Natural Duality, Modality, and Coalgebra

Yoshihiro Maruyama
Oxford University Computing Laboratory
http://researchmap.jp/ymaruyama

Abstract

The theory of natural dualities is a general theory of Stone-Priestley-type categorical dualities based on the machinery of universal algebra. Such dualities play a fundamental role in recent developments of coalgebraic logic. At the same time, however, natural duality theory has not subsumed important dualities in coalgebraic logic, including Jónsson-Tarski's topological duality and Abramsky-Kupke-Kurz-Venema's coalgebraic duality for the class of all modal algebras. By introducing a new notion of $\mathbb{ISP}_{\mathbb{M}}$, in this paper, we aim to extend the theory of natural dualities so that it encompasses Jónsson-Tarski duality and Abramsky-Kupke-Kurz-Venema duality. The main results are topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a semi-primal algebra with a bounded lattice reduct. These dualities are shown building upon Keimel-Werner's semi-primal duality theorem. Our general theory subsumes both Jónsson-Tarski and Abramsky-Kupke-Kurz-Venema dualities. Moreover, it provides new coalgebraic dualities for algebras of many-valued modal logics and certain insights into a category-equivalence problem for categories of algebras involved. It also follows from our dualities that the corresponding categories of coalgebras have final coalgebras and cofree coalgebras. $\mathbb{ISP}_{\mathbb{M}}$ provides a natural framework for the universal algebra of modalities, and as such, for the theory of modal natural dualities.

Keywords: natural duality theory; Stone duality; Jónsson-Tarski duality; Keimel-Werner's semi-primal duality; coalgebraic logic; Vietoris hyperspace

1 Introduction

By proposing a new notion of $\mathbb{ISP}_{\mathbb{M}}$ as the modalization of $\mathbb{ISP}$, in this paper, we attempt to extend the theory of natural dualities (see [10, 37]) so that it encompasses Jónsson-Tarski's topological duality (see [24, 5, 17, 19]) and Abramsky-Kupke-Kurz-Venema's coalgebraic duality (see [2, 26]) for the class of all modal algebras. Such dualities play a fundamental role in recent developments of coalgebraic logic (see [9]), which allows us to unify different kinds of modal logics, based on the theory of coalgebras.

A typical story in coalgebraic logic is as follows (see [27]). A dual adjunction induced by a schizophrenic object (see [39]) represents the syntax and semantics of a propositional logic (some researchers call such an adjunction a logical connection; see [28]). The Stone adjunction between Boolean algebras and sets is a typical example of this. We then fix an endofunctor on one category in the dual adjunction, which in turn induces an endofunctor on the other. The algebras and coalgebras of the endofunctors give rise to the syntax and semantics of the propositional logic equipped with modality. In particular, the standard modal logic K and Kripke semantics arise from the Stone adjunction by taking the power-set functor as an endofunctor on sets. Many modal logics such as monotone modal logic and probabilistic modal logic fall into this picture. In "good" cases, we can finally obtain duality between the corresponding algebras and coalgebras, as we are able to lead from Stone duality to Abramsky-Kupke-Kurz-Venema duality via Jónsson-Tarski duality.

In relation to this picture of coalgebraic logic, we start with Keimel-Werner's semi-primal duality theorem (see [10, Theorem 3.3.14] and [25]) in natural duality theory. Keimel-Werner's theorem is a universal-algebraic generalization of Stone duality for Boolean algebras and can be seen as dual adjunctions induced by schizophrenic objects, representing the syntax and semantics of propositional logics. Our aim is then to establish the corresponding Jónsson-Tarski-type duality and Abramsky-Kupke-Kurz-Venema-type duality by introducing the new notion of $\mathbb{ISP}_{\mathbb{M}}$. At least under the assumption of semi-primality, categories arising from $\mathbb{ISP}_{\mathbb{M}}$ can be considered as categories of algebras for certain "free-generation" endofunctors on categories obtained via $\mathbb{ISP}$, and the duals of categories induced by $\mathbb{ISP}_{\mathbb{M}}$ can be described as categories of coalgebras for certain "Vietoris-style" endofunctors on the duals of categories corresponding to $\mathbb{ISP}$.

In the following, let us first review an aspect of natural duality theory and a certain difficulty in incorporating into it Jónsson-Tarski duality and Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras. We shall then see that the difficulty can be overcome with the help of $\mathbb{ISP}_{\mathbb{M}}$.

The theory of natural dualities by Davey et al. is a powerful general theory of Stone-Priestley-type dualities based on the machinery of universal algebra. It basically considers duality theory for $\mathbb{ISP}(M)$ where M is a finite algebra. It is useful for obtaining new dualities and actually encompasses many known dualities, including Stone duality for Boolean algebras (see [42]), Priestley duality for distributive lattices (see [40]), and Cignoli duality for MV_n-algebras, i.e., algebras of Łukasiewicz n-valued logic (see [8, 35]), to name but a few (for more instances, see [10, 37]).

At the same time, however, it has not encompassed Jónsson-Tarski duality or Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras, any of which is important in coalgebraic logic. We consider that this is mainly because the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ for a finite algebra M, in contrast to the fact that any of the class of Boolean algebras, the class of distributive lattices, and the class of MV_n-algebras can be expressed as $\mathbb{ISP}(M)$ for a suitable finite algebra M.

We should note here that, given a modal algebra, the Boolean operations of the function algebra on its spectrum (i.e., the space of prime filters) can be defined pointwise, while only the modal operation of the function algebra cannot be defined pointwise (recall that it is defined depending on the canonical relation induced by the modal operation of the original modal algebra). In a nutshell, modality is not a pointwise operation unlike the other Boolean operations. For the very reason, the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ (all the operations of $A \in \mathbb{ISP}(M)$ are pointwise by definition), and we have to pay a special attention to modality when developing natural duality theory for algebras with modal operations. We remark that the same thing can be said also for the implication operation of a Heyting algebra, which is not pointwise (on the spectrum of the Heyting algebra). And this actually tells us a duality-theoretic reason why Gödel failed to capture intuitionistic logic as a many-valued logic (broadly speaking, $\mathbb{ISP}(M)$ amounts to algebras of M-valued logic).

In this paper, we introduce a new notion of $\mathbb{ISP}_{\mathbb{M}}$ in order to extend the theory of natural dualities so that it encompasses Jónsson-Tarski's topological duality and Abramsky-Kupke-Kurz-Venema's coalgebraic duality. It is crucial here that the class of all modal algebras coincides with

$\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ for the two-element Boolean algebra $\mathbf{2}$. Moreover, we have the following facts: for $\mathbf{n}$ defined in Definition 2.9 below, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ coincides with the class of all algebras of Łukasiewicz n-valued modal logic (for this logic, see, e.g., [6, 20, 44]); a similar thing holds also for algebras of a version of Fitting's many-valued modal logic (for this logic, see, e.g., [15, 16, 29, 32]). Thus, the notion of $\mathbb{ISP}_{\mathbb{M}}$ seems to be natural and useful for our goal.

Our main results (Theorem 3.24 and Theorem 4.11) are topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a semi-primal algebra with a bounded lattice reduct. Our results encompass both Jónsson-Tarski and Abramsky-Kupke-Kurz-Venema dualities as the case $L = \mathbf{2}$. They also encompass topological dualities in [44, 32] for algebras of many-valued modal logics. Our dualities are developed based on Keimel-Werner's semi-primal duality theorem in the theory of natural dualities, and may be considered as modalized extensions of the semi-primal duality theorem on algebras with bounded lattice reducts. As applications, we obtain new coalgebraic dualities for algebras of Łukasiewicz n-valued modal logic and for algebras of a version of Fitting's many-valued modal logic. With the help of the duality results, we can also show the existence of final coalgebras and cofree coalgebras in the cateogries of coalgebras involved. Note that final coalgebras are significantly used for the semantics of programming languages (see [45]). We finally provide a duality-based criterion for the equivalence of categories of algebras concerned.

Several authors have developed duality theories for those classes of modal algebras (in the wider sense) that can be expressed as $\mathbb{ISP}(M)$ for finite algebras M. (see, e.g., [41, 43]). However, they do not encompass Jónsson-Tarski duality or Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras, since the class of all modal algebras cannot be expressed as $\mathbb{ISP}(M)$ for a finite algebra M. By modalizing the notion of $\mathbb{ISP}$, this paper makes it possible to incorporate both Jónsson-Tarski and Abramsky-Kupke-Kurz-Venema dualities into the theory of natural dualities.

As a (rough) historical note, we remark that the duality of modal algebras and coalgebras for the Vietoris functor (or Stone coalgebras) was essentially discovered by Abramsky, and his relevant talk was given at the 1988 British Colloquium on Theoretical Computer Science as mentioned in [2]. The paper version [2] of the 1988 talk, however, had remained unpublished until 2005. On the other hand, in 2003, Kupke, Kurz, and Venema published their paper [26] providing a detailed description of the duality. Their work was done independently of Abramsky's. Taking all this into consideration, we call the duality "Abramsky-Kupke-Kurz-Venema duality" in this paper. It could also be called just "Abramsky duality" (especially if a shorter term is preferred). At the same time, we emphasize that Esakia first mentioned the use of Vietoris spaces in the context of non-classical logics in his paper [14], as early as in 1974.

The paper is organized as follows. In Section 2, we introduce the notions of $\mathbb{ISP}_{\mathbb{M}}$ and Kripke condition. The Kripke condition may be considered as completeness in logical terms and plays an important role in our duality theory. In Section 3, we show the first result, i.e., a topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$. In Section 4, we show the second result, i.e., a coalgebraic duality for $\mathbb{ISP}_{\mathbb{M}}(L)$, which implies coalgebraic dualities for algebras of Łukasiewicz n-valued modal logic and for algebras of a version of Fitting's many-valued modal logic. As an application of our dualities, we obtain a result on the equivalence of categories of algebras involved. It also follows from our dualities that the corresponding categories of coalgebras have cofree coalgebras and final coalgebras. In Section 5, we conclude the paper by discussing several future directions of research, including a coalgebraic extension of the notion of $\mathbb{ISP}_{\mathbb{M}}$, and by comparing natural duality theory with categorical duality theory. Furthermore, we briefly discuss broader philosophical backgrounds behind the idea of Stone duality in a wider sense, at the end of the paper.

2 The Notion of $\mathbb{ISP}_{\mathbb{M}}$

For universal algebra and lattice theory, we refer the reader to [7, 11]. For category theory, we refer to [4], which contains categorical universal algebra and categorical universal topology (especially, categorical Birkhoff theorems and its topological analogues).

