FUNDAMENTA
MATHEMATICAE
Online First version

The universal factorial Hall–Littlewood P- and Q-functions

by

Masaki Nakagawa (Okayama) and **Hiroshi Naruse** (Kofu)

Dedicated to the memory of Professor Piotr Pragacz

Abstract. We introduce *factorial* analogues of the ordinary Hall–Littlewood P- and Q-polynomials, which we call the *factorial Hall–Littlewood P- and Q-polynomials*. Using the *universal* formal group law, we further generalize these polynomials to the *universal factorial Hall–Littlewood P- and Q-functions*. We show that these functions satisfy the *vanishing property* which the ordinary factorial Schur S-, P-, and Q-polynomials have. By the vanishing property, we derive the Pieri-type formula and a certain generalization of the classical hook formula. We then characterize our functions in terms of Gysin maps from flag bundles in complex cobordism theory. Using this characterization and Gysin formulas for flag bundles, we obtain generating functions for the universal factorial Hall–Littlewood P- and Q-functions. Using our generating functions, we show that our factorial Hall–Littlewood P- and Q-polynomials have a certain *cancellation property*. Further applications such as Pfaffian formulas for K-theoretic factorial Q-polynomials are also given.

1. Introduction. Let $\boldsymbol{x}_n = (x_1, \ldots, x_n)$ and t be independent indeterminates over $\mathbb{Z}$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition of length $\leq n$. Then the ordinary *Hall–Littlewood P-* and *Q-polynomials*, denoted by $P_\lambda(\boldsymbol{x}_n; t)$ and $Q_\lambda(\boldsymbol{x}_n; t)$ respectively, are symmetric polynomials with coefficients in $\mathbb{Z}[t]$. When $t = 0$, both $P_\lambda(\boldsymbol{x}_n; t)$ and $Q_\lambda(\boldsymbol{x}_n; t)$ reduce to the ordinary Schur S-polynomial $s_\lambda(\boldsymbol{x}_n)$, and when $t = -1$, to the ordinary Schur P-polynomial $P_\lambda(\boldsymbol{x}_n)$ and Q-polynomial $Q_\lambda(\boldsymbol{x}_n)$ respectively. Thus the polynomials $P_\lambda(\boldsymbol{x}_n; t)$, $Q_\lambda(\boldsymbol{x}_n; t)$ serve to interpolate between the Schur polynomials and the Schur P- and Q-polynomials, and play a crucial role in the symmetric function theory, representation theory, and combinatorics. In the context

2020 *Mathematics Subject Classification*: Primary 05E05; Secondary 55N20, 55N22, 57R77.
Key words and phrases: factorial Hall–Littlewood P- and Q-functions, generating functions, formal group laws, complex cobordism theory, Gysin formulas.
Received 17 November 2022; revised 15 May 2023.
Published online 16 August 2023.

DOI: 10.4064/fm257-5-2023

of *Schubert calculus*, it is well-known that the ordinary Schur S-, P-, and Q-polynomials appear as the Schubert classes in the ordinary cohomology rings of the various Grassmannians (Fulton [7, §9.4], Fulton–Pragacz [8, Chapters II and III], Pragacz [31, §6]). Moreover, their *factorial* analogues, the factorial Schur S-, P-, and Q-polynomials, play an analogous role in *equivariant* Schubert calculus (Knutson–Tao [17], Ikeda [11], Ikeda–Naruse [13]).

As for the Hall–Littlewood polynomials, it is known that there are some geometric or representation-theoretic interpretations of them related to flag varieties or flag bundles (the readers are referred to e.g. De Concini–Procesi [5], Garsia–Procesi [9], Pragacz [32]). In the context of Schubert calculus, there seems to be no obvious geometric meaning of the Hall–Littlewood polynomials at present. In [35], Totaro considered the *coinvariant ring* $F(e,n)$ of the complex reflection group $G(e,1,n) = \mathbb{Z}/e\mathbb{Z} \wr S_n$ (the wreath product) for $e \geq 2$, and suggested thinking of $F(e,n)$ as the cohomology of a certain "flag manifold". He also considered a subring $C(e,n)$ of $F(e,n)$, and described a basis for the ring $C(e,n)$ given by the Hall–Littlewood Q-polynomials. For $e = 2$, $C(2,n) \subset F(2,n)$ is the inclusion of the cohomology of the Lagrangian Grassmannian in that of the isotropic flag manifold of the symplectic group, and Totaro's result is interpreted as a generalization of the classical result in Schubert calculus for Lagrangian Grassmannians (Józefiak [16], Pragacz [31, §6]). It is natural to consider a generalization of the above theory to the *double coinvariant rings* (or *equivariant coinvariant rings*) of complex reflection groups (cf. recent work of McDaniel [20]). From a geometric or topological point of view, one expects that these rings would be related to *torus-equivariant cohomology* of certain "flag manifolds", and factorial versions of the Hall–Littlewood polynomials would play a crucial role.

Moreover, we notice that all the results stated above are formulated in the ordinary cohomology theory $H^*(-)$. In topology, it is classical that a *complex-oriented generalized cohomology theory* $h^*(-)$ gives rise to a *formal group law* $F^h(u,v)$ over the coefficient ring $h^* := h^*(\mathrm{pt})$, where pt is a single point. Three typical examples are the ordinary cohomology theory $H^*(-)$, the (topological) complex K-theory $K^*(-)$, and the complex cobordism theory $MU^*(-)$, which correspond to the *additive* formal group law $F_a(u,v) = u + v$, the *multiplicative* formal group law $F_m(u,v) = u \oplus v = u + v - \beta uv$, and the *universal* formal group law $F_{\mathbb{L}}(u,v) = u +_{\mathbb{L}} v$, respectively. By the classical result of Quillen [34, Proposition 1.10], complex cobordism theory is *universal* among all complex-oriented generalized cohomology theories. Therefore it is also quite natural to ask whether one can generalize the above results formulated in the ordinary cohomology theory to complex cobordism theory.

Motivated by these facts and the preceding results, we introduce *factorial* and *universal* analogues of the ordinary Hall–Littlewood P- and Q-polynomials, which we call the *universal factorial Hall–Littlewood P- and Q-functions* (for notation, see §2.1):

DEFINITION 1.1 (Definition 3.1, cf. Naruse [28]). For a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers with $r \le n$, we define

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right],$$

$$HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[[[\boldsymbol{x}; t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right].$$

To the best of our knowledge, even a factorial version of the ordinary Hall–Littlewood polynomials has not appeared in the literature. Here we emphasize the importance of these factorial Hall–Littlewood polynomials. In fact, they will be needed in describing the torus-equivariant cohomology of the *p-compact flag variety* corresponding to $G(e, 1, n)$ (cf. recent work of Ortiz [30]). In this context, the "deformation parameters" $\boldsymbol{b}$ are interpreted as the torus-equivariant parameters. We will discuss this new aspect of the Hall–Littlewood functions in more detail in our forthcoming paper [27].

We will show that our factorial Hall–Littlewood P- and Q-functions have the so-called *vanishing property* (see Propositions 3.7, 3.8). This property will be useful in the so-called *GKM description* of the torus-equivariant cohomology ring of the p-compact flag variety corresponding to $G(e, 1, n)$ ([27]). By the vanishing property, we can derive a Pieri-type formula for factorial Hall–Littlewood P-polynomials (see Proposition 3.9). Moreover, by a simple recursive argument based on the associativity of factorial Hall–Littlewood P-polynomials, we can derive a certain generalization of the hook formula (see Proposition 3.10). We then give a characterization of them in terms of Gysin maps from full flag bundles in complex cobordism theory (Proposition 3.5). Using this characterization, we derive *generating functions* for the universal factorial Hall–Littlewood P- and Q-functions. The idea of getting our result is to apply the Gysin formula for a projective bundle repeatedly to the full flag bundle since a full flag bundle is constructed as a sequence of projective bundles. However, the existence of the deformation parameter $\boldsymbol{b} = (b_1, b_2, \ldots)$ precludes a direct application of the Gysin formula. To circumvent this difficulty, we develop a specific modification in each step (for details, see §4.1). Then, by a careful argument, we succeed in getting the required result. To state it, we need some notation from §§2.1, 2.2, and 4.1: For a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers with $r \le n$, we set

$$
\begin{aligned}
\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}) &:= \frac{u_i}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)} \\
&\times \left(\prod_{j=1}^{n} \frac{u_i +_{\mathbb{L}} [t](\overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j} \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)} \prod_{j=1}^{\lambda_i} \frac{u_i +_{\mathbb{L}} b_j}{u_i} - t^{n-i+1} \prod_{j=1}^{\lambda_i} \frac{b_j}{u_i} \right),
\end{aligned}
$$

$$
\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b}) := \prod_{i=1}^{r} \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}).
$$

Then our main result is as follows:

THEOREM 1.2 (Theorem 4.3). *For a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers with $r \le n$, the universal factorial Hall–Littlewood P-function $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1} \cdots u_r^{-\lambda_r}$ in $\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b})$:*

$$
HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}](\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})).
$$

Using a similar, but simpler technique, we can also obtain a generating function for $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$ (see Theorem 4.5). Here we stress the usefulness of the technique of generating functions. For instance, it is easy to derive Pfaffian formulas for factorial K-theoretic Q-polynomials in a simple and uniform manner (see Theorem 5.3). Moreover, a certain *cancellation property* (cf. Pragacz [31, §2]) of the factorial Hall–Littlewood P- and Q-polynomials can be verified immediately (see Proposition 5.1). For further applications of generating functions to obtain the so-called Pieri rule for K-theoretic P- and Q-polynomials, see also Naruse [28].

