational, formal frameworks.

This phenomenon seems to be well understood building upon Kreisel's distinction between foundations and organization of mathematics, which was introduced in [7]. Most pure mathematicians are not interested in philosophical or logical foundations of mathematics any more, though they were so during the period of the crisis of mathematics. At the same time, however, they even now need or want to have some conceptual frameworks (sets, categories, etc.), according to which they organize mathematical structures and theorems. In this sense, we may say that organization of mathematics is the basis on which mathematics actually stands, since they indeed practice mathematics (especially, its structural aspects) using those conceptual frameworks. We thus have two conceptions of foundations of mathematics, according to the difference between the basis on which mathematics stands and that on which it can stand.

In an nutshell, one conception of foundations of mathematics is epistemological and/or ontological basis of mathematics in an ideal sense, which is usually meant in philosophy of mathematics, and correspond to what Kreisel just called foundations of mathematics. The other is conceptual, methodological machinery for organizing actual mathematics (as both problem solving and theory building), which shall simply be called foundations of mathematical practice, and correspond to what Kreisel called organization of mathematics. The issue of sets versus categories as foundations of mathematics in the first sense has already been discussed much (see, e.g., Feferman [4] or Marquis [9]); in the following sections, thus, I would like to shed light on aspects of the issue as foundations of mathematical practice.

2 Sets and Categories in Modernization

The concept of sets was introduced by Cantor, and then developed and exploited by Dedekind in his ideal theory. Some "constructively" inclined mathematicians, including Gordon and Kronecker, were strongly against set theory, which looked like "theology" to Gordon. In spite of such objections, set theory eventually became quite popular, but why did it happen? The main reason would be that the use of set theory turned out to be useful and effective to deal with computationally complex problems such as Gordon's problem in the theory of invariants, which was elegantly solved using set-theoretical, conceptual methods by Hilbert. Hilbert promoted the use of set theory in developments of mathematics (especially, algebra) not because it is foundations of mathematics, but because it is powerful, conceptual machinery for practicing mathematics, even though theorems obtained via set-theoretical means sometimes lack computational contents (probably, the problem of computational complexity was more serious than this). In that way, algebra became "set-theoretical" algebra in Göttingen, with Nöther playing a leading role after Hilbert. It should be emphasized again that, in those developments, set theory was not foundations of mathematics in the usual sense, but it was means for "conceptual mathematics", and foundations of mathematical practice (especially algebraic practice for Hilbert).[1]

Subsequently, Bourbaki systematically used set theory in order to "axiomatize" (not "formalize") various structures in mathematics from a unifying point of view: their basic idea is to start with point sets; equip them with fundamental structures (topology, algebraic operations, etc.); endow the resulting structures with further structures (e.g., differential structures). In [2], they explicitly state that their aim is not to formalize mathematics in the sense of formal logic; rather, it's to capture structural aspects of mathematics via their non-formal-logical, "axiomatic" methods. They also say that set theory may be replaced with another framework if doing so turns out to be better; thus they have no real, foundational claim. After Bourbaki, category theory finally came into the play, further deepening Bourbakian structuralism, and thus depriving it of primarily existing, point sets. After all, structures only remained without any set-theoretical basis; thus we could say that structures precede point sets in category theory.[2]

When Grothendieck introduced his categorical approach to algebraic geometry and number theory (or arithmetic geometry), André Weil and Carl Siegel expressed strong objections to it; for them, mathematics was something of more concrete nature. As is well known, however, categorical methods have eventually dominated quite some part of mainstream mathematics (algebraic and arithmetic geometry, algebraic topology, and so on). Again, the reason does not come from foundations of mathematics. It's because categorical methods enabled us to organize mathematical concepts with great conceptual clarity and unifying power, and to solve many outstanding problems (e.g., the Weil conjecture); it does come from the viewpoint of foundations of mathematical practice.[3]

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[1] For these historical developments, see [3, 5].

[2] This is strange from the viewpoint of Bourbaki's structuralism, but would actually be natural from the perspective of mathematics before modernization, in which there were just various, naively defined objects or structures with no unifying way to reduce them to common grounds like sets or to define them in a rigorous manner.

[3] Although Lawvere's work on topos theory was on the side of foundations of mathe-

Both set theory and category theory have thus been developed, through the modernization of mathematics, in the pursuit of effective, conceptual machinery for mathematical theory building and problem solving. From this perspective, set theory and category theory share the same spirit of conceptual mathematics. In the following section, I shall point out that sets and categories are not really in conflict even in a technical sense; for example, set-theoretical structures play essential roles in certain developments of category theory.