Throughout this paper, let L denote a finite algebra with a bounded lattice reduct (it is natural from a logical point of view to suppose the existence of a bounded lattice reduct, since most logics are equipped with the lattice connectives $\wedge$ and $\vee$ and the truth constants 0 and 1). Let $\mathbf{2}$ denote the two-element Boolean algebra.

From a logical point of view, we may see L as an algebra of truth values. Since the lattice reduct of L turns out to be a complete Heyting algebra (note that any finite distributive lattice is a Heyting algebra), the lattice reduct of L is actually a so-called truth-value object Ω in an elementary topos. The case that $L = \mathbf{2}$ amounts to classical logic, and $\mathbb{ISP}(\mathbf{2})$ coincides with the class of all Boolean algebras.

We define the notion of modal power as follows. For a set S, L^S denotes the set of all functions from S to L. A Kripke frame is defined as a tuple (S, R) such that S is a non-empty set and R is a binary relation on S.

Definition 2.1. *For a Kripke frame (S, R), the modal power of L with respect to (S, R) is defined as $L^S \in \mathbb{ISP}(L)$ equipped with a unary operation $\Box_R$ on L^S defined by*

$$(\Box_R f)(w) = \bigwedge \{f(w') \,;\, wRw'\}$$

where $f \in L^S$ and $w \in S$. Then, a modal power of L is defined as the modal power of L with respect to (S, R) for some Kripke frame (S, R). (To be precise about the order of quantifiers, this means that, for any modal power A of L, there is some Kripke frame (S, R) such that A is a modal power of L with respect to (S, R).)

For a Kripke frame (S, R), let $L^{(S,R)}$ denote the modal power of L with respect to (S, R).

The notion of $\mathbb{ISP}_{\mathbb{M}}$ is then defined as follows.

Definition 2.2. $\mathbb{ISP}_{\mathbb{M}}(L)$ *denotes the class of all isomorphic copies of subalgebras of modal powers of L.*

We often denote by $(A, \Box)$ an element of $\mathbb{ISP}_{\mathbb{M}}(L)$. Note that $\Box(x \wedge y) = \Box x \wedge \Box y$ for $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$ and $x, y \in A$.

Definition 2.3. $\mathbf{ISP}(L)$ *denotes the category of algebras in $\mathbb{ISP}(L)$ and homomorphisms where a homomorphism is defined as a function which preserves all the operations of L.*

$\mathbf{ISP_M(L)}$ *denotes the category of algebras in $\mathbb{ISP}_{\mathbb{M}}(L)$ and modal homomorphisms where a modal homomorphism is defined as a function which preserves $\Box$ and all the operations of L.*

The modalization of $\mathbb{ISP}$ preserves the closedness under $\mathbb{I}$, $\mathbb{S}$, and $\mathbb{P}$ as follows.

Proposition 2.4. $\mathbb{ISP}_{\mathbb{M}}(L)$ *is closed under $\mathbb{I}$, $\mathbb{S}$, and $\mathbb{P}$.*

Proof. It is clear that $\mathbb{ISP}_{\mathbb{M}}(L)$ is closed under $\mathbb{I}$ and $\mathbb{S}$. In order to show that it is closed under direct products, let I be a set and $(A_i, \Box_i) \in \mathbb{ISP}_{\mathbb{M}}(L)$ for $i \in I$. Then it follows that for each $i \in I$

there is a Kripke frame (S_i, R_i) such that $(A_i, \Box_i)$ is embedded into $L^{(S_i,R_i)}$, i.e., the modal power of L with respect to (S_i, R_i). Define a Kripke frame (S, R) by

$$S = \coprod_{i\in I} S_i \text{ and } R = \coprod_{i\in I} R_i.$$

We claim that $\prod_{i\in I}(A_i, \Box_i)$ can be embedded into $L^{(S,R)}$. To show this, we define a function

$$e : \prod_{i\in I}(A_i, \Box_i) \to L^{(S,R)}$$

as follows. Given $x \in S$ and $f_i : A_i \to L$ for $i \in I$, define $(e((f_i)_{i\in I}))(x) = f_k(x)$ where k is the unique $j \in I$ such that $x \in S_j$. Let $\Box$ denote the modal operation of $\prod_{i\in I}(A_i, \Box_i)$. Note that $\Box$ is defined pointwise. We show that $e(\Box(f_i)_{i\in I}) = \Box_R e((f_i)_{i\in I})$. Let $x \in S$. It follows from the definition of (S, R) that if $x \in S_k$ for $k \in I$ then

$$(\Box_R e((f_i)_{i\in I}))(x) = \bigwedge\{e((f_i)_{i\in I})(y) \,;\, xR_k y\} = (\Box_k f_k)(x).$$

It also holds that if $x \in S_k$ then

$$e(\Box(f_i)_{i\in I})(x) = e((\Box_i f_i)_{i\in I})(x) = (\Box_k f_k)(x).$$

Thus, we have shown that e preserves $\Box$. It is straightforward to see that e also preserves the other operations of $\prod_{i\in I}(A_i, \Box_i)$. Hence, $\mathbb{ISP}_{\mathbb{M}}(L)$ is closed under direct products. $\Box$

According to the theory of free algebras in universal algebra, the above proposition gives us the following.

Corollary 2.5. $\mathbb{ISP}_{\mathbb{M}}(L)$ *has free algebras.*

Given $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, we define the corresponding canonical relation $R_\Box$ on $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$.

Definition 2.6. *For $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, we define a binary relation $R_\Box$ on $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ as follows: For $v, u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$, $vR_\Box u$ iff the following holds:*

$$\forall a \in L\ \forall x \in A\ (v(\Box x) \geq a \textit{ implies } u(x) \geq a).$$

Definition 2.7. $\mathbb{ISP}_{\mathbb{M}}(L)$ *(or L) satisfies the Kripke condition iff, for any $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$, any $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$, and any $x \in A$, the following holds:*

$$v(\Box x) = \bigwedge\{u(x) \,;\, vR_\Box u\}.$$

The Kripke condition may be considered as completeness in logical terms.

In this paper, the Kripke condition can be seen as a condition on L rather than $\mathbb{ISP}_{\mathbb{M}}(L)$, since we concentrate on "normal" modal logic induced by L. If we also consider other types of modal logics, however, it seems that $\mathbb{ISP}_{\mathbb{M}}$ is not a unique way to generate the corresponding classes of modal algebras (in the wider sense). In that case, the Kripke condition depends on the way of generating modal algebras as well as the basic structure L.

The notions of $\mathbb{ISP}_{\mathbb{M}}$ and Kripke condition are motivated by Proposition 2.8 and Proposition 2.10 below.

Proposition 2.8. $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ *coincides with the class of all modal algebras and satisfies the Kripke condition.*

Proof. By Jónsson-Tarski representation (see, e.g., [5, Theorem 5.43]), any modal algebra can be embedded into a modal power of **2**. It is straightforward to see that any $A \in \mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ is a modal algebra. Thus, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ coincides with the class of all modal algebras. It follows from Proposition 3.14 below that $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ satisfies the Kripke condition (a direct proof of this fact can also be given in a similar way to the completeness proof of classical modal logic K). □

The algebra **n** of truth values in Łukasiewicz n-valued logic is defined as follows (see, e.g., [18]):

Definition 2.9. *Let* **n** *denote* $\{0, 1/(n-1), ..., (n-2)/(n-1), 1\}$ *equipped with the operations* $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1)$ *defined by*

$$\begin{aligned} x \wedge y &= \min(x, y); \\ x \vee y &= \max(x, y); \\ x * y &= \max(0, x + y - 1); \\ x \wp y &= \min(1, x + y); \\ x \rightarrow y &= \min(1, 1 - (x - y)); \\ x^{\perp} &= 1 - x. \end{aligned}$$

An $\mathcal{MMV}_n$-algebra introduced in [44, Definition 3.1] is an algebra of Łukasiewicz n-valued modal logic. We then have the following.

Proposition 2.10. $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ *coincides with the class of all* $\mathcal{MMV}_n$*-algebras and satisfies the Kripke condition.*

Proof. By Teheux representation following from [44, Theorem 4.11], any $\mathcal{MMV}_n$-algebra can be embedded into a modal power of **n**. It is straightforward to see that any $A \in \mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ is an $\mathcal{MMV}_n$-algebra. Thus, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ coincides with the class of all modal algebras. It follows from Proposition 3.14 below that $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ satisfies the Kripke condition (or this also follows from the completeness of Łukasiewicz n-valued modal logic). □

A similar proposition can be shown also for L-**ML**-algebras, which are algebras of a version of Fitting's many-valued modal logic (see [29, 32]).

Thus, the notion of $\mathbb{ISP}_{\mathbb{M}}$ seems to be natural and useful.

3 Modal Semi-Primal Duality Theorem

In the remaining part of the paper, we assume that L is semi-primal. A semi-primal algebra is a useful concept in universal algebra and is defined as follows.

Definition 3.1. *Let A be an algebra (in the sense of universal algebra) and n a positive integer. A function $f : A^n \rightarrow A$ is called conservative iff, for any $a_1, ..., a_n \in A$, $f(a_1, ..., a_n)$ is in the subalgebra of A generated by $\{a_1, ..., a_n\}$.*

A semi-primal algebra is a finite algebra A such that, for any positive integer n, every conservative function $f : A^n \rightarrow A$ is a term function of A. (Note that a term function is called a polynomial in some literature.)

Intuitively, we may say that a conservative function on an algebra is a function preserving the subalgebra structure of the algebra. For characterizations of semi-primality and term-definable operations on semi-primal algebras, we refer the reader to [38, 12].

We remark that, under the assumption of the semi-primality of L, $\mathbf{ISP_M}(L)$ actually forms a variety (or a monadic category in categorical terms), which shall be shown in a subsequent paper on the finite axiomatizability of $\mathbb{ISP}(L)$ and $\mathbb{ISP}_{\mathbb{M}}(L)$.

Now it is straightforward to verify the following lemmas by cheking that each function is conservative.

Lemma 3.2. *Define a function* $q : L^4 \to L$ *by*

$$q(x, y, z, w) = \begin{cases} w & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}$$

where $x, y, z, w \in L$*. Then,* $q : L^4 \to L$ *is a term function of* L*.*

The function $q : L^4 \to L$ is called the quaternary discriminator.

Lemma 3.3. *Let* $a \in L$*. Define a function* $\mathrm{T}_a : L \to L$ *by*

$$\mathrm{T}_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

where $x \in L$*. Then,* T_a *is a term function of* L*.*

To verify the proposition above, note that any subalgebra of L contains constants 0 and 1 by the definition of a subalgebra.