1.1. Organization of the paper. The paper is organized as follows: In Section 2, we prepare the notation and conventions concerning the universal formal group law, a Gysin formula for a projective bundle, which will be used throughout the paper. In Section 3, the universal factorial Hall–Littlewood P- and Q-functions are introduced, and a characterization of them by means of a Gysin map is given. The vanishing property of these functions is also discussed. By the vanishing property, a Pieri-type formula and a generalization of the hook formula are derived. Using Gysin formulas for flag bundles and characterizations of the Hall–Littlewood functions by means of Gysin maps, in Section 4, we obtain generating functions for these universal factorial Hall–Littlewood functions. In Section 5, using the generating functions, we show that the factorial Hall–Littlewood P- and Q-polynomials satisfy a certain cancellation property. Pfaffian formulas for factorial K-theoretic Q-polynomials can be obtained as a by-product. In the Appendix (Section 6), we deal with the topic closely related to the current work, namely, generating functions for the *dual Grothendieck polynomials* and the *dual K-theoretic Schur Q-polynomials*.

2. Notation, conventions, and preliminary results. For notation and conventions, we shall follow our previous papers [24, 26]. However, to make the exposition self-contained as much as possible, we collect some of them below.

2.1. Lazard ring $\mathbb{L}$ and the universal formal group law $F_{\mathbb{L}}$. Let

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j \geq 1} a^{\mathbb{L}}_{i,j} u^i v^j \in \mathbb{L}[[u,v]]$$

be the *universal formal group law*, where $\mathbb{L}$ is the *Lazard ring*. Namely, $F_{\mathbb{L}}(u,v)$ is a formal power series in two indeterminates u, v with coefficients $a^{\mathbb{L}}_{i,j} \in \mathbb{L}$ which satisfies the axioms of the formal group law. For the universal formal group law, we shall use the following notation:

$$\begin{aligned} &u +_{\mathbb{L}} v = F_{\mathbb{L}}(u,v) && \text{(formal sum)}, \\ &\overline{u} = [-1]_{\mathbb{L}}(u) && \text{(formal inverse of } u), \\ &u -_{\mathbb{L}} v = u +_{\mathbb{L}} [-1]_{\mathbb{L}}(v) = u +_{\mathbb{L}} \overline{v} && \text{(formal subtraction)}. \end{aligned}$$

Furthermore, we define $[0]_{\mathbb{L}}(u) := 0$, and inductively, $[n]_{\mathbb{L}}(u) := [n-1]_{\mathbb{L}}(u) +_{\mathbb{L}} u$ for a positive integer $n \geq 1$. We also define $[-n]_{\mathbb{L}}(u) := [n]_{\mathbb{L}}([-1]_{\mathbb{L}}(u))$ for $n \geq 1$. We call $[n]_{\mathbb{L}}(u)$ the *n-series*. Denote by $\ell_{\mathbb{L}}(u) \in \mathbb{L} \otimes \mathbb{Q}[[u]]$ the *logarithm* of $F_{\mathbb{L}}$, i.e., the unique formal power series with leading term u such that

$$\ell_{\mathbb{L}}(u +_{\mathbb{L}} v) = \ell_{\mathbb{L}}(u) + \ell_{\mathbb{L}}(v).$$

Using the logarithm $\ell_{\mathbb{L}}(u)$, one can rewrite the n-series $[n]_{\mathbb{L}}(u)$ for a non-negative integer n as $\ell_{\mathbb{L}}^{-1}(n \cdot \ell_{\mathbb{L}}(u))$, where $\ell_{\mathbb{L}}^{-1}(u)$ is the formal power series inverse to $\ell_{\mathbb{L}}(u)$. This formula allows us to define

$$[t]_{\mathbb{L}}(x) = [t](x) := \ell_{\mathbb{L}}^{-1}(t \cdot \ell_{\mathbb{L}}(x))$$

for an indeterminate t. This is a natural extension of $t \cdot x$ as well as of the n-series $[n]_{\mathbb{L}}(x)$.

Next we shall introduce various generalizations of the ordinary power of variables. Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a countably infinite sequence of independent variables. We also introduce another set of independent variables $\boldsymbol{b} = (b_1, b_2, \ldots)$. Then, for a positive integer $k \geq 1$, we define a generalization of the ordinary kth power x^k of one variable x by

$$[x|\boldsymbol{b}]^k_{\mathbb{L}} := \prod_{j=1}^{k} (x +_{\mathbb{L}} b_j) = (x +_{\mathbb{L}} b_1) \cdots (x +_{\mathbb{L}} b_k).$$

We set $[x|\boldsymbol{b}]^0_{\mathbb{L}} := 1$. For a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers, we set

$$[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}} := \prod_{i=1}^{r} [x_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}} = \prod_{i=1}^{r} \prod_{j=1}^{\lambda_i} (x_i +_{\mathbb{L}} b_j).$$

Similarly, we define

$$[[x|\boldsymbol{b}]]^k_{\mathbb{L}} := (x +_{\mathbb{L}} x)[x|\boldsymbol{b}]^{k-1}_{\mathbb{L}} = (x +_{\mathbb{L}} x)(x +_{\mathbb{L}} b_1) \cdots (x +_{\mathbb{L}} b_{k-1}).$$

For a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers, we set

$$[[\boldsymbol{x}|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} := \prod_{i=1}^{r} [[x_i|\boldsymbol{b}]]^{\lambda_i}_{\mathbb{L}} = \prod_{i=1}^{r} (x_i +_{\mathbb{L}} x_i)[x_i|\boldsymbol{b}]^{\lambda_i - 1}_{\mathbb{L}}.$$

Moreover, for indeterminates x and t, we define

$$[[x;t|\boldsymbol{b}]]^k_{\mathbb{L}} := (x +_{\mathbb{L}} [t](\overline{x}))[x|\boldsymbol{b}]^{k-1}_{\mathbb{L}}$$

for a positive integer $k \geq 1$. For a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers, we define

$$[[\boldsymbol{x};t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} := \prod_{i=1}^{r} [[x_i;t|\boldsymbol{b}]]^{\lambda_i}_{\mathbb{L}} = \prod_{i=1}^{r} (x_i +_{\mathbb{L}} [t](\overline{x}_i))[x_i|\boldsymbol{b}]^{\lambda_i - 1}_{\mathbb{L}}.$$

2.2. Gysin formula for a projective bundle in complex cobordism. Recall from Quillen [33, Theorem 1] the Gysin formula for a projective bundle in complex cobordism. We shall state his result in a manner suitable for our purpose (for more details, see Nakagawa–Naruse [26, §3.1]): Let $E \to X$ be a complex vector bundle of rank n. For any $m \in \mathbb{Z}$, denote by $\mathscr{S}^{\mathbb{L}}_m(E) = \mathscr{S}^{MU}_m(E)$ the Segre class of E in complex cobordism, and

$$\mathscr{S}^{\mathbb{L}}(E;u) := \sum_{m \in \mathbb{Z}} \mathscr{S}^{\mathbb{L}}_m(E) u^m$$

its Segre series. The explicit expression of $\mathscr{S}^{\mathbb{L}}(E;u)$ is

(2.1)

$$\mathscr{S}^{\mathbb{L}}(E;u) = \frac{1}{\mathscr{P}^{\mathbb{L}}(z)} \prod_{j=1}^{n} \frac{z}{z +_{\mathbb{L}} \overline{x}_j} \bigg|_{z=u^{-1}} = \frac{1}{\mathscr{P}^{\mathbb{L}}(z)} \frac{z^n}{\prod_{j=1}^{n}(z +_{\mathbb{L}} \overline{x}_j)} \bigg|_{z=u^{-1}},$$

where $\mathscr{P}^{\mathbb{L}}(z) := 1 + \sum_{i=1}^{\infty} a^{\mathbb{L}}_{i,1} z^i$, and $x_1, \dots, x_n$ are the Chern roots of E in complex cobordism.