3 Sets in Categories, Categories in Sets

No doubt that set-theoretical structures yield ample examples of categorical structures, without which category theory would lose quite some of its significance. Moreover, set-theoretical models of a specific concept of categories can be used to prove the consistency of that concept of categories.[4] Yoneda's lemma tells us that any (locally small) category $\mathbf{C}$ can be embedded into the category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, which is a category of set-theoretical nature. In this sense, any (locally small) categorical structure can be realized in a set-theoretical way.[5]

Yoneda's lemma is widely used and highly important in category theory, but actually rely on set-theoretical structures in an essential way. It could not be established without a suitable concept of the category of sets. Especially, the concept of Hom sets (i.e., the set of arrows between two objects) plays a crucial role in developments of category theory, including Yoneda's lemma and the definition of adjunction via the isomorphism of Hom sets.[6] Furthermore, the set-theoretical distinction between small and large categories is essential to avoid Russell-type paradoxes. The concept of sets is thus indispensable in a non-trivial manner in substantial part of category

matics, and had some influence on mainstream mathematics as well, it is still true that toposes for geometers usually mean Grothendieck toposes, rather than Lawvere-Tierney's elementary toposes.

[4]For example, the concept of Cartesian closed categories in which every object has the fixpoint property can be shown to be consistent by constructing set-theoretical models. However, adding the existence of coproducts to the conditions makes the concept inconsistent.

[5]This should not be confused with the fact that some categories, including the category of topological spaces and homotopy equivalence classes of continuous maps, have no faithful functor into **Sets**.

[6]This dependence on sets can be avoided to some extent if we work with enriched categories; however, Hom sets inevitably appear in the base category of enrichment even in that case.

theory.

Categorical duality between various sorts of algebras and spaces is one of the central topics of mathematics (e.g., Gelfand duality, Pontryagin duality, Stone duality, etc.).[7] As Lawvere said a potential duality arises when one object lives in two categories, the concept of an object that can be seen as objects of two categories has been exploited much in general theory of categorical dualities.[8] For example, Gelfand duality between C^*-algebras and compact Hausdorff spaces is established by seeing the complex numbers $\mathbb{C}$ both as a space and as an algebra (and by constructing two Hom functors using the two structures on $\mathbb{C}$).

However, why can we say "one" object lives in two categories? From a categorical point of view, objects only make sense inside the category in which they live, and we cannot say that an object in one category and an object in another category are the same "one" object. In the case of Gelfand duality, $\mathbb{C}$ in the category of algebras and $\mathbb{C}$ in the category of spaces cannot be regarded as (different appearances of) the same object. Set-theoretically, we can of course say that they have the same underlying set, and so they are two faces of the same object; however, speaking in this way is not acceptable from a categorical perspective, since objects cannot live outside categories (no object outside categories) in category theory. What allows us to see one object as objects of two categories is the set-theoretic perspective, which can give us the absolute context in which mathematical objects exist, while objects can only exist in a relative context (i.e., relative to a category) in category theory. We may thus say that the main idea of categorical duality theory is actually of set-theoretical nature.

We can also recognize the importance of non-categorical thinking in categorical theories in recent developments of categorical quantum mechanics and topos foundations of physics. And higher-dimensional categorical structures are relevant to a sort of a set-theoretical, cumulative hierarchy in some sense.

[7]Lawvere discusses duality between the formal and the conceptual in [8]. We may consider that algebras are formal objects like polynomials and spaces play a conceptual or semantic role as algebraic varieties are sets of "models" or points in which polynomials are "false" or vanish (in this interpretation, Hilbet Nullstellensatz amounts to logical completeness).

[8]For details, see Johnstone [6], which is a main reference in this area.

4 Concluding Remarks

Beginning with Kreisel's distinction between foundations and organization of mathematics, I first clarified two different senses of foundations of mathematics. One is foundations of mathematics in the traditional sense, which is the basis on which mathematics can stand (with good explanation of its epistemology and ontology). The other is foundations of mathematical practice, which is the basis on which mathematics actually stand.

Set theory and category theory tend to be seen as in conflict, and it would indeed be true when they are conceived of as foundations of mathematics in the former sense. At the same time, however, they are actually in harmony if regarded as foundations of mathematics in the latter sense. As discussed in Section 2, both have been developed in the process of the modernization of mathematics in order to overcome a vast amount of computation via conceptualization, which is probably at the heart of the power of modern mathematics. They shared the same spirit of conceptual mathematics in such historical developments. And, as discussed in Section 3, the concept of sets and the set-theoretical way of thinking play vital roles in the practice of category theory, for example, in Yoneda's lemma, a definition of adjunctions, and categorical duality theory.

These should justify or at least provide some justification for the main claim of the present paper that, when sets and categories are regarded as foundations of mathematical practice, they are not in conflict any more, but indeed interact and complement each other.[9]

References

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