From a logical point of view, $\mathrm{T}_a(p)$ intuitively means that the truth value of a proposition p is exactly a for an element a of the algebra L of truth values, which may be seen as a truth-value object in a topos, since the lattice reduct of L is a complete Heyting algebra.

Lemma 3.4. *Let* $a \in L$*. Define a function* $\mathrm{U}_a : L \to L$ *by*

$$\mathrm{U}_a(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x \not\geq a \end{cases}$$

where $x \in L$*. Then,* U_a *is a term function of* L*.*

We can also define the function $\mathrm{U}_a : L \to L$ by using T_a in the following way:

$$\mathrm{U}_a(x) = \bigvee \{\mathrm{T}_b(x)\ ;\ a \leq b \text{ and } b \in L\}.$$

It is straightforward to see that U_a and $\wedge$ are commutative, i.e.,

$$\mathrm{U}_a(x \wedge y) = \mathrm{U}_a(x) \wedge \mathrm{U}_a(y).$$

Moreover, $\Box$ and U_a are commutative, i.e.,

$$\Box \mathrm{U}_a(x) = \mathrm{U}_a(\Box x)$$

for any $x \in A$ where $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$. This can be verified using the fact that U_a and $\wedge$ are commutative (note that $\Box$ is defined via $\wedge$). We also remark that $\mathrm{U}_1(x) = \mathrm{T}_1(x)$.

Lemma 3.5. *Let $a \in L$. Define a function $(\text{-}) \to (\text{-}) : L^2 \to L$ by*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x \not\leq y \end{cases}$$

where $x, y \in L$. Then, $\to$ is a term function of L.

The function $(\text{-}) \to (\text{-}) : L^2 \to L$ can also be defined by $x \to y = q(x \wedge y, x, 1, y)$.

We can apply Keimel-Werner's semi-primal duality theorem [10, Theorem 3.3.14] to obtain a topological duality for $\mathbb{ISP}(L)$, which is explained in the following subsection. We shall later build a duality theory for $\mathbb{ISP}_{\mathbb{M}}(L)$ based on the semi-primal duality theorem.

3.1 Semi-primal duality for $\mathbb{ISP}(L)$

Let $\mathrm{SubAlg}(L)$ denote the set of all subalgebras of L. For a Boolean space S, let $\mathrm{SubSp}(S)$ denote the set of all closed subspaces of S, where a Boolean space means a zero-dimensional compact Hausdorff space.

Definition 3.6. *We define a category $\mathbf{BS}_L$. An object in $\mathbf{BS}_L$ is a tuple (S, α) such that S is a Boolean space and that a function $\alpha : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(S)$ satisfies:*

1. *$S = \alpha(L)$;*
2. *if $L_3 = L_1 \cap L_2$ for $L_1, L_2, L_3 \in \mathrm{SubAlg}(L)$, then $\alpha(L_3) = \alpha(L_1) \cap \alpha(L_2)$.*

An arrow $f : (S, \alpha) \to (S', \beta)$ in L-BS is a continuous map $f : S \to S'$ that satisfies the condition that, for any $M \in \mathrm{SubAlg}(L)$, if $x \in \alpha(M)$ then $f(x) \in \beta(M)$. We call a map satisfying the condition a subspace-preserving map.

Having an object in $\mathbf{BS}_L$ is equivalent to having a meet-preserving function $\alpha : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(S)$. This provides another definition of an object in $\mathbf{BS}_L$ as a $\mathrm{SubAlg}(L)$-indexed family of Boolean spaces satisfying the condition of meet-preservation.

Note also that the condition 2 above implies that, if $L_1 \subset L_2$ for $L_1, L_2 \in \mathrm{SubAlg}(L)$, then $\alpha(L_1) \subset \alpha(L_2)$.

We equip L and its subalgebras with the discrete topologies. Define $\alpha_L : \mathrm{SubAlg}(L) \to \mathrm{SubSp}(L)$ by $\alpha_L(M) = M$ for $M \in \mathrm{SubAlg}(L)$. Then, (L, α_L) is an object in $\mathbf{BS}_L$.

For $A \in \mathbb{ISP}(L)$, we equip $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ with the topology generated by $\{\langle x \rangle\,;\, x \in A\}$ where

$$\langle x \rangle = \{v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)\,;\, v(x) = 1\}$$

for $x \in A$. Note that for $x \in A$, $\langle x \rangle$ is clopen, since $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L) \setminus \langle x \rangle = \langle \mathrm{T}_1(x) \to 0 \rangle$ by Lemma 3.3 and Lemma 3.5.

Definition 3.7. *We define a contravariant functor* $\mathrm{Spec} : \mathbf{ISP}(L) \to \mathbf{BS}_L$*. For an object A in $\mathbf{ISP}(L)$, let*

$$\mathrm{Spec}(A) = (\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L), \alpha_A)$$

where α_A is defined by

$$\alpha_A(M) = \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$$

for $M \in \mathrm{SubAlg}(L)$. For an arrow $f : A \to B$ in $\mathbf{ISP}(L)$, $\mathrm{Spec}(f)$ *is defined by* $\mathrm{Spec}(f)(v) = v \circ f$ *for $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(B, L)$.*

The functor Spec can be defined also for $\mathbf{ISP_M}(L)$ (by considering modality-free reducts). The domain of Spec is defined to be $\mathbf{ISP}(L)$ just because it is an ingredient of duality between $\mathbf{ISP}(L)$ and $\mathbf{BS}_L$.

Definition 3.8. *We define a contravariant functor* $\mathrm{Cont} : \mathbf{BS}_L \to \mathbf{ISP}(L)$. *For an object* (S, α) *in* $\mathbf{BS}_L$, *define* $\mathrm{Cont}(S, \alpha)$ *as*

$$\mathrm{Hom}_{\mathbf{BS}_L}((S, \alpha), (L, \alpha_L))$$

equipped with the pointwise operations. For an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ *in* $\mathbf{BS}_L$, $\mathrm{Cont}(f)$ *is defined by* $\mathrm{Cont}(f)(g) = g \circ f$ *for* $g \in \mathrm{Cont}(S_2, \alpha_2)$.

Later we shall extend Spec and Cont to the modal setting (RSpec and MCont respectively).

By Keimel-Werner's semi-primal duality theorem [10, Theorem 3.3.14], we obtain the following.

Theorem 3.9. $\mathbf{ISP}(L)$ *and* $\mathbf{BS}_L$ *are dually equivalent via* Spec *and* Cont.

The semi-primal duality theorem is essentially due to [25].

Based on the above duality, we shall show that $\mathbf{ISP_M}(L)$ is dually equivalent to $\mathbf{RBS}_L$, which is defined in Definition 3.15 below. In order to prove this duality, we first verify the Kripke condition for $\mathbb{ISP}_{\mathbb{M}}(L)$ in the next subsection.

3.2 The verification of the Kripke condition

In order to show that $\mathbb{ISP}_{\mathbb{M}}(L)$ satisfies the Kripke condition, we use the prime filter theorem for Boolean algebras (see, e.g., [23]). We first introduce the notion of the Boolean core $\mathcal{B}(A)$ of $A \in \mathbb{ISP}(L)$.

Definition 3.10. *For* $A \in \mathbb{ISP}(L)$, *define*

$$\mathcal{B}(A) = \{x \in A\, ;\ \mathrm{T}_1(x) = x\}.$$

Note that $\mathrm{T}_a(x), \mathrm{U}_a(x) \in \mathcal{B}(A)$ for any $x \in A$ and $a \in L$.

Lemma 3.11. *For* $A \in \mathbb{ISP}(L)$, $(\mathcal{B}(A), \wedge, \vee, \mathrm{T}_0, 0, 1)$ *forms a Boolean algebra.*

Proof. This follows from the the two facts that $(\mathcal{B}(A), \wedge, \vee, \mathrm{T}_0, 0, 1)$ is a subalgebra of a direct power of $(\mathcal{B}(L), \wedge, \vee, \mathrm{T}_0, 0, 1)$ and that $(\mathcal{B}(L), \wedge, \vee, \mathrm{T}_0, 0, 1)$ is the two-element Boolean algebra (note that T_0 is the complement operation). □

Lemma 3.12. *For* $A \in \mathbb{ISP}(L)$, *let* P *be a prime filter of a Boolean algebra* $\mathcal{B}(A)$. *Define* $v_P : A \to L$ *by*

$$v_P(x) = a \Leftrightarrow \mathrm{T}_a(x) \in P.$$

Then, v_P *is an element of* $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$.

Proof. Since $\bigvee_{a \in L} \mathrm{T}_a(x) = 1$ for $x \in A$ and since $\mathrm{T}_a(x) \wedge \mathrm{T}_b(x) = 0$ for $a, b \in L$ with $a \neq b$, v_P is well defined as a function from A to L. Let $t : A^n \to A$ be an n-ary operation of A. Let $x_i \in A$ and $a_i = v_P(x_i)$ for $i = 1, ..., n$. Then it follows by definition that

$$\mathrm{T}_{a_1}(x_1) \wedge ... \wedge \mathrm{T}_{a_n}(x_n) \in P.$$

It is straightforward to show the following inequality (note that it is enough to verify the inequality in L):

$$\mathrm{T}_{a_1}(x_1) \wedge ... \wedge \mathrm{T}_{a_n}(x_n) \leq \mathrm{T}_{t(a_1,...,a_n)}(t(x_1,...,x_n)).$$

Thus we have $\mathrm{T}_{t(a_1,...,a_n)}(t(x_1,...,x_n)) \in P$, which implies that

$$v_P(t(x_1,...,x_n)) = t(a_1,...,a_n) = t(v_P(x_1),...,v_P(x_n)).$$

This completes the proof. □

The following lemma is crucial for the verification of the Kripke condition.

Lemma 3.13. *Let* $(A,\Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$*,* $x \in A$*,* $a \in L$*, and* $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A,L)$*. Then the following holds:*

$$v(\Box x) \geq a \text{ iff for any } u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A,L),\ vR_\Box u \text{ implies } u(x) \geq a.$$

Proof. It is easily verified that the left-hand side implies the right-hand side. We show the converse by proving the contrapositive. Assume that $v(\Box x) \not\geq a$. This means that $v(\mathrm{U}_a(\Box x)) \neq 1$. Let

$$X = \{\mathrm{U}_b(y)\ ;\ v(\mathrm{U}_b(\Box y)) = 1\}.$$

Note that $X \subset \mathcal{B}(A)$. Let F be the filter of $\mathcal{B}(A)$ generated by X.