Now consider the Grassmann bundle $\pi^1 : G^1(E) \to X$ of *hyperplanes* in E. Denote by Q^1 the tautological *quotient* bundle on $G^1(E)$. Put $x_1 := c^{MU}_1(Q^1) \in MU^2(G^1(E))$. For a monomial m of a formal Laurent series F, we denote by $[m](F)$ the coefficient of m in F. Note that the Grassmann bundle $G^1(E)$ of hyperplanes in E is canonically isomorphic to the projective bundle $P(E^{\vee}) = G_1(E^{\vee})$ of lines in the dual bundle $E^{\vee}$. Then, by dualizing the formula [26, (3.4)], we have the following form of Quillen's Gysin formula:

Proposition 2.1. *For a polynomial $f(u) \in MU^*(X)[u]$, the Gysin map $\pi^1_* : MU^*(G^1(E)) \to MU^*(X)$ is described by*

$$\pi^1_*(f(x_1)) = [u^{n-1}](f(u) \cdot \mathscr{S}^{\mathbb{L}}(E;1/u)). \tag{2.2}$$

This is the fundamental formula for establishing more general Gysin formulas for general flag bundles.

Here we shall fix some notation concerning flag bundles (1): Let $E \to X$ be a complex vector bundle of rank n. For $r = 1, \ldots, n$, denote by $\pi^{r,r-1,\ldots,1}$: $\mathcal{F}\ell^{r,r-1,\ldots,1}(E) = \mathcal{F}\ell_{n-r,n-r+1,\ldots,n-1}(E) \to X$ the associated flag bundle. Thus a point in $\mathcal{F}\ell^{r,r-1,\ldots,1}(E)$ is written as a pair $(x, (W_\bullet)_x)$, where $(W_\bullet)_x$ is a flag, i.e., nested subspaces of the form $(W_1)_x \subset \cdots \subset (W_r)_x$, $\operatorname{codim}(W_i)_x = r + 1 - i$, in the fiber E_x of E over each point $x \in X$. Following Darondeau–Pragacz [4, §1.2], we shall call the flag bundle of the form $\pi^{r,r-1,\ldots,1}$: $\mathcal{F}\ell^{r,r-1,\ldots,1}(E) \to X$ the *full* flag bundle. When $r = n$, we call $\pi^{n,n-1,\ldots,1}$: $\mathcal{F}\ell^{n,n-1,\ldots,1}(E) \to X$ the *complete* flag bundle, and just write $\pi : \mathcal{F}\ell(E) \to X$. On $\mathcal{F}\ell(E)$, there is the universal flag of subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = \pi^*(E),$$

where $\operatorname{rank} U_i = i$ $(i = 0, 1, \ldots, n)$ and we put

$$x_i := c_1^{MU}(U_{n+1-i}/U_{n-i}) \in MU^2(\mathcal{F}\ell(E)) \quad (i = 1, \ldots, n), \tag{2.3}$$

which are the MU^*-theory Chern roots of E. It is well-known (see e.g., Darondeau–Pragacz [4, §1.2]) that the full flag bundle $\mathcal{F}\ell^{r,r-1,\ldots,1}(E)$ is constructed as a sequence of Grassmann bundles of codimension 1 hyperplanes (2):

$$\pi^{r,\ldots,1} : \mathcal{F}\ell^{r,r-1,\ldots,1}(E) = G^1(U_{n-r+1}) \xrightarrow{\pi^r} \cdots \to G^1(U_{n-1}) \xrightarrow{\pi^2} G^1(E) \xrightarrow{\pi^1} X. \tag{2.4}$$

3. Universal factorial Hall–Littlewood P- and Q-functions. In this section, we shall introduce our main object of study, the *universal factorial Hall–Littlewood P- and Q-functions*, which are universal as well as factorial analogues of the ordinary Hall–Littlewood polynomials.

3.1. Universal factorial Hall–Littlewood P- and Q-functions

3.1.1. *Definition of universal factorial Hall–Littlewood P- and Q-functions.* We shall use the notation introduced in §2.1. We consider the variables $\boldsymbol{x} = (x_1, x_2, \ldots)$ and $\boldsymbol{b} = (b_1, b_2, \ldots)$ with $\deg(x_i) = \deg(b_i) = 1$ for $i = 1, 2, \ldots$. Then we make the following definition:

Definition 3.1 (Universal factorial Hall–Littlewood P- and Q-functions). For a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers with $r \le n$,

(1) The notation concerning flag bundles or flag manifolds varies depending on the authors. We basically follow that used by Nakagawa–Naruse [24, §4.1] and Darondeau–Pragacz [4, §1].

(2) Note that, in [4, §1.2], the full flag bundle is constructed as a sequence of projective bundles of lines.

we define

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b}) := \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i+_{\mathbb{L}}[t](\overline{x}_j)}{x_i+_{\mathbb{L}}\overline{x}_j}\right],$$

$$HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b}) := \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[[[\boldsymbol{x};t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i+_{\mathbb{L}}[t](\overline{x}_j)}{x_i+_{\mathbb{L}}\overline{x}_j}\right],$$

where the symmetric group S_n acts on the variables $\boldsymbol{x}_n=(x_1,\dots,x_n)$ by permuting them. We also define

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t) := HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{0}) \quad\text{and}\quad HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t) := HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{0}).$$

In what follows, $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t)$ and $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t)$ will be called the *universal Hall–Littlewood P- and Q-functions* respectively.

In the above definition, the action of the subgroup $(S_1)^r\times S_{n-r}$ of S_n on the first factors $[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}$ and $[[\boldsymbol{x};t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}}$ is trivial, and the second factor

$$\prod_{1\le i\le r,\, i<j\le n}\frac{x_i+_{\mathbb{L}}[t](\overline{x}_j)}{x_i+_{\mathbb{L}}\overline{x}_j}$$

is invariant under this action. Therefore, the action of the symmetric group does not depend on the choice of a representative w of the coset $\overline{w}\in S_n/(S_1)^r\times S_{n-r}$. Note that for $t=-1$ in the definition, $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;-1|\boldsymbol{b})$ (resp. $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;-1|\boldsymbol{b})$) coincides with the universal factorial Schur P-function $P^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$ (resp. Q-function $Q^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$), for a strict partition λ, which have been introduced in our previous paper [23, Definition 4.1]. By contrast, for $t=0$, both $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;0|\boldsymbol{b})$ and $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;0|\boldsymbol{b})$ are different from the universal factorial Schur functions $s^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$ [23, Definition 4.10] and $\mathbb{S}^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$ [24, Definition 5.1].

3.1.2. *Factorial Hall–Littlewood P- and Q-polynomials.* The specialization from $F_{\mathbb{L}}(u,v)=u+_{\mathbb{L}}v$ to $F_a(u,v)=u+v$ is of particular importance. Under this specialization, the generalized powers $[x|\boldsymbol{b}]^k_{\mathbb{L}}$, $[[x;t|\boldsymbol{b}]]^k_{\mathbb{L}}$ reduce to

$$[x|\boldsymbol{b}]^k=\prod_{j=1}^{k}(x+b_j),\qquad [[x;t|\boldsymbol{b}]]^k=(x-tx)[x|\boldsymbol{b}]^{k-1},$$

and we obtain *new* symmetric polynomials denoted by $HP_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$ and $HQ_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$ respectively. More explicitly, these are defined as follows:

Definition 3.2 (Factorial Hall–Littlewood P- and Q-polynomials). For a sequence $\lambda=(\lambda_1,\dots,\lambda_r)$ of positive integers with $r\le n$, we define

$$\begin{aligned}
HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) &:= \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[[\boldsymbol{x}|\boldsymbol{b}]^\lambda \prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i-tx_j}{x_i-x_j}\right],\\
&= \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[\prod_{i=1}^{r}\prod_{j=1}^{\lambda_i}(x_i+b_j)\times\prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i-tx_j}{x_i-x_j}\right],\\
HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) &:= \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[[[\boldsymbol{x};t|\boldsymbol{b}]]^\lambda \prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i-tx_j}{x_i-x_j}\right]\\
&= (1-t)^r\times\\
&\sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[\prod_{i=1}^{r}\prod_{j=1}^{\lambda_i-1}x_i(x_i+b_j)\times\prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i-tx_j}{x_i-x_j}\right].
\end{aligned}$$

We also define

$$HP_\lambda(\boldsymbol{x}_n;t) := HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{0}) \quad\text{and}\quad HQ_\lambda(\boldsymbol{x}_n;t) := HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{0}),$$

and call them the *Hall–Littlewood P- and Q-polynomials* respectively.

Note that, by definition, $HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) = (1-t)^{\ell(\lambda)}HP_\lambda(\boldsymbol{x}_n;t|0,\boldsymbol{b})$. For a strict partition λ, if $t=-1$, then $HP_\lambda(\boldsymbol{x}_n;-1|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n;-1|\boldsymbol{b}) = 2^{\ell(\lambda)}HP_\lambda(\boldsymbol{x}_n;-1|0,\boldsymbol{b})$ coincide with the factorial Schur P- and Q-polynomials (by replacing $\boldsymbol{b}$ with $-\boldsymbol{b} = (-b_1,-b_2,\dots)$) (for their definition, see Ikeda–Mihalcea–Naruse [12, §4.2]). However, for any partition λ, neither $HP_\lambda(\boldsymbol{x}_n;0|\boldsymbol{b})$ nor $HQ_\lambda(\boldsymbol{x}_n;0|\boldsymbol{b})$ coincides with the factorial Schur polynomial (for its definition, see Molev–Sagan [21, §2, (3)]).

Example 3.3. Direct computation using Definition 3.2 gives some examples:

$$\begin{aligned}
&HP_{(1)}(\boldsymbol{x}_n;t|\boldsymbol{b}) = x_1+\cdots+x_n+\frac{1-t^n}{1-t}b_1,\\
&HP_{(1^2)}(\boldsymbol{x}_n;t|\boldsymbol{b})\\
&\qquad = (1+t)\left[m_{(1^2)}(\boldsymbol{x}_n)+\frac{1-t^{n-1}}{1-t}b_1m_{(1)}(\boldsymbol{x}_n)+\frac{(1-t^{n-1})(1-t^n)}{(1-t)(1-t^2)}b_1^2\right],\\
&HP_{(2)}(\boldsymbol{x}_n;t|\boldsymbol{b}) = s_{(2)}(\boldsymbol{x}_n)-ts_{(1^2)}(\boldsymbol{x}_n)+(b_1+b_2)s_{(1)}(\boldsymbol{x}_n)+b_1b_2\frac{1-t^n}{1-t}.
\end{aligned}$$

Here $m_\lambda(\boldsymbol{x}_n)$ and $s_\lambda(\boldsymbol{x}_n)$ are respectively the monomial symmetric polynomials and Schur polynomials corresponding to λ.

If λ is a partition of length $\ell(\lambda)=r\le n$, i.e., $\lambda_1\ge\cdots\ge\lambda_r>0$, our factorial Hall–Littlewood P- and Q-polynomials are related to Macdonald's Hall–Littlewood P- and Q-polynomials in the following way: We rewrite λ as $\lambda=(n_1^{p_1}\cdots n_d^{p_d})$, where $n_1>\cdots>n_d=0$, $p_i>0$ for each i, $p_d=n-r$, and $\sum_{i=1}^{d} p_i = n$. We put $\nu(k) := \sum_{i=1}^{k} p_i$ for $k = 1, \dots, d$ and $\nu(0) := 0$. Denote by S_{p_k} the symmetric group on p_k letters $\nu(k-1)+1, \dots, \nu(k)$ for $k = 1, \dots, d$. Thus the stabilizer subgroup S_n^{λ} of λ under the action of S_n on λ is given by $S_n^{\lambda} = \prod_{k=1}^{d} S_{p_k}$. For an integer $k \geq 0$, let $v_k(t) := \prod_{i=1}^{k} \frac{1-t^i}{1-t}$, and for the above partition λ, we define (3)

$$v_{\lambda>0}(t) := \prod_{k=1}^{d-1} v_{p_k}(t).$$

Using the identity

$$\sum_{w \in S_n} w \cdot \left[\prod_{1 \leq i < j \leq n} \frac{x_i - t x_j}{x_i - x_j} \right] = v_n(t) \tag{3.1}$$

of [19, Chapter III, (1.4)], one can prove the following fact along the same lines as for the usual Hall–Littlewood polynomials in [19, Chapter III, (1.5)]:

$$HP_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) = v_{\lambda>0}(t) \times \sum_{\overline{w} \in S_n/S_n^{\lambda}} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^{\lambda} \cdot \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i > \lambda_j}} \frac{x_i - t x_j}{x_i - x_j} \right]. \tag{3.2}$$

Thus $HP_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$ is divisible by $v_{\lambda>0}(t)$. Taking this fact into account, we define

$$P_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \frac{1}{v_{\lambda>0}(t)} HP_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}), \tag{3.3}$$

or equivalently

$$P_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/S_n^{\lambda}} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^{\lambda} \cdot \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i > \lambda_j}} \frac{x_i - t x_j}{x_i - x_j} \right]. \tag{3.4}$$

It is this polynomial that can be considered as a factorial version of Macdonald's Hall–Littlewood P-polynomial $P_{\lambda}(\boldsymbol{x}_n; t)$. Putting $\boldsymbol{b} = \boldsymbol{0}$ in (3.3), we have $HP_{\lambda}(\boldsymbol{x}_n; t) = v_{\lambda>0}(t) P_{\lambda}(\boldsymbol{x}_n; t)$. In particular, for λ strict, $HP_{\lambda}(\boldsymbol{x}_n; t)$ coincides with $P_{\lambda}(\boldsymbol{x}_n; t)$. On the other hand, by the argument in Macdonald's book [19, pp. 210–211], we see that $HQ_{\lambda}(\boldsymbol{x}_n; t)$ equals the ordinary Hall–Littlewood Q-polynomial $Q_{\lambda}(\boldsymbol{x}_n; t)$.

Remark 3.4. (1) The universal analogue of the left-hand side of (3.1), namely,

$$\sum_{w \in S_n} w \cdot \left[\prod_{1 \leq i < j \leq n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right]$$

(3) Do not confuse $v_{\lambda>0}(t)$ with $v_{\lambda}(t) := \prod_{i \geq 0} v_{m_i}(t)$ in Macdonald [19, Chapter III, §1], where $m_i = m_i(\lambda)$ means the *multiplicity* for each $i \geq 0$.

is no longer a polynomial in t alone (it contains the variables $x_1, \ldots, x_n$). Therefore a formula analogous to (3.2) does not hold in this case.

(2) For a general sequence λ of positive integers, $HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ may not be divisible by $v_{\lambda>0}(t)$.

3.2. Characterization of the universal factorial Hall–Littlewood P- and Q-functions. Geometrically, the universal factorial Hall–Littlewood P- and Q-functions are characterized by means of the Gysin map for certain flag bundles. (We learned this idea from Pragacz's work [32].) Let $E \to X$ be a complex vector bundle of rank n, and $x_1, \ldots, x_n$ the MU^*-theory Chern roots of E as in (2.3). Consider the associated full flag bundle $\pi^{r,r-1,\ldots,1} : \mathcal{F}\ell^{r,r-1,\ldots,1}(E) \to X$. Then the Gysin homomorphism $(\pi^{r,\ldots,1})_* : MU^*(\mathcal{F}\ell^{r,\ldots,1}(E)) \to MU^*(X)$ is described as the following type of a symmetrizing operator (see Nakagawa–Naruse [24, Theorem 4.10], and also Brion [3, Proposition 1.1] for cohomology): For an $(S_1)^r \times S_{n-r}$-invariant polynomial $f(X_1, \ldots, X_n) \in MU^*(X)[X_1, \ldots, X_n]^{(S_1)^r \times S_{n-r}}$, one has

$$(\pi^{r,\ldots,1})_*(f(x_1, \ldots, x_n)) = \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[\frac{f(x_1, \ldots, x_n)}{\prod_{1 \le i \le r,\, i<j \le n} (x_i +_{\mathbb{L}} \overline{x}_j)} \right].$$

Then it follows from Definition 3.1 and the above description of the Gysin homomorphism $(\pi^{r,\ldots,1})_*$ that the following formula holds:

Proposition 3.5 (Characterization of the universal factorial Hall–Littlewood P- and Q-functions).

$$(\pi^{r,\ldots,1})_* \Big([\boldsymbol{x}|\boldsymbol{b}]^\lambda_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} (x_i +_{\mathbb{L}} [t](\overline{x}_j)) \Big) = HP^{\mathbb{L}}_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}), \tag{3.5}$$

$$(\pi^{r,\ldots,1})_* \Big([[\boldsymbol{x}; t|\boldsymbol{b}]]^\lambda_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} (x_i +_{\mathbb{L}} [t](\overline{x}_j)) \Big) = HQ^{\mathbb{L}}_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}). \tag{3.6}$$

Here $\boldsymbol{b} = (b_1, b_2, \ldots)$ is a sequence of elements in $MU^*(X)$.