We claim that $\mathrm{U}_a(x) \notin F$. Suppose for contradiction that $\mathrm{U}_a(x) \in F$. Then there is $\varphi \in A$ such that $\varphi \leq \mathrm{U}_a(x)$ and φ is constructed from $\wedge$ and elements of X. Since the equation $\mathrm{U}_b(y \wedge y') = \mathrm{U}_b(y) \wedge \mathrm{U}_b(y')$ holds in general, we may assume that for some $\{\mathrm{U}_b(x_b)\ ;\ b \in L\} \subset X$,

$$\varphi = \bigwedge\{\mathrm{U}_b(x_b)\ ;\ b \in L\}.$$

By $\varphi \leq \mathrm{U}_a(x)$, it follows from the definition of modal power that $\Box\varphi \leq \Box\mathrm{U}_a(x)$. We also have

$$\Box\varphi = \bigwedge\{\mathrm{U}_b(\Box x_b)\ ;\ b \in L\}.$$

Since $\mathrm{U}_b(x_b) \in X$, we have $v(\mathrm{U}_b(\Box x_b)) = 1$ for any $b \in L$, whence it follows that $v(\Box\varphi) = 1$. Thus, we have

$$v(\mathrm{U}_a(\Box x)) = v(\Box\mathrm{U}_a(x)) = 1,$$

which is a contradiction. Hence, we have $\mathrm{U}_a(x) \notin F$.

By the prime filter theorem for Boolean algebras, there is a prime filter P of $\mathcal{B}(A)$ such that $F \subset P$ and $\mathrm{U}_a(x) \notin P$. Define $v_P : A \to L$ as in Lemma 3.12 and then we have

$$v_P \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A,L).$$

Since $\mathrm{U}_a(x) \notin P$ and since $\mathrm{T}_1(\mathrm{U}_a(x)) = \mathrm{U}_a(x)$, it follows that

$$v_P(\mathrm{U}_a(x)) \neq 1, \text{ i.e., } v_P(x) \not\geq a.$$

To complete the proof, it remains to show that $vR_\Box v_P$. By using $X \subset P$, this follows from the fact that $v_P(\mathrm{U}_b(y)) = 1$ for any $\mathrm{U}_b(y) \in X$ (i.e., $v(\Box y) \geq b$ implies $v_P(y) \geq b$). □

By the above lemma we obtain the following proposition.

Proposition 3.14. $\mathbb{ISP}_{\mathbb{M}}(L)$ *satisfies the Kripke condition, i.e., for any* $(A,\Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$*, any* $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A,L)$*, and any* $x \in A$*, the following holds:*

$$v(\Box x) = \bigwedge\{u(x)\ ;\ vR_\Box u\}.$$

The above proposition plays an important role in establishing our duality result.

3.3 Category $\mathbf{RBS}_L$

For a Kripke frame (S, R) and $X \subset S$, define $R^{-1}[X] = \{w \in S\,;\, \exists w' \in X\ wRw'\}$. For $w \in S$, define $R[w] = \{w' \in S\,;\, wRw'\}$.

Definition 3.15. *We define a category $\mathbf{RBS}_L$. An object in $\mathbf{RBS}_L$ is a triple (S, α, R) such that (S, α) is an object in $\mathbf{BS}_L$ and that a binary relation R on S satisfies:*

1. *$R[w]$ is closed in S for any $w \in S$;*
2. *if $X \subset S$ is clopen in S, then $R^{-1}[X]$ is clopen in S;*
3. *for any $M \in \mathrm{SubAlg}(L)$, if $w \in \alpha(M)$ then $R[w] \subset \alpha(M)$.*

An arrow $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ in $\mathbf{RBS}_L$ is defined as an arrow $f : (S_1, \alpha_1) \to (S_2, \alpha_2)$ in $\mathbf{BS}_L$ which satisfies:

4. *if wR_1w' then $f(w)R_2f(w')$;*
5. *if $f(w_1)R_2w_2$ then there is $w' \in S_1$ such that w_1R_1w' and $f(w') = w_2$.*

In order to show a dual equivalence between the categories $\mathbf{ISP_M}(L)$ and $\mathbf{RBS}_L$, we introduce functors RSpec and MCont in the next subsection.

3.4 Functors RSpec and MCont

Definition 3.16. *We define a contravariant functor* $\mathrm{RSpec} : \mathbf{ISP_M}(L) \to \mathbf{RBS}_L$*. For an object $(A, \Box)$ in $\mathbf{ISP_M}(L)$, let*

$$\mathrm{RSpec}(A) = (\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L), \alpha_L, R_\Box)$$

where $R_\Box$ is defined in Definition 2.6. For an arrow $f : A \to B$ in $\mathbf{ISP_M}(L)$, define $\mathrm{RSpec}(f)$ *by*

$$\mathrm{RSpec}(f)(v) = v \circ f$$

for $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(B, L)$.

The well-definedness of RSpec is shown by the following two lemmas.

Lemma 3.17. *Let $(A, \Box) \in \mathbb{ISP}_{\mathbb{M}}(L)$. Then,* $\mathrm{RSpec}(A)$ *is an object in $\mathbf{RBS}_L$.*

Proof. By Theorem 3.9, $\mathrm{RSpec}(A)$ without $R_\Box$ is an object in $\mathbf{BS}_L$.

We first show that $\mathrm{RSpec}(A)$ satisfies item 1 in Definition 3.15. Let $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. Assume $u \notin R_\Box[v]$ for $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. It suffices to show that there is an open subset O of $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ such that

$$u \in O \text{ and } R_\Box[v] \cap O = \emptyset.$$

Since $u \notin R_\Box[v]$, there is $x_0 \in A$ such that $\exists a \in L\ (v(\Box x_0) \geq a$ and $u(x_0) \ngeq a)$. Then it follows from Lemma 3.4 and Lemma 3.5 that

$$u \in \langle \mathrm{U}_a(x_0) \to 0 \rangle \text{ and } R_\Box[v] \cap \langle \mathrm{U}_a(x_0) \to 0 \rangle = \emptyset.$$

We next show that $\mathrm{RSpec}(A)$ satisfies item 2 in Definition 3.15. Since $R_\Box^{-1}$ preserves unions of sets and since $\{\langle x \rangle\,;\, x \in A\}$ forms a base of the topology of $\mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ (note that it is closed under finite intersections), it suffices to show that $R_\Box^{-1}(\langle x\rangle)$ is clopen in S for any $x \in A$. We claim that

$$R_\Box^{-1}(\langle x\rangle) = \langle \neg\Box\neg \mathrm{T}_1(x)\rangle$$

where $\neg\varphi$ is the abbreviation of $\varphi \to 0$. Note that the right-hand side is clopen. To show the claim, we first assume $v \in \langle \neg\Box\neg \mathrm{T}_1(x)\rangle$. By Lemma 3.5, we have $v(\Box\neg \mathrm{T}_1(x)) = 0$. Then it follows from the Kripke condition that

$$0 = v(\Box\neg \mathrm{T}_1(x)) = \bigwedge\{u(\neg \mathrm{T}_1(x))\ ;\ vR_\Box u\}.$$

Since $u(\neg \mathrm{T}_1(x))$ is either 0 or 1 by Lemma 3.3 and Lemma 3.5, there is $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$ with $vR_\Box u$ such that $u(\neg \mathrm{T}_1(x)) = 0$. Then we have $u \in \langle x\rangle$. Therefore we conclude $v \in R_\Box^{-1}(\langle x\rangle)$. The converse is similarly proved by using the Kripke condition.

We finally show that $\mathrm{RSpec}(A)$ satisfies item 3 in Definition 3.15. Assume for contradiction that $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$ and $R_\Box[u] \setminus \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M) \neq \emptyset$ for $M \in \mathrm{SubAlg}(L)$. Then there is $v \in R_\Box[u] \setminus \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$, which means that $uR_\Box v$ and there is $z_0 \in A$ such that $v(z_0) \notin M$. Let $a = v(z_0)$. Then we have the following: for any $w \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$,

$$w(\mathrm{T}_a(z_0) \to z_0) = \begin{cases} 1 & \text{if } w(z_0) \neq a \\ a & \text{if } w(z_0) = a. \end{cases}$$

Thus it follows from the Kripke condition and $uR_\Box v$ that

$$u(\Box(\mathrm{T}_a(z_0) \to z_0)) = \bigwedge\{w(\mathrm{T}_a(z_0) \to z_0)\ ;\ uR_\Box w\} = a.$$

This contradicts $u \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, M)$, since $a \notin M$. Thus, $\mathrm{RSpec}(A)$ satisfies item 3. □

The following lemma is shown using the prime filter theorem for Boolean algebras.

Lemma 3.18. *For $(A_1, \Box_1), (A_2, \Box_2) \in \mathbb{ISP}_{\mathbb{M}}(L)$, let f be a modal homomorphism from $(A_1, \Box_1)$ to $(A_2, \Box_2)$. Then,* $\mathrm{RSpec}(f)$ *is an arrow in* $\mathbf{RBS}_L$.

Proof. By Theorem 3.9, $\mathrm{RSpec}(f)$ is an arrow in $\mathbf{BS}_L$. Thus it remains to show that $\mathrm{RSpec}(f)$ satisfies items 4 and 5 in Definition 3.15. We first verify item 4. For $v, u \in \mathrm{RSpec}(A_2)$, assume $(v, u) \in R_{\Box_2}$. Then it suffices to show that

$$(v \circ f, u \circ f) \in R_{\Box_1}.$$

To show this, suppose that $v \circ f(\Box_1 x) \geq a$ for $x \in A_1$ and $a \in L$. Then we have $v(\Box_2 f(x)) \geq a$. It follows from assumption that $u(f(x)) \geq a$. Thus we have $(v \circ f, u \circ f) \in R_{\Box_1}$.

We next verify item 5. Assume that $(\mathrm{RSpec}(f)(v), u) \in R_{\Box_1}$ for $v \in \mathrm{RSpec}(A_2)$ and $u \in \mathrm{RSpec}(A_1)$. Define

$$\begin{aligned} X_v &= \{\mathrm{U}_a(x)\ ;\ v(\Box_2 \mathrm{U}_a(x)) = 1\}; \\ X_u &= \{f(\mathrm{U}_a(x))\ ;\ u(\mathrm{U}_a(x)) = 1\}. \end{aligned}$$

Let $X = X_v \cup X_u$. We claim that X has the finite intersection property. Suppose for contradiction that X does not have the finite intersection property. Then, since by $\mathrm{U}_a(x) = \mathrm{U}_1(\mathrm{U}_a(x))$ we have

$$X_v = \{\mathrm{U}_1(x)\ ;\ v(\Box_2 \mathrm{U}_1(x)) = 1\} \text{ and } X_u = \{f(\mathrm{U}_1(x))\ ;\ u(\mathrm{U}_1(x)) = 1\}$$

and since U_1 distributes over $\wedge$, there are $\mathrm{U}_1(x), f(\mathrm{U}_1(y)) \in A_2$ such that $v(\Box_2\mathrm{U}_1(x)) = 1$, $u(\mathrm{U}_1(y)) = 1$, and $\mathrm{U}_1(x) \leq \neg f(\mathrm{U}_1(y))$ where $\neg\varphi$ is the abbreviation of $\varphi \to 0$. Then we have

$$\Box_2\mathrm{U}_1(x) \leq \Box_2\neg f(\mathrm{U}_1(y)) = f(\Box_2\neg\mathrm{U}_1(y)).$$

It follows from $v(\Box_2\mathrm{U}_1(x)) = 1$ that

$$v(f(\Box_2\neg\mathrm{U}_1(y))) = 1, \text{ i.e., } (\mathrm{RSpec}(f)(v))(\Box_2\neg\mathrm{U}_1(y)) = 1.$$

By assumption, we have $u(\neg\mathrm{U}_1(y)) = 1$, which contradicts $u(\mathrm{U}_1(y)) = 1$. Thus X has the finite intersection property. By the prime filter theorem for Boolean algebras, there is a prime filter P of $\mathcal{B}(A_2)$ such that $X \subset P$. Define $v_P : A_2 \to L$ as in Lemma 3.12 and then we have

$$v_P \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A_2, L).$$

It follows from $X_v \subset P$ that $vR_{\Box_2}v_P$. It follows from $X_u \subset P$ that $\mathrm{RSpec}(f)(v_P) = u$. This completes the proof. □

Thus we have shown that RSpec is well defined.

Definition 3.19. *We define a contravariant functor* $\mathrm{MCont} : \mathbf{RBS}_L \to \mathbf{ISP_M}(L)$. *For an object* (S, α, R) *in* $\mathbf{RBS}_L$, *define*

$$\mathrm{MCont}(S, \alpha, R) = (\mathrm{Cont}(S, \alpha), \Box_R)$$

(for the definition of $\Box_R$*, see Definition 2.1). For an arrow* $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ *in* $\mathbf{RBS}_L$*, define* $\mathrm{MCont}(f)$ *by*

$$\mathrm{MCont}(f)(g) = g \circ f$$

for $g \in \mathrm{Cont}(S_2, \alpha_2)$.

The well-definedness of MCont is shown by the following two lemmas.

Lemma 3.20. *Let* (S, α, R) *be an object in* $\mathbf{RBS}_L$. *Then,* $\mathrm{MCont}(S, \alpha, R)$ *is in* $\mathbb{ISP}_{\mathbb{M}}(L)$.

Proof. By Theorem 3.9, $\mathrm{MCont}(S, \alpha, R)$ without $\Box$ is in $\mathbb{ISP}(L)$. We first verify that $\Box_R$ is well defined on $\mathrm{MCont}(S, \alpha, R)$, i.e., if $f \in \mathrm{MCont}(S, \alpha, R)$ then $\Box_R f \in \mathrm{MCont}(S, \alpha, R)$. Let $f \in \mathrm{MCont}(S, \alpha, R)$. We then have the following: For $a \in L$,

$$(\Box_R f)^{-1}(a) = R^{-1}[(\mathrm{T}_a(f))^{-1}(1)] \cap (S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)]),$$

where note that $w \in R^{-1}[(\mathrm{T}_a(f))^{-1}(1)]$ means that there is $w' \in S$ such that wRw' and $f(w') = a$; and $w \in S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)]$ means that there is no $w \in S$ such that wRw' and $f(w') \not\geq a$. Since

$$R^{-1}[(\mathrm{T}_a(f))^{-1}(1)] \cap (S \setminus R^{-1}[(\mathrm{U}_a(f))^{-1}(0)])$$

is clopen in S, $\Box_R f$ is a continuous map from S to L. It follows from the condition 3 in Definition 3.15 that $\Box_R f$ is subspace-preserving. Thus we have $\Box_R f \in \mathrm{MCont}(S, \alpha, R)$, whence $\Box_R$ is well defined. It follows from the definition of $\Box_R$ that $\mathrm{MCont}(S, \alpha, R)$ is a subalgebra of a modal power L^S of L, whence we have $\mathrm{MCont}(S, \alpha, R) \in \mathbb{ISP}_{\mathbb{M}}(L)$. □

Lemma 3.21. *Let* $f : (S_1, \alpha_1, R_1) \to (S_2, \alpha_2, R_2)$ *be an arrow in* $\mathbf{RBS}_L$. *Then,* $\mathrm{MCont}(f)$ *is a modal homomorphism.*

Proof. By Theorem 3.9, $\mathrm{MCont}(f)$ is an arrow in $\mathbf{ISP}(L)$. It suffices to show that $\mathrm{MCont}(f)(\Box g_2) = \Box(\mathrm{MCont}(f)(g_2))$ for $g_2 \in \mathrm{Cont}(S_2, \alpha_2)$. Let $w_1 \in S_1$. Then, we have

$$(\mathrm{MCont}(f)(\Box g_2))(w_1) = \Box g_2 \circ f(w_1) = \bigwedge\{g_2(w_2)\ ;\ f(w_1)R_2 w_2\}.$$

Let a denote the rightmost side of the above equation. We also have

$$(\Box(\mathrm{MCont}(f)(g_2)))(w_1) = (\Box(g_2 \circ f))(w_1) = \bigwedge\{g_2(f(w'))\ ;\ w_1 R_1 w'\}.$$

Let b denote the rightmost side of the above equation. Since f satisfies item 4 in Definition 3.15, we have $a \leq b$. Since f satisfies item 5 in Definition 3.15, we have $a \geq b$. Hence we have $a = b$. □

Thus we have shown that MCont is well defined.

3.5 Topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$

In this subsection, we show a topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$, thus generalizing Jónsson-Tarski duality for modal algebras from the viewpoint of universal algebra.

Theorem 3.22. *Let $A \in \mathbb{ISP}_{\mathbb{M}}(L)$. Then, A is isomorphic to* $\mathrm{MCont} \circ \mathrm{RSpec}(A)$ *in the category* $\mathbf{ISP_M}(L)$*.*

Proof. Define $\varepsilon_A : A \to \mathrm{MCont} \circ \mathrm{RSpec}(A)$ by

$$\varepsilon_A(x)(v) = v(x)$$

for $x \in A$ and $v \in \mathrm{Hom}_{\mathbf{ISP}(L)}(A, L)$. It follows from Theorem 3.9 that ε_A is an isomorphism in $\mathbf{ISP}(L)$. Thus it remains to show that ε_A preserves $\Box$, i.e., $\varepsilon_A(\Box x) = \Box_{R_\Box}\varepsilon_A(x)$ for $x \in A$. For $v \in \mathrm{RSpec}(A)$, we have the following:

$$\begin{aligned}
(\Box_{R_\Box}\varepsilon_A(x))(v) &= \bigwedge\{\varepsilon_A(x)(u)\ ;\ vR_\Box u\} \\
&= \bigwedge\{u(x)\ ;\ vR_\Box u\} \\
&= v(\Box x) \quad \text{(by the Kripke condition)} \\
&= \varepsilon_A(\Box x)(v).
\end{aligned}$$

This completes the proof. □

Theorem 3.23. *Let (S, α, R) be an object in $\mathbf{RBS}_L$. Then, (S, α, R) is isomorphic to* $\mathrm{RSpec} \circ \mathrm{MCont}(S, \alpha, R)$ *in the category $\mathbf{RBS}_L$.*

Proof. Define $\eta_{(S,\alpha,R)} : (S, \alpha, R) \to \mathrm{RSpec} \circ \mathrm{MCont}(S, \alpha, R)$ by

$$\eta_{(S,\alpha,R)}(x)(f) = f(x)$$

for $x \in S$ and $f \in \mathrm{Cont}(S, \alpha)$. By Theorem 3.9, $\eta_{(S,\alpha,R)}$ is an isomorphism in the category $\mathbf{BS}_L$. Below, we denote $\eta_{(S,\alpha,R)}$ by η_S. We first show that, for any $w, w' \in S$, wRw' iff $\eta_S(w)R_{\Box_R}\eta_S(w')$. Recall that the right-hand side holds iff the following condition holds: $\forall a \in L \forall f \in \mathrm{Cont}(S, \alpha)(\eta_S(w)(\Box_R f) \geq a$ implies $\eta_S(w')(f) \geq a)$.

Assume that wRw'. We verify the above condition. Let $a \in L$ and $f \in \mathrm{Cont}(S, \alpha)$. Assume $\eta_S(w)(\Box_R f) \geq a$. Since

$$a \leq \eta_S(w)(\Box_R f) = (\Box_R f)(w) = \bigwedge\{f(z)\,;\, wRz\},$$

we have $\eta_S(w')(f) = f(w') \geq a$.

The converse is shown as follows. To prove the contrapositive, assume that $(w, w') \notin R$. It follows from Definition 3.15 that there is a clopen subset O of S such that $w' \in O$ and $R[w] \cap O = \emptyset$. Define $f : S \to L$ by $f(w) = 0$ for $w \in O$ and $f(w) = 1$ for $w \notin O$. Then we have $f \in \mathrm{Cont}(S, \alpha)$, $(\Box_R f)(w) = 1$, and $f(w') \neq 1$. Thus we have

$$\eta_S(w)(\Box_R f) \geq 1 \text{ and } \eta_S(w')(f) \not\geq 1,$$

whence the above condition does not hold.

It remains to show that η_S and η_S^{-1} satisfy item 5 in Definition 3.15. This follows immediately from the facts that wRw' iff $\eta_S(w) R_{\Box_R} \eta_S(w')$ and that η_S is bijective. □

Finally we obtain the modal semi-primal duality theorem.

Theorem 3.24. *The categories* $\mathbf{ISP_M}(L)$ *and* $\mathbf{RBS}_L$ *are dually equivalent via the functors* RSpec *and* MCont.

Proof. Let Id_1 denote the identity functor on $\mathbf{ISP_M}(L)$ and Id_2 denote the identity functor on $\mathbf{RBS}_L$. It is sufficient to show that there are natural isomorphisms $\varepsilon : \mathrm{Id}_1 \to \mathrm{MCont} \circ \mathrm{RSpec}$ and $\eta : \mathrm{Id}_2 \to \mathrm{RSpec} \circ \mathrm{MCont}$. For an L-**ML**-algebra A, define ε_A as in the proof of Theorem 3.22. For an object (S, α, R) in $\mathbf{RBS}_L$, define $\eta_{(S,\alpha,R)}$ as in the proof of Theorem 3.23. Then it is straightforward to verify that η and ε are natural transformations. It follows from Theorem 3.22 and Theorem 3.23 that η and ε are natural isomorphisms. □

The original Jónsson-Tarski duality can be recovered by letting L be the two-element Boolean algebra in the above theorem.