At first sight, this characterization seems to be merely a paraphrase of Definition 3.1. However, this geometric interpretation will be crucial to our current work. In fact, as shown in §4, a careful application of the fundamental Gysin formula (2.2) to the left hand sides of (3.5), (3.6) enables us to obtain the generating functions for the universal factorial Hall–Littlewood P- and Q-functions.

Remark 3.6. As a special case of the above result, the factorial Hall–Littlewood P-polynomial $HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ is characterized by the cohomology

Gysin map:

$$(\pi^{r,\dots,1})_*\Big([\boldsymbol{x}|\boldsymbol{b}]^\lambda \prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i - tx_j)\Big) = HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}).$$

A factorial version of Macdonald's Hall–Littlewood P-polynomial $P_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ can also be characterized by the Gysin map: Consider the partial flag bundle $\pi^\lambda : \mathcal{F}\ell^\lambda(E) := \mathcal{F}\ell^{\nu(d-1),\nu(d-2),\dots,\nu(1)}(E) \to X$. Here we write $\lambda = (n_1^{p_1}\cdots n_d^{p_d})$ and $\nu(k) = \sum_{i=1}^k p_i$ as in §3.1. Then

$$(\pi^\lambda)_*\Big([\boldsymbol{x}|\boldsymbol{b}]^\lambda \prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i - tx_j)\Big) = P_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}).$$

3.3. Vanishing properties of factorial Hall–Littlewood P- and Q-polynomials. It is known that the factorial Schur S-, P-, and Q-polynomials have a remarkable property called the *vanishing property* (see Molev–Sagan [21, Theorem 2.1], Ivanov [15, Theorem 5.3]). In this subsection, we shall show that our factorial Hall–Littlewood P- and Q-polynomials also have this property. Let $\boldsymbol{b} = (b_1, b_2, \dots)$ be a sequence of indeterminates, and t be an indeterminate. For a partition $\mu = (\mu_1, \mu_2, \dots)$, let $m_i = m_i(\mu)$ be the multiplicity of i $(1 \le i \le \mu_1)$, i.e., the number of components in μ of size i. We define

$$-\boldsymbol{b}_\mu(t) := (-\boldsymbol{b}_{\mu_1}^{m_{\mu_1}}(t), \dots, -\boldsymbol{b}_2^{m_2}(t), -\boldsymbol{b}_1^{m_1}(t)),$$

where $-\boldsymbol{b}_i^k(t) := (-b_i, -tb_i, \dots, -t^{k-1}b_i)$ (we set $-\boldsymbol{b}_i^0(t) = (\,)$, the empty sequence). Let us consider substituting the variables $\boldsymbol{x}_n = (x_1, \dots, x_n)$ with the sequence $-\boldsymbol{b}_\mu(t)$ for a partition μ of length $\ell(\mu) \le n$. We sometimes write $\boldsymbol{x}_n \to -\boldsymbol{b}_\mu(t)$, or more specifically, say, $x_1 \to -b_{\mu_1}$ when we make such a substitution. After the substitution $\boldsymbol{x}_n \to -\boldsymbol{b}_\mu(t)$, denote by $\mathrm{ev}_\mu(x_i)$ $(i = 1, \dots, n)$ the ith entry of $-\boldsymbol{b}_\mu(t)$. Then we have

$$(\mathrm{ev}_\mu(x_1), \dots, \mathrm{ev}_\mu(x_n)) = -\boldsymbol{b}_\mu(t).$$

We also use the notation $\mathrm{ev}_\mu(f(x_1, \dots, x_n)) = f(\mathrm{ev}_\mu(x_1), \dots, \mathrm{ev}_\mu(x_n))$. For example, if $\mu = (5,5,5,4,1,1)$, then $m_1(\mu) = 2$, $m_2(\mu) = 0$, $m_3(\mu) = 0$, $m_4(\mu) = 1$, $m_5(\mu) = 3$, and $-\boldsymbol{b}_\mu(t) = (-b_5, -tb_5, -t^2b_5, -b_4, -b_1, -tb_1)$, $\mathrm{ev}_\mu(x_1) = -b_5$, $\mathrm{ev}_\mu(x_2) = -tb_5$, $\mathrm{ev}_\mu(x_2 - tx_1) = -tb_5 - t\cdot(-b_5) = 0$, etc. With these notations, we can prove the following:

Proposition 3.7 (Vanishing property). *Let λ, μ be partitions of length at most n and set $\hat{\mu} := \mu + (1^n) = (\mu_1 + 1, \dots, \mu_n + 1)$. Then the factorial Hall–Littlewood P- and Q-polynomials have the following vanishing property:*

(1) *If $\mu \not\supset \lambda$, then*

$$HQ_\lambda(-\boldsymbol{b}_\mu(t), \underbrace{0, \dots, 0}_{n-\ell(\mu)}; t|\boldsymbol{b}) = 0, \qquad HP_\lambda(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) = 0.$$

(2) *If* $\mu = \lambda$, *then*

$$HQ_\lambda(-\boldsymbol{b}_\lambda(t), \underbrace{0, \dots, 0}_{n-\ell(\lambda)}; t|\boldsymbol{b}) = \prod_{q=1}^{\lambda_1} \prod_{k=1}^{m_q(\lambda)} \Big(\prod_{p=1}^{q} (-t^{k-1} b_q + t^{m_p(\lambda)} b_p) \Big),$$

$$HP_\lambda(-\boldsymbol{b}_{\hat{\lambda}}(t); t|\boldsymbol{b}) = v_{\lambda>0}(t) \prod_{q=2}^{\hat{\lambda}_1} \prod_{k=1}^{m_q(\hat{\lambda})} \Big(\prod_{p=1}^{q-1} (-t^{k-1} b_q + t^{m_p(\hat{\lambda})} b_p) \Big).$$

Proof. We only handle the case of $HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$; the case of $HQ_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ can be proved similarly.

(1) Assuming $\lambda \not\subset \mu$, we can find the minimal k such that $\lambda_k > \mu_k$ $(1 \leq k \leq \ell(\lambda) = r)$. For each choice w of $\overline{w} \in S_n/(S_1)^r \times S_{n-r}$, we will show that the corresponding summand in (3.2) vanishes, i.e.,

$$\left(w \cdot \left[[x_1|\boldsymbol{b}]^{\lambda_1} \cdots [x_r|\boldsymbol{b}]^{\lambda_r} \prod_{1 \leq i \leq r,\, i<j \leq n} \frac{x_i - t x_j}{x_i - x_j} \right] \right)_{\boldsymbol{x}_n \to -\boldsymbol{b}_{\hat{\mu}}(t)} = 0.$$

For the permutation w, take the minimal d $(1 \leq d \leq k)$ such that $w(d) \geq k$. Then we divide the discussion into two cases:

Case 1: $w(d) = 1$ or $[w(d) > 1$ and $\mu_{w(d)-1} > \mu_{w(d)}]$. In this case,

$$([x_{w(d)}|\boldsymbol{b}]^{\lambda_d})_{x_{w(d)} \to \mathrm{ev}_{\hat{\mu}}(x_{w(d)})} = 0$$

because $\mathrm{ev}_{\hat{\mu}}(x_{w(d)}) = -b_{\mu_{w(d)}+1}$ and $\lambda_d \geq \lambda_k > \mu_k \geq \mu_{w(d)}$.

Case 2: $w(d) > 1$ and $\mu_{w(d)-1} = \mu_{w(d)}$. In this case, we claim that

$$\mathrm{ev}_{\hat{\mu}} \left(\prod_{1 \leq i \leq r,\, i<j \leq n} \frac{x_{w(i)} - t x_{w(j)}}{x_{w(i)} - x_{w(j)}} \right) = 0.$$

First note that, by the minimality of k, we have $w(d) > k$. Let p $(1 \leq p \leq n)$ be an integer such that $w(p) = w(d) - 1$. Then, by the minimality of d, we have $d < p \leq n$. Since $\mu_{w(p)} = \mu_{w(d)}$ and $w(d) = w(p) + 1$, we have $\mathrm{ev}_{\hat{\mu}}(x_{w(d)}) = t \cdot \mathrm{ev}_{\hat{\mu}}(x_{w(p)})$. As $1 \leq d \leq r$ and $d < p \leq n$, the factor $\mathrm{ev}_{\hat{\mu}}(x_{w(d)} - t x_{w(p)})$ vanishes, and our claim follows.

(2) When $\mu = \lambda$, we first show that each summand corresponding to $\overline{w} \in S_n/(S_1)^r \times S_{n-r}$ vanishes under the evaluation $\mathrm{ev}_{\hat{\lambda}}$, except for $\overline{w} = \overline{e}$ (where e is the identity element). In fact, if $\overline{w} \neq \overline{e}$, we can find minimal d such that $1 \leq d \leq r$ and $w(d) > d$. Then, by dividing the argument into the cases of $\lambda_{w(d)-1} > \lambda_{w(d)}$ and of $\lambda_{w(d)-1} = \lambda_{w(d)}$, we can show that the corresponding summand vanishes under $\mathrm{ev}_{\hat{\lambda}}$.

For $w = e$, we can evaluate the term as follows. For each i $(1 \leq i \leq r)$, we can write $\mathrm{ev}_{\hat{\lambda}}(x_i) = t^{k-1} b_q$ $(k \geq 1,\ q = \lambda_i + 1 \geq 2)$. Then the direct computation yields

$$\mathrm{ev}_{\hat{\lambda}}\left([x_i|\boldsymbol{b}]^{\lambda_i}\prod_{j=i+1}^{n}\frac{x_i-tx_j}{x_i-x_j}\right)=\frac{1-t^{m_q(\hat{\lambda})-k+1}}{1-t}\prod_{p=1}^{q-1}(-t^{k-1}b_q+t^{m_p(\hat{\lambda})}b_p).$$

We then take the product of all these evaluations for $1\le i\le r$. Since

$$\prod_{q=2}^{\hat{\lambda}_1}\prod_{k=1}^{m_q(\hat{\lambda})}\frac{1-t^{m_q(\hat{\lambda})-k+1}}{1-t}=v_{\lambda>0}(t),$$

we get the desired formula. ■

More generally, we can prove the vanishing property of the universal factorial Hall–Littlewood P- and Q-functions in a similar way. We only state the result, for which we need some notations. For a partition μ, we define

$$\overline{\boldsymbol{b}}_\mu[t]:=(\overline{\boldsymbol{b}}_{\mu_1}^{m_{\mu_1}}[t],\overline{\boldsymbol{b}}_{\mu_1-1}^{m_{\mu_1-1}}[t],\dots,\overline{\boldsymbol{b}}_2^{m_2}[t],\overline{\boldsymbol{b}}_1^{m_1}[t]),$$

where $\overline{\boldsymbol{b}}_i^k[t]:=(\bar{b}_i,[t](\bar{b}_i),\dots,[t^{k-1}](\bar{b}_i))$ (we set $\overline{\boldsymbol{b}}_i^0[t]=(\,)$, the empty sequence).

Proposition 3.8 (Vanishing property). *Let λ, μ be partitions of length at most n and set $\hat{\mu}=\mu+(1^n)=(\mu_1+1,\dots,\mu_n+1)$. Then the universal factorial Hall–Littlewood P- and Q-functions have the following vanishing property:*

(1) *If $\mu\not\supset\lambda$, then*

$$HQ_\lambda^{\mathbb{L}}(\overline{\boldsymbol{b}}_\mu[t],\underbrace{0,\dots,0}_{n-\ell(\mu)};t|\boldsymbol{b})=0,\qquad HP_\lambda^{\mathbb{L}}(\overline{\boldsymbol{b}}_{\hat{\mu}}[t];t|\boldsymbol{b})=0.$$

(2) *If $\mu=\lambda$, then*

$$HQ_\lambda^{\mathbb{L}}(\overline{\boldsymbol{b}}_\lambda[t],\underbrace{0,\dots,0}_{n-\ell(\lambda)};t|\boldsymbol{b})=\prod_{q=1}^{\lambda_1}\prod_{k=1}^{m_q(\lambda)}\left(\prod_{p=1}^{q}([t^{k-1}](\bar{b}_q)+_{\mathbb{L}}[t^{m_p(\lambda)}](b_p))\right),$$

$$HP_\lambda^{\mathbb{L}}(\overline{\boldsymbol{b}}_{\hat{\lambda}}[t];t|\boldsymbol{b})=v_{\lambda>0}(t)\prod_{q=2}^{\hat{\lambda}_1}\prod_{k=1}^{m_q(\hat{\lambda})}\left(\prod_{p=1}^{q-1}([t^{k-1}](\bar{b}_q)+_{\mathbb{L}}[t^{m_p(\hat{\lambda})}](b_p))\right).$$

3.4. Pieri-type formula and hook formula. The vanishing property established in the previous section is useful in that one can derive from it several interesting results on factorial Hall–Littlewood polynomials. Denote by $\Lambda(\boldsymbol{x}_n)=\mathbb{Z}[x_1,\dots,x_n]^{S_n}$ the ring of symmetric polynomials of n variables, and by $\mathcal{P}_n$ the set of partitions of length $\le n$. Then it is known that the usual Hall–Littlewood P-polynomials $P_\lambda(\boldsymbol{x}_n;t)$ ($\lambda\in\mathcal{P}_n$) form a $\mathbb{Z}[t]$-basis of $\Lambda(\boldsymbol{x}_n)[t]\cong\mathbb{Z}[t]\otimes_{\mathbb{Z}}\Lambda(\boldsymbol{x}_n)$ (cf. Macdonald [19, III, (2.7)]). Therefore there exist polynomials $c^{\nu}_{\lambda,\mu}(t) = c^{\nu,(n)}_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ such that

$$P_\lambda(\boldsymbol{x}_n;t)P_\mu(\boldsymbol{x}_n;t) = \sum_\nu c^{\nu}_{\lambda,\mu}(t)P_\nu(\boldsymbol{x}_n;t) \quad (\lambda,\mu,\nu \in \mathcal{P}_n).$$

It is known (see Macdonald [19, III, (5.7)]) that the following Pieri-type formula holds:

$$P_{(1)}(\boldsymbol{x}_n;t)P_\mu(\boldsymbol{x}_n;t) = \sum_{\mu\subset\nu,\,|\nu/\mu|=1} \alpha_{\nu/\mu}(t)P_\nu(\boldsymbol{x}_n;t), \tag{3.7}$$

where the polynomial $\alpha_{\nu/\mu}(t) = \alpha^{(n)}_{\nu/\mu}(t)$ is given by $\frac{1-t^{m_j(\nu)}}{1-t}$ if ν/μ has a box in the jth column. As for the factorial version of Macdonald's Hall–Littlewood P-polynomials $P_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ (see (3.3)), one can consider a similar problem: First we see that the factorial Hall–Littlewood P-polynomials $P_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ ($\lambda \in \mathcal{P}_n$) form a $\mathbb{Z}[t]\otimes_{\mathbb{Z}}\mathbb{Z}[\boldsymbol{b}]$-basis of $\Lambda(\boldsymbol{x}_n|\boldsymbol{b})[t] := \mathbb{Z}[t]\otimes_{\mathbb{Z}}\mathbb{Z}[\boldsymbol{b}]\otimes_{\mathbb{Z}} \Lambda(\boldsymbol{x}_n)$, where $\mathbb{Z}[\boldsymbol{b}] = \mathbb{Z}[b_1,b_2,\ldots]$ is the polynomial ring in indeterminates $\boldsymbol{b} = (b_1,b_2,\ldots)$. Therefore there exist polynomials $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = c^{\nu,(n)}_{\lambda,\mu}(t|\boldsymbol{b}) \in \mathbb{Z}[t]\otimes\mathbb{Z}[\boldsymbol{b}]$ such that

$$P_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})P_\mu(\boldsymbol{x}_n;t|\boldsymbol{b}) = \sum_\nu c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b})P_\nu(\boldsymbol{x}_n;t|\boldsymbol{b}) \quad (\lambda,\mu,\nu \in \mathcal{P}_n). \tag{3.8}$$