We have extended Keimel-Werner's semi-primal duality without modality:

$$\mathbf{ISP}(L) \simeq \mathbf{BS}_L{}^{\mathrm{op}}$$

to the duality with modality:

$$\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L{}^{\mathrm{op}}.$$

This was accomplished via the new notion of $\mathbb{ISP}_{\mathbb{M}}$, without which it would be difficult to obtain such a modalized analogue of the semi-primal duality theorem.

In the next section, we shall show how to describe the category $\mathbf{RBS}_L$ in terms of the theory of coalgebras, thus obtaining a coalgebraic description of the duality $\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L{}^{\mathrm{op}}$.

4 Coalgebraic Duality Theorem

Let us recall the definitions of coalgebra and its morphism (for the basics of coalgebras, we refer the reader to [3, 22]).

Definition 4.1. *Let* $\mathbf{C}$ *be a category and* T *an endofunctor on* $\mathbf{C}$*. A* T*-coalgebra is defined as a tuple* (C, δ) *for an object* C *in* $\mathbf{C}$ *and an arrow* $\delta : C \to \mathrm{T}(C)$ *in* $\mathbf{C}$*. For* T*-coalgebras* (C_1, δ_1) *and* (C_2, δ_2)*, a* T*-coalgebra morphism from* (C_1, δ_1) *to* (C_2, δ_2) *is defined as an arrow* $f : C_1 \to C_2$ *in* $\mathbf{C}$ *that satisfies* $\delta_2 \circ f = \mathrm{T}(f) \circ \delta_1$*. Then,* $\mathbf{Coalg}(\mathrm{T})$ *denotes the category of* T*-coalgebras and* T*-coalgebra morphisms.*

Let us recall the definition of Vietoris topology.

Definition 4.2. *Let* S *be a topological space,* $\mathcal{O}_S$ *the set of all open subsets of* S*, and* $\mathcal{C}_S$ *the set of all closed subsets of* S*. For a subset* U *of* S*, define*

$$B_S(U) = \{F \in \mathcal{C}_S \,;\, F \subset U\} \text{ and } D_S(U) = \{F \in \mathcal{C}_S \,;\, F \cap U \neq \emptyset\}.$$

The Vietoris space $\mathrm{V}(S)$ *of* S *is defined as a topological space whose underlying set is* $\mathcal{C}_S$ *and whose topology is generated by*

$$\{B_S(U) \,;\, U \in \mathcal{O}_S\} \cup \{D_S(U) \,;\, U \in \mathcal{O}_S\}.$$

Then we have the following proposition (see [34]).

Proposition 4.3. *If* S *is a Boolean space, then* $\mathrm{V}(S)$ *is a Boolean space whose topology is generated by the following set of clopen subsets of* $\mathrm{V}(S)$*:*

$$\{B_S(U) \,;\, U \in \mathcal{O}_S \cap \mathcal{C}_S\} \cup \{D_S(U) \,;\, U \in \mathcal{O}_S \cap \mathcal{C}_S\}.$$

We now introduce the concept of L-Vietoris functor.

Definition 4.4. *We define the* L*-Vietoris functor* $\mathrm{V}_L : \mathbf{BS}_L \to \mathbf{BS}_L$ *as follows. For an object* (S, α) *in* $\mathbf{BS}_L$*, define*

$$\mathrm{V}_L(S, \alpha) = (\mathrm{V}(S), \mathrm{V} \circ \alpha),$$

where, for $M \in \mathrm{SubAlg}(L)$*,* $\mathrm{V} \circ \alpha(M)$ $(= \mathrm{V}(\alpha(M)))$ *is the Vietoris space of a subspace* $\alpha(M)$ *of* S*. For an arrow* $f : (S, \alpha) \to (S', \alpha')$ *in* $\mathbf{BS}_L$*,* $\mathrm{V}_L(f) : (\mathrm{V}(S), \mathrm{V} \circ \alpha) \to (\mathrm{V}(S'), \mathrm{V} \circ \alpha')$ *is defined by*

$$\mathrm{V}_L(f)(F) = f(F) \ (= \{f(x) \,;\, x \in F\})$$

for $F \in \mathrm{V}(S)$*.*

The well-definedness of the L-Vietoris functor is shown by the following two lemmas. We use the notations of Definition 4.2 in the following proofs of them.

Lemma 4.5. *Let* (S, α) *be an object in* $\mathbf{BS}_L$*. Then,* $\mathrm{V}_L(S, \alpha)$ *is an object in* $\mathbf{BS}_L$*.*

Proof. By Proposition 4.3, $\mathrm{V}(S)$ is a Boolean space.

We show that for $M \in \mathrm{SubAlg}(L)$, $\mathrm{V} \circ \alpha(M)$ is a closed subspace of $\mathrm{V}(S)$. Since an element of $\mathrm{V} \circ \alpha(M)$ is of the form $F \cap \alpha(M)$ for $F \in \mathcal{C}_S$ and since by $\alpha(M) \in \mathcal{C}_S$ we have $F \cap \alpha(M) \in \mathcal{C}_S$ for $F \in \mathcal{C}_S$, $\mathrm{V} \circ \alpha(M)$ is a subset of $\mathrm{V}(S)$. Since for $U \in \mathcal{O}_S$ we have both

$$B_S(U) \cap \mathrm{V} \circ \alpha(M) = \{F \in \mathrm{V} \circ \alpha(M) \,;\, F \subset U\} = B_{\alpha(M)}(U \cap \alpha(M))$$

and

$$D_S(U) \cap \mathrm{V} \circ \alpha(M) = \{F \in \mathrm{V} \circ \alpha(M) \,;\, F \cap U \neq \emptyset\} = D_{\alpha(M)}(U \cap \alpha(M)),$$

$\mathrm{V} \circ \alpha(M)$ is a subspace of $\mathrm{V}(S)$. In order to show that $\mathrm{V} \circ \alpha(M)$ is closed in $\mathrm{V}(S)$, assume that $F \in \mathrm{V}(S)$ and $F \notin \mathrm{V} \circ \alpha(M)$. Then, there is $x \in F$ such that $x \notin \alpha(M)$. Since $\alpha(M)$ is closed in S, $D_S(S \setminus \alpha(M))$ is open in $\mathrm{V}(S)$. Moreover, we have

$$F \in D_S(S \setminus \alpha(M)) \text{ and } \mathrm{V} \circ \alpha(M) \cap D_S(S \setminus \alpha(M)) = \emptyset.$$

Hence, $\mathrm{V} \circ \alpha(M)$ is closed in $\mathrm{V}(S)$.

We next show that $\mathrm{V} \circ \alpha$ satisfies the three conditions in Definition 3.6. It follows from $\alpha(L) = S$ that $\mathrm{V} \circ \alpha(L) = \mathrm{V}(S)$. If $L_1 \subset L_2$ for $L_1, L_2 \in \mathrm{SubAlg}(L)$, then $\alpha(L_1) \subset \alpha(L_2)$ and, since $\alpha(L_1)$ is closed in $\alpha(L_2)$, we have $\mathrm{V} \circ \alpha(L_1) \subset \mathrm{V} \circ \alpha(L_2)$. Assume that $L_1 = L_2 \cap L_3$ for $L_1, L_2, L_3 \in \mathrm{SubAlg}(L)$. Then, we have

$$\mathrm{V} \circ \alpha(L_1) = \mathrm{V} \circ \alpha(L_2 \cap L_3) = \mathrm{V}(\alpha(L_2) \cap \alpha(L_3)).$$

An element of $\mathrm{V}(\alpha(L_2) \cap \alpha(L_3))$ is of the form

$$F \cap \alpha(L_2) \cap \alpha(L_3)$$

for $F \in \mathcal{C}_S$. An element of $\mathrm{V}(\alpha(L_2)) \cap \mathrm{V}(\alpha(L_3))$ is of the form

$$(F_1 \cap \alpha(L_2)) \cap (F_2 \cap \alpha(L_3))$$

for $F_1, F_2 \in \mathcal{C}_S$, which follows from the fact that for $X \subset S$ we have

$$\exists F_1, F_2 \in \mathcal{C}_S \; X = F_1 \cap \alpha(L_2) = F_2 \cap \alpha(L_3) \;\Leftrightarrow\; \exists F_1, F_2 \in \mathcal{C}_S \; X = (F_1 \cap \alpha(L_2)) \cap (F_2 \cap \alpha(L_3)).$$

Hence we have $\mathrm{V}(\alpha(L_2) \cap \alpha(L_3)) = \mathrm{V}(\alpha(L_2)) \cap \mathrm{V}(\alpha(L_3))$ and so $\mathrm{V} \circ \alpha(L_1) = \mathrm{V} \circ \alpha(L_2) \cap \mathrm{V} \circ \alpha(L_3)$. □

Lemma 4.6. *Let $f : (S, \alpha) \to (S', \alpha')$ be an arrow in $\mathbf{BS}_L$. Then, $\mathrm{V}_L(f)$ is an arrow in $\mathbf{BS}_L$.*

Proof. Since f is a continuous map between Boolean spaces, it follows from [13, Theorem 3.1.8] that $\mathrm{V}_L(f)$ maps a closed subset of S to a closed subset of S'. In order to show that $\mathrm{V}_L(f)$ is continuous, let $U \in \mathcal{O}_{S'}$. Then we have

$$\mathrm{V}_L(f)^{-1}(B_{S'}(U)) = \{F \in \mathcal{C}_S \; ; \; f(F) \subset U\} = \{F \in \mathcal{C}_S \; ; \; F \subset f^{-1}(U)\} = B_S(f^{-1}(U))$$

and also

$$\mathrm{V}_L(f)^{-1}(D_{S'}(U)) = \{F \in \mathcal{C}_S \; ; \; f(F) \cap U \neq \emptyset\} = \{F \in \mathcal{C}_S \; ; \; F \cap f^{-1}(U) \neq \emptyset\} = B_S(f^{-1}(U)).$$

Thus, $\mathrm{V}_L(f)$ is continuous. It remains to show that $\mathrm{V}_L(f)$ is subspace-preserving. Assume that $F \in \mathrm{V} \circ \alpha(M)$ for $M \in \mathrm{SubAlg}(L)$. Then we have $F \subset \alpha(M)$. Since f is subspace-preserving, we have $f(F) \subset \alpha'(M)$. Thus it follows that

$$\mathrm{V}_L(f)(F) = f(F) \subset \alpha'(M).$$

Hence we have $\mathrm{V}_L(f)(F) \in \mathrm{V} \circ \alpha'(M)$. □

In order to show that $\mathbf{Coalg}(\mathrm{V}_L)$ is isomorphic to $\mathbf{RBS}_L$, we introduce two functors RC and CR between the two categories.