By definition, the "structure constant" $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b})$ is a homogeneous polynomial of degree $|\lambda|+|\mu|-|\nu|$ in the indeterminates $\boldsymbol{b} = (b_1,b_2,\ldots)$ with coefficients in $\mathbb{Z}[t]$. Comparing the highest homogeneous components in $\boldsymbol{x}_n = (x_1,\ldots,x_n)$ on both sides of (3.8), we see that

$$c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = \begin{cases} c^{\nu}_{\lambda,\mu}(t) & \text{if } |\lambda|+|\mu| = |\nu|, \\ 0 & \text{if } |\lambda|+|\mu| < |\nu|. \end{cases}$$

From the commutativity of the product on the left hand side of (3.8), the symmetry $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = c^{\nu}_{\mu,\lambda}(t|\boldsymbol{b})$ holds obviously. Furthermore, using the vanishing property ([4]), Proposition 3.7, we claim that $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b})$ is zero unless $\lambda \subset \nu$ and $\mu \subset \nu$. The proof proceeds as follows (cf. Molev–Sagan [21, p. 4434]): Fix λ,μ and let ν be a minimal partition with respect to the containment relation such that $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) \neq 0$ in (3.8). Suppose that $\mu \not\subset \nu$. We set $\boldsymbol{x}_n = -\boldsymbol{b}_{\hat{\nu}}(t)$ in (3.8). Then, by Proposition 3.7(1), we have

$$0 = c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b})P_\nu(-\boldsymbol{b}_{\hat{\nu}}(t);t|\boldsymbol{b}).$$

By Proposition 3.7(2), we have $P_\nu(-\boldsymbol{b}_{\hat{\nu}}(t);t|\boldsymbol{b}) \neq 0$, and hence $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = 0$. However, this contradicts $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) \neq 0$, and hence $\mu \subset \nu$ holds. From this and the symmetry relation $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = c^{\nu}_{\mu,\lambda}(t|\boldsymbol{b})$, our claim follows.

Now we consider the case where $\lambda = (1)$ in (3.8). Then, by the known properties of the structure constants, we only need to consider those ν with

([4]) By the definition (3.3), $P_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$'s also satisfy the vanishing property.

$\mu \subset \nu$ and $|\nu| \le |\mu| + 1$, Thus (3.8) takes the following form:

$$P_{(1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) P_\mu(\boldsymbol{x}_n; t|\boldsymbol{b}) = c^{\mu}_{(1),\mu}(t|\boldsymbol{b}) P_\mu(\boldsymbol{x}_n; t|\boldsymbol{b}) + \sum_{\mu\subset\nu,\, |\nu/\mu|=1} c^{\nu}_{(1),\mu}(t|\boldsymbol{b}) P_\nu(\boldsymbol{x}_n; t|\boldsymbol{b}).$$

Setting $\boldsymbol{x}_n = -\boldsymbol{b}_{\hat{\mu}}(t)$ and using the vanishing property, we see that $c^{\mu}_{(1),\mu}(t|\boldsymbol{b}) = P_{(1)}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b})$. On the other hand, for degree reasons, we have $c^{\nu}_{(1),\mu}(t|\boldsymbol{b}) = c^{\nu}_{(1),\mu}(t) = \alpha_{\nu/\mu}(t)$ when $\mu \subset \nu$ and $|\nu/\mu| = 1$. Thus we obtain the following formula:

Proposition 3.9 (Pieri-type formula for factorial Hall–Littlewood P-polynomials).

$$P_{(1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) P_\mu(\boldsymbol{x}_n; t|\boldsymbol{b}) = P_{(1)}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) P_\mu(\boldsymbol{x}_n; t|\boldsymbol{b}) + \sum_{\mu\subset\nu,\, |\nu/\mu|=1} \alpha_{\nu/\mu} P_\nu(\boldsymbol{x}_n; t|\boldsymbol{b}).$$

Using Proposition 3.9, we can derive a generalization of the so-called *hook* (*length*) *formula*. We argue as follows (the following argument is essentially the same as that given in Molev–Sagan [21, Proposition 3.2] for factorial Schur polynomials, although they did not mention the relation to the hook formula; for this type of argument, see also Naruse–Okada [29, Lemma 4.5]). For simplicity, we shall use the abbreviated notation P_λ, $c^{\nu}_{\lambda,\mu}$, and $\alpha_{\lambda/\mu}$ for $P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$, $c^{\nu,(n)}_{\lambda,\mu}(t|\boldsymbol{b})$, and $\alpha^{(n)}_{\lambda/\mu}(t)$ respectively. Then our hook formula can be stated as follows:

Proposition 3.10 (Hook formula for factorial Hall–Littlewood P-polynomials). *Let μ be a partition of length $\ell(\mu) \le n$ and size $|\mu| = k$, a positive integer. Then*

(3.9)
$$\sum_{\mu=\mu^{(0)}\supsetneq\mu^{(1)}\supsetneq\cdots\supsetneq\mu^{(k)}=\emptyset} \frac{\alpha_{\mu^{(k-1)}/\mu^{(k)}}}{c^{\mu}_{(1),\mu} - c^{\mu^{(k)}}_{(1),\mu^{(k)}}} \cdots \frac{\alpha_{\mu^{(1)}/\mu^{(2)}}}{c^{\mu}_{(1),\mu} - c^{\mu^{(2)}}_{(1),\mu^{(2)}}} \cdot \frac{\alpha_{\mu^{(0)}/\mu^{(1)}}}{c^{\mu}_{(1),\mu} - c^{\mu^{(1)}}_{(1),\mu^{(1)}}} = \frac{1}{P_\mu(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b})}.$$

Proof. We use the associativity of the product

$$(P_{(1)} P_\lambda) P_\mu = P_{(1)} (P_\lambda P_\mu),$$

and take the coefficient of P_μ on both sides. Using the fact that $c^{\gamma}_{\alpha,\beta}$ is zero unless $\alpha \subset \gamma$ and $\beta \subset \gamma$, and Proposition 3.9, we have

$$c^{\lambda}_{(1),\lambda} c^{\mu}_{\lambda,\mu} + \sum_{\mu\supset\nu\supsetneq\lambda,\, |\nu/\lambda|=1} \alpha_{\nu/\lambda}\, c^{\mu}_{\nu,\mu} = c^{\mu}_{(1),\mu} c^{\mu}_{\lambda,\mu},$$

and therefore

$$(c^{\mu}_{(1),\mu} - c^{\lambda}_{(1),\lambda}) c^{\mu}_{\lambda,\mu} = \sum_{\mu \supset \nu \supsetneq \lambda,\, |\nu/\lambda|=1} \alpha_{\nu/\lambda}\, c^{\mu}_{\nu,\mu}.$$

By definition and Example 3.3, we know that $P_{(1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) = x_1 + \cdots + x_n + \frac{1-t^n}{1-t} b_1$. Therefore, if $\mu \supsetneq \lambda$, we see that

$$c^{\mu}_{(1),\mu} - c^{\lambda}_{(1),\lambda} = P_{(1)}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) - P_{(1)}(-\boldsymbol{b}_{\hat{\lambda}}(t); t|\boldsymbol{b}) \neq 0.$$

Thus we have the following recurrence formula:

$$c^{\mu}_{\lambda,\mu} = \sum_{\mu \supset \nu \supsetneq \lambda,\, |\nu/\lambda|=1} \frac{\alpha_{\nu/\lambda}}{c^{\mu}_{(1),\mu} - c^{\lambda}_{(1),\lambda}} c^{\mu}_{\nu,\mu}.$$

Using this recurrence formula repeatedly, we obtain

$$c^{\mu}_{\emptyset,\mu} = \sum_{\mu=\mu^{(0)} \supsetneq \mu^{(1)} \supsetneq \cdots \supsetneq \mu^{(k)} = \emptyset} \frac{\alpha_{\mu^{(k-1)}/\mu^{(k)}}}{c^{\mu}_{(1),\mu} - c^{\mu^{(k)}}_{(1),\mu^{(k)}}} \cdots \frac{\alpha_{\mu^{(0)}/\mu^{(1)}}}{c^{\mu}_{(1),\mu} - c^{\mu^{(1)}}_{(1),\mu^{(1)}}} c^{\mu}_{\mu,\mu}.$$

The fact that $c^{\mu}_{\emptyset,\mu} = 1$ is obvious from the definition of structure constants. The value of $c^{\mu}_{\mu,\mu}$ equals $P_{\mu}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b})$ by the vanishing property, Proposition 3.7. Therefore, we have the desired formula. ∎

As mentioned before the proposition, one can obtain a similar hook formula from [21, Proposition 3.2]. More concretely, in their notation,

$$\sum_{\emptyset=\rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l)} = \nu} \frac{1}{(|a_\nu| - |a_{\rho^{(0)}}|) \cdots (|a_\nu| - |a_{\rho^{(l-1)}}|)} = \frac{1}{s_\nu(a_\nu|a)}. \tag{3.10}$$

We remark that this formula can be interpreted as a special case of Nakada's *colored hook formula* [22, Corollary 7.2], which is a generalization of the famous hook formula due to Frame–Robinson–Thrall [6]. As an example, let us take $\nu = (2,2)$ and $n = 2$, the number of variables. Then the above formula leads to

$$\begin{aligned} &\frac{1}{(a_3 - a_2)(a_3 - a_1)(a_4 - a_1)(a_4 + a_3 - a_2 - a_1)} \\ &\qquad + \frac{1}{(a_3 - a_2)(a_4 - a_2)(a_4 - a_1)(a_4 + a_3 - a_2 - a_1)} \\ &\qquad = \frac{1}{(a_3 - a_2)(a_3 - a_1)(a_4 - a_2)(a_4 - a_1)}. \end{aligned}$$

Now consider the simple root system $\{\alpha_1, \alpha_2, \alpha_3\}$ of type A_3. If one represents the simple root α_i as $a_i - a_{i+1}$ for $i = 1, 2, 3$, then the above identity becomes

$$\begin{aligned}(3.11)\qquad &\frac{1}{\alpha_2(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)} \\ &\qquad + \frac{1}{\alpha_2(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)} \\ &= \frac{1}{\alpha_2(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)},\end{aligned}$$

which agrees with the example given in [22, p. 1088]. When we specialize t to be 0, our factorial Hall–Littlewood P-polynomial $HP_\lambda(\boldsymbol{x}_n; 0|\boldsymbol{b}) = P_\lambda(\boldsymbol{x}_n; 0|\boldsymbol{b})$ does not coincide with the factorial Schur polynomial $s_\lambda(\boldsymbol{x}_n|\boldsymbol{b})$ ([5]). Thus the $t = 0$ specialization of our hook formula (3.9) yields *another* colored hook formula (see the example below). It is well-known that the classical hook formula and its *shifted* analogue have geometric background known as Schubert calculus, and are closely related to the combinatorics of Grassmannians, root systems, and Weyl groups (see, e.g., Hiller [10]). In our forthcoming paper [27], we shall discuss the geometric or topological background of our hook formula, in relation to the complex reflection groups $G(e, 1, n)$ and $G(e, e, n)$ (for the root systems of these groups, see Bremke–Malle [1, 2]).

Example 3.11. For the partition $\mu = (2, 2)$, the explicit form of our hook length formula is as follows: First note that there exist two "paths" from $\mu = (2, 2)$ to $\emptyset = (\ )$:

$$\begin{aligned}\mu &= (2,2) \supsetneq (2,1) \supsetneq (2) \supsetneq (1) \supsetneq (\ ), \\ \mu &= (2,2) \supsetneq (2,1) \supsetneq (1,1) \supsetneq (1) \supsetneq (\,).\end{aligned}$$

From the fact that $c^{\nu}_{(1),\nu} = c^{\nu,(n)}_{(1),\nu}(t|\boldsymbol{b}) = P^{(n)}_{(1)}(-\boldsymbol{b}_{\hat{\nu}}(t); t|\boldsymbol{b})$, we get the following result directly:

$$\begin{aligned} c^{(\,)}_{(1),(\,)} &= 0, \\ c^{(1)}_{(1),(1)} &= -b_2 + t^{n-1}b_1, \\ c^{(1,1)}_{(1),(1,1)} &= (1+t)(-b_2 + t^{n-2}b_1), \\ c^{(2)}_{(1),(2)} &= -b_3 + t^{n-1}b_1, \\ c^{(2,1)}_{(1),(2,1)} &= -b_3 - b_2 + (1+t)t^{n-2}b_1, \\ c^{(2,2)}_{(1),(2,2)} &= (1+t)(-b_3 + t^{n-2}b_1). \end{aligned}$$

([5]) In the definition of the factorial Schur polynomial $s_\lambda(x|a)$ given by Molev–Sagan [21, §2, (3)], we replaced a doubly-infinite variable sequence $a = (a_i)$, $i \in \mathbb{Z}$, by $\boldsymbol{b} = (b_1, b_2, \ldots)$.

Similarly, $\alpha_{\nu/\lambda} = \alpha^{(n)}_{\nu/\lambda}(t)$ can be computed directly from the definition, and we get

$$\alpha_{(2,2)/(2,1)} = 1+t, \quad \alpha_{(2,1)/(2)} = 1, \quad \alpha_{(2,1)/(1,1)} = 1, \quad \alpha_{(2)/(1)} = 1,$$
$$\alpha_{(1,1)/(1)} = 1+t, \quad \alpha_{(1)/()} = 1.$$

By Proposition 3.7, we have, for $\mu = (2,2)$,

$$P^{(n)}_{\mu}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) = (-b_3 + t^{n-2}b_1)(-tb_3 + t^{n-2}b_1)(-b_3+b_2)(-tb_3+b_2).$$

Therefore our hook formula gives the following identity:

$$\begin{aligned}
\frac{1+t}{-tb_3+b_2}\cdot\Bigg(&\frac{1}{-tb_3+t^{n-2}b_1}\cdot\frac{1}{-b_3-tb_3+b_2+t^{n-2}b_1}\\
&+\frac{1}{-b_3-tb_3+b_2+tb_2}\cdot\frac{1+t}{-b_3-tb_3+b_2+t^{n-2}b_1}\Bigg)\\
&\times\frac{1}{-b_3-tb_3+t^{n-2}b_1+t^{n-1}b_1}\\
&\qquad=\frac{1}{(-b_3+t^{n-2}b_1)(-tb_3+t^{n-2}b_1)(-b_3+b_2)(-tb_3+b_2)}.
\end{aligned}$$

4. Generating functions for the universal factorial Hall–Littlewood P- and Q-functions. In this section, by utilizing a Gysin formula in complex cobordism (Proposition 2.1) we shall derive the generating functions for the universal factorial Hall–Littlewood P- and Q-functions.

4.1. Generating function for $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$. The basic idea is to apply the fundamental formula (2.2) repeatedly to the characterization (3.5) to obtain the generating function. Here we remark that formula (2.2) still holds for a formal power series $f(u) \in MU^*(X)[[u]]$, and we shall use such an extended form of (2.2). However, we will be confronted with some difficulty when we apply the formula to (3.5). In order to clarify the difficulty, let us consider the simplest case $\lambda = (\lambda_1)$ with $\lambda_1 \geq 1$ (and hence $r = 1$) of (3.5). We wish to push forward the expression $[x_1|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}} \prod_{j=2}^{n}(x_1 +_{\mathbb{L}} [t](\overline{x}_j))$ via the Gysin map $\pi^1_* : MU^*(G^1(E)) \to MU^*(X)$. Naively, setting

$$f(u) := [u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}} \cdot \prod_{j=2}^{n}(u +_{\mathbb{L}} [t](\overline{x}_j)),$$

we wish to compute $\pi^1_*(f(x_1))$. However, one cannot regard $f(u)$ as an element of $MU^*(X)[[u]]$. Therefore we consider the following expression instead:

$$f_1(u) := \frac{[u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}}}{u +_{\mathbb{L}} [t](\overline{u})} \cdot \prod_{j=1}^{n}(u +_{\mathbb{L}} [t](\overline{x}_j)).$$

Since symmetric functions in $x_1, \ldots, x_n$ can be regarded as elements of $MU^*(X)$ ($x_1, \ldots, x_n$ are the Chern roots of E), the coefficients of $f_1(u)$ with respect to u are actually in $MU^*(X)$. Moreover, obviously $f(x_1) = f_1(x_1)$. However, it is not a formal power series in u because of the constant term $b_1 \cdots b_{\lambda_1}$ in the numerator, and therefore formula (2.2) does not apply directly. We further modify $f_1(u)$ to

$$f_2(u) := \frac{[u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}}}{u +_{\mathbb{L}} [t](\overline{u})} \Big\{ \prod_{j=1}^{n} (u +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^{n} [t](u +_{\mathbb{L}} \overline{x}_j) \Big\}. \tag{4.1}$$

The effect of subtracting $\prod_{j=1}^{n}[t](u +_{\mathbb{L}} \overline{