Definition 4.7. *A functor* RC : $\mathbf{RBS}_L \to \mathbf{Coalg}(\mathrm{V}_L)$ *is defined as follows. For an object* (S, α, R) *in* $\mathbf{RBS}_L$*,* $\mathrm{RC}(S, \alpha, R)$ *is defined as a* V_L*-coalgebra*

$$((S, \alpha), R[\text{-}])$$

where $R[\text{-}] : (S, \alpha) \to \mathrm{V}_L(S, \alpha)$ *is defined by* $R[x] = \{y \in S\ ;\ xRy\}$ *for* $x \in S$*. For an arrow* f *in* $\mathbf{RBS}_L$*, define* $\mathrm{RC}(f) = f$*.*

In the above definition, $\mathrm{RC}(S, \alpha, R)$ is a V_L-coalgebra, since $R[\text{-}] : (S, \alpha) \to \mathrm{V}_L(S, \alpha)$ is an arrow in $\mathbf{BS}_L$ by items 1, 2 and 3 in Definition 3.15 and by Proposition 4.3. It is straightforward to verify that $\mathrm{RC}(f)$ is an arrow in $\mathbf{Coalg}(\mathrm{V}_L)$ for an arrow f in $\mathbf{RBS}_L$. Thus, the functor RC is well defined.

Definition 4.8. *A functor* CR : $\mathbf{Coalg}(\mathrm{V}_L) \to \mathbf{RBS}_L$ *is defined as follows. For an object* $((S, \alpha), \delta)$ *in* $\mathbf{Coalg}(\mathrm{V}_L)$*, define*

$$\mathrm{CR}((S, \alpha), \delta) = (S, \alpha, R_\delta)$$

where a binary relation R_δ *on* S *is defined by*

$$xR_\delta y \Leftrightarrow y \in \delta(x)$$

for $x, y \in S$*. For an arrow* f *in* $\mathbf{Coalg}(\mathrm{V}_L)$*, define* $\mathrm{CR}(f) = f$*.*

The well-definedness of the functor CR is shown by the following lemma.

Lemma 4.9. *For an object* $((S, \alpha), \delta)$ *in* $\mathbf{Coalg}(\mathrm{V}_L)$*,* $\mathrm{CR}((S, \alpha), \delta)$ *is an object in* $\mathbf{RBS}_L$*.*

Proof. It suffices to show that (S, α, R_δ) satisfies the three conditions in Definition 3.15. First, for $x \in S$ we have $R_\delta[x] = \delta(x) \in \mathrm{V}(S)$ and so $R_\delta[x]$ is a closed subset of $\mathrm{V}(S)$. Second, for a clopen subset O of S, the following holds:

$$\begin{aligned} R_\delta^{-1}[O] &= \{x \in S\ ;\ \exists y \in O\ xR_\delta y\} = \{x \in S\ ;\ \exists y \in O\ y \in \delta(x)\} \\ &= \{x \in S\ ;\ O \cap \delta(x) \neq \emptyset\} = \{x \in S\ ;\ \delta(x) \in D_S(O)\} \\ &= \delta^{-1}(D_S(O)). \end{aligned}$$

Since O is clopen in S, $D_S(O)$ is clopen in $\mathrm{V}(S)$ by Proposition 4.3. Thus, since δ is continuous, $R_\delta^{-1}[O]$ is clopen. Since S is a Boolean space, this implies that R_δ is a continuous map from S to $\mathrm{V}(S)$. Third, if $x \in \alpha(M)$ for $M \in \mathrm{SubAlg}(L)$, then we have $R_\delta[x] = \delta(x) \in \mathrm{V} \circ \alpha(M)$ (recall that δ is subspace-preserving by definition). This completes the proof. ☐

It is straightforward to verify that $\mathrm{CR}(f)$ is an arrow in $\mathbf{RBS}_L$ for an arrow f in $\mathbf{Coalg}(\mathrm{V}_L)$. Thus we obtain the following theorem.

Theorem 4.10. *The categories* $\mathbf{Coalg}(\mathrm{V}_L)$ *and* $\mathbf{RBS}_L$ *are isomorphic via* CR *and* RC*.*

Proof. Clearly we have $\mathrm{CR} \circ \mathrm{RC}(f) = f$ for an arrow f in $\mathbf{RBS}_L$ and $\mathrm{RC} \circ \mathrm{CR}(f) = f$ for an arrow f in $\mathbf{Coalg}(\mathrm{V}_L)$. Let (S, α, R) be an object in $\mathbf{RBS}_L$. Then we have:

$$xR_{R[\text{-}]}y \ \Leftrightarrow\ y \in R[x] \ \Leftrightarrow\ xRy.$$

Thus, (S, α, R) is exactly the same as $\mathrm{CR} \circ \mathrm{RC}(S, \alpha, R)$. Let $((S, \alpha), \delta)$ be an object in $\mathbf{Coalg}(\mathrm{V}_L)$. Then we have:

$$y \in \delta_{R_\delta}(x) \ \Leftrightarrow\ xR_\delta y \ \Leftrightarrow\ y \in \delta(x).$$

Thus, $((S, \alpha), \delta)$ is exactly the same as $\mathrm{RC} \circ \mathrm{CR}((S, \alpha), \delta)$. ☐

By Theorem 3.24 and Theorem 4.10, we obtain the following coalgebraic duality theorem, which generalizes Abramsky-Kupke-Kurz-Venema duality for modal algebras from the viewpoint of universal algebra.

Theorem 4.11. *The categories* $\mathbf{ISP_M}(L)$ *and* $\mathbf{Coalg}(\mathrm{V}_L)$ *are dually equivalent.*

Thus, the modal semi-primal duality $\mathbf{ISP_M}(L) \simeq \mathbf{RBS}_L{}^{\mathrm{op}}$ can be described in terms of the theory of coalgebras. Abramsky-Kupke-Kurz-Venema duality can be recovered by letting L be the two-element Boolean algebra in the above theorem.

Since $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ coincides with the class of all $\mathcal{MMV}_n$-algebras and since $\mathbf{n}$ forms a semi-primal algebra with a lattice reduct, the above theorem gives us a coalgebraic duality for $\mathcal{MMV}_n$-algebras (i.e., algebras of Łukasiewicz n-valued modal logic):

Corollary 4.12. *The category of* $\mathcal{MMV}_n$*-algebras and their homomorphisms is dually equivalent to* $\mathbf{Coalg}(\mathrm{V}_\mathbf{n})$.

In a similar way, we obtain a coalgebraic duality for L-**ML**-algebras (i.e., algebras of a version of Fitting's many-valued modal logic). We remark that [30, Lemma 2.6] is useful when proving that $\mathbf{n}$ is semi-primal (this can be shown in a similar way to [32, Lemma 2.3] via [30, Lemma 2.6]).

With the help of Corollary 2.5 and Theorem 4.11, we obtain the following.

Corollary 4.13. $\mathbf{Coalg}(\mathrm{V}_L)$ *has cofree coalgebras.*

Since $\mathbf{ISP_M}(L)$ has the initial algebra, we obtain the final coalgebra theorem for V_L.

Corollary 4.14. *The endofunctor* V_L *has a final coalgebra.*

If L is not only semi-primal but also primal (for its definition, see [10]), then by Hu theorem (see [10, Theorem 4.1.1]) $\mathbf{ISP}(L)$ is dually equivalent to the category of Boolean spaces (i.e., $\mathbf{BS}_2$), whence $\mathbf{ISP_M}(L)$ is dually equivalent to the category of descriptive general frames (i.e., $\mathbf{RBS}_2$) and is also dually equivalent to the category of Stone coalgebras (i.e., $\mathbf{Coalg}(\mathrm{V}_2)$).

This implies that if L and L' are primal then the categories $\mathbf{ISP_M}(L)$ and $\mathbf{ISP_M}(L')$ are equivalent. More generally, since the definition of $\mathbf{RBS}_L$ depends only on the order structure of subalgebras of L, Theorem 3.24 gives us the following.

Corollary 4.15. *If* L *and* L' *are semi-primal algebras with lattice reducts and if* $\mathrm{SubAlg}(L)$ *and* $\mathrm{SubAlg}(L')$ *are order isomorphic, then the categories* $\mathbf{ISP_M}(L)$ *and* $\mathbf{ISP_M}(L')$ *are equivalent.*

Similarly, if L and L' are semi-primal and if $\mathrm{SubAlg}(L)$ and $\mathrm{SubAlg}(L')$ are order isomorphic, then the categories $\mathbf{Coalg}(\mathrm{V}_L)$ and $\mathbf{Coalg}(\mathrm{V}_{L'})$ are equivalent. Note that if L and L' are primal, then $\mathrm{SubAlg}(L)$ and $\mathrm{SubAlg}(L')$ are always order isomorphic.

5 Conclusions and Future Work

We have introduced the new notion of $\mathbb{ISP}_{\mathbb{M}}$ and extended the theory of natural dualities so as to encompass Jónsson-Tarski duality and Abramsky-Kupke-Kurz-Venema duality for the class of all modal algebras, which are becoming more and more important in coalgebraic logic. Whereas $\mathbb{ISP}(M)$ cannot be the class of all modal algebras, crucially, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ coincides with the class of all modal algebras, and furthermore, there are similar facts for algebras of many-valued modal logics.

$\mathbb{ISP}_{\mathbb{M}}$ thus provides a natural framework for the universal algebra of modalities, and as such, for the theory of modal natural dualities. From a technical point of view, our starting point was Keimel-Werner's semi-primal duality for $\mathbb{ISP}(L)$ in natural duality theory. Having shifted our focus from $\mathbb{ISP}(L)$ to $\mathbb{ISP}_{\mathbb{M}}(L)$, we verified the Kripke condition for $\mathbb{ISP}_{\mathbb{M}}(L)$ where L is a semi-primal algebra with a bounded lattice reduct. The Kripke condition is completeness in logical terms, and we needed a weaker form of the axiom of choice for the verification of it. As main results, we obtained topological and coalgebraic dualities for $\mathbb{ISP}_{\mathbb{M}}(L)$ with three kinds of applications of them: coalgebraic dualities for many-valued modal logics; the existence of a final coalgebra and cofree coalgebras in $\mathbf{Coalg}(\mathrm{V}_L)$; and a criterion for the equivalence of categories of the form $\mathbf{ISP_M}(L)$.

We conclude the paper by mentioning several future directions of research and by placing Stone duality in a wider context of the interaction between mathematics and philosophy.

Firstly, it would be fruitful to generalize the notion of $\mathbb{ISP}_{\mathbb{M}}$ from the viewpoint of coalgebraic logic, since a number of modal logics (e.g., monotone modal logic and graded modal logic) can be described in coalgebraic terms. This is expected to allow us to develop natural duality theory for coalgebraic modal logics. Now, how can we generalize $\mathbb{ISP}_{\mathbb{M}}$ to a coalgebraic-logical setting? Let us begin with an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ and fix a function $\heartsuit : T(L^n) \to L$ where $n \in \omega$ and L is an algebra (possibly with some conditions on $T, \heartsuit, L$). Then, given a T-coalgebra $\delta : X \to T(X)$, we can define an n-ary modal operation on $\mathrm{Hom}_{\mathbf{Set}}(X, L)$ by

$$f \in \mathrm{Hom}_{\mathbf{Set}}(X, L)^n \mapsto \heartsuit \circ T(f) \circ \delta \in \mathrm{Hom}_{\mathbf{Set}}(X, L)$$

where f is considered as an element of $\mathrm{Hom}_{\mathbf{Set}}(X, L^n)$ via the isomorphism $\mathrm{Hom}_{\mathbf{Set}}(X, L)^n \simeq \mathrm{Hom}_{\mathbf{Set}}(X, L^n)$. We can recover $\Box_R : \mathrm{Hom}_{\mathbf{Set}}(X, L) \to \mathrm{Hom}_{\mathbf{Set}}(X, L)$ in Definition 2.1 by letting $T : \mathbf{Set} \to \mathbf{Set}$ be the power-set functor and $\heartsuit : T(L) \to L$ the meet operation of L. Thus, this yields an extended notion of modal power parametrized by T and $\heartsuit$, and hence a generalization of $\mathbb{ISP}_{\mathbb{M}}(L)$ from the viewpoint of coalgebraic logic. In future work, we will attempt to develop natural duality theory for this coalgebraic generalization of $\mathbb{ISP}_{\mathbb{M}}(L)$.

Another important direction of research would be to establish an intuitionistic analogue of the theory presented in this paper, which involves a universal-algebraic generalization of Esakia duality for Heyting algebras. First of all, the class of all Heyting algebras cannot be expressed as $\mathbb{ISP}(L)$ for any single algebra L. As already mentioned in Section 1, this is nothing but a duality-theoretic expression of the reason why Gödel failed to capture intuitionistic logic as a many-valued logic. Hence we have to consider a new way to generate a class of algebras. Given an intuitionistic frame $(X, \leq)$, we can define an implication operation $\to: \mathrm{Hom}_{\mathbf{Set}}(X, L)^2 \to \mathrm{Hom}_{\mathbf{Set}}(X, L)$ by

$$(f \to g)(x) = \bigwedge\{f(y) \to g(y)\ ;\ x \leq y\}.$$

In this way, we obtain the concept of an intuitionistic power of L and so the concept of $\mathbb{ISP}_{\mathbb{I}}(L)$ i.e., the class of isomorphic copies of subalgebras of intuitionistic powers of L. We can show that, for the two-element distributive lattice $\mathbf{2}$, $\mathbb{ISP}_{\mathbb{I}}(\mathbf{2})$ coincides with the class of all Heyting algebras and that, for $\mathbf{n}$ without $\to$ or $\neg$, $\mathbb{ISP}_{\mathbb{I}}(\mathbf{n})$ coincides with the class of all algebras of intuitionistic Łukasiewicz n-valued logic (which is naturally defined via n-valued Kripke semantics). In future work, we will attempt to develop natural duality theory for $\mathbb{ISP}_{\mathbb{I}}(L)$, in order to make it possible to incorporate Esakia duality for Heyting algebras into the theory of natural dualities. At the same time, however, we have to remark that there is a different perspective on intuitionistic logic, i.e., we can see it as distributive lattices with residuation or the right adjoints of meets. This point of view leads us to the notion of $\mathbb{IS}_{\mathbb{R}}\mathbb{P}$, and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2})$ coincides with the class of all Heyting algebras. Although we do not describe a precise definition here, $\mathbb{S}_{\mathbb{R}}(M)$ is the class of "residuated" subalgebras of a given ordered algebra M. Interestingly, it does not hold in general that $\mathbb{ISP}_{\mathbb{I}}(M) = \mathbb{IS}_{\mathbb{R}}\mathbb{P}(M)$. Hence, the two perspectives on intuitionistic logic (i.e., the former, Kripke-semantics-based one and the latter, residuation-based one) are really different in that sense.

While natural duality theory is based on universal algebra and general topology (possibly with relational structures), which are of set-theoretical character, we can also develop duality theory building upon category theory, especially categorical algebra and categorical topology (see [4]). Because universal algebra is well developed for finitary algebras (though not for infinitary ones), we consider that natural duality theory is suitable for "finitary Stone-type dualities", by which we mean Stone-type dualities concerning finitary operations and so compact spectrums. On the other hand, the theory of monads, which is categorical universal algebra, naturally encompass infinitary algebras such as frames (or locales) and continuous lattices (both are the Eilenberg-Moore algebras of certain monads). Accordingly, categorical duality theories (see, e.g., [33, 39]) seem suitable for "infinitary Stone-type dualities", a typical example of which is Isbell-Papert's dual adjunction between frames and topological spaces. Note that finitary Stone-type dualies often require a weaker form of the axiom of choice, whereas infinitary ones sometimes avoid such a non-deterministic principle, as is the case in Isbell-Papert duality or duality between point-free spaces and point-set spaces in general (see [31, 33]).

Categorical duality theories (including those cited above) are usually more general than natural duality theory, subsuming both finitary and infinitary ones. At the same time, however, they are less substantial than natural duality theory, especially in the sense that they often lack the "adequate" treatment of dual equivalences. Category theory can lead us to dual adjunctions in a significant way, but not to dual equivalences. Although there is a mechanical way to derive equivalences from adjunctions, it is quite trivial, and, at the moment, there appears to be no general, substantial way to do it categorically as [39, p.102] says (roughly, categories $\mathbf{A}_i$ and $\mathbf{B}_i$ below amount to trivial descriptions of a dual equivalence derived from a given dual adjunction):

> The main task for establishing a duality in a concrete situation is now to identify $\mathbf{A}_i$ and $\mathbf{B}_i$. This can be a very hard problem, and this is where categorical guidance comes to an end.

The real issue thus lies in providing substantial characterizations of $\mathbf{A}_i$ and $\mathbf{B}_i$. In contrast to this situation in categorical duality theories, natural duality theory does yield non-trivial descriptions of $\mathbf{A}_i$ and $\mathbf{B}_i$ involved, thus revealing how dual equivalences can be developed in various concrete situations. We consider that this is an important strength of natural duality theory, gained by restricting its scope more than categorical duality theories. By focusing on less general situations, natural duality theory succeeds in giving a more nuanced understanding of Stone duality.

Finally, we briefly touch upon the fundamental question: why do we study Stone duality (in a wider sense) at all? Stone-type dualites are theoretically elegant, and there would be no doubt that they are highly beneficial in practice, since they have indeed had numerous applications in logic, mathematics, and computer science. This is not what we really want to say here, however. Facing the question, we dare to say that Stone duality is duality between human knowledge and the reality of the world, or duality between epistemology and ontology, the two fundamental disciplines of philosophy. This nature of Stone duality is particularly striking in the case of duality between point-set spaces and point-free spaces, as points are ontological ingredients of the notion of space, and regions (or properties of space) are its epistemological ingredients (for more details, see [33]).

The idea of Stone duality as duality between ontology and epismotelogy is not merely a philosophical doctrine, but also a crucial notion lurking behind practical applications of Stone duality. For example, the main idea of [1] was to see Stone duality as duality between observable properties and denotational meanings of computational processes. Obviously, observable properties of computational processes are human knowledge in the context of computer science, and their denotational meanings are a matter of reality and not that of human knowledge (of course, computational processes are the "world" in computer science; we do not necessarily mean this real world by "world").

Duality between algebras and coalgebras, including those relevant to this paper, may also be considered as an expression of duality between the epistemological and the ontological, via the idea of coalgebraic logic that coalgebras represent some sort of systems (e.g., computer systems) and algebras the (observable) properties of them. Here recall that usually we can only know about computer systems through their (observable) properties; evidently, the former is on the side of reality, and the latter on the side of our knowledge. Broadly speaking, most Stone-type dualities in mathematical logic are expressions of duality between syntax and semantics, which is in turn a specific kind of duality between the epistemological and the ontological.

Such a dichotomy (or duality) between epistemological and ontological things or perspectives can actually be observed in a much broader context, and so is the relation of the epistemological with the ontological. We mention only one case here. Kitaro Nishida, a philosopher of the Kyoto School, considered experience as having a person, rather than a person as having experience, saying (see [36]):

> Over time I came to realize that it is not that experience exists because there is an individual, but that an individual exists because there is experience.

That is, a person is (or at least may be identified with) a bundle of experiences, which is conceived of as being more primary than the notion of a person, in the Nishida philosophy. Its family resemblance to point-free geometry could be clarified in analogy with the leading idea of point-free geometry that a point is a bundle of shrinking regions (or certain properties of space).

Philosophical dichotomies can evolve into mathematical (categorical) dualities, as the case of point-free geometry shows. Indeed, foundational ideas of point-free geometry were first proposed by philosophers including Whitehead and Husserl, and then they were implemented in mathematical fashions, giving rise to categorical dualites between point-free spaces and point-set spaces as mentioned above. It would thus be fruitful to pursuit categorical dualities corresponding to given philosophical dichotomies, which may even have practical impacts as Stone duality was applied to computer science.

With such evidence in mind, we believe that Stone duality can form a sgnificant theme of philosophy as well as mathematics. From the viewpoint of the history of ideas, it would also be worth noting that the 20th century was the time when the emphasis drastically shifted from the ontological to the epistemological in diverse disciplines, ranging from mathematics (e.g., noncommutative geometry), to physics (e.g., algebraic quantum field theory), to computer science (e.g., domain theory in logical form; logic in general is of such nature), and to philosophy (e.g., the theory of meaning; we wonder if we could add phenomenology here).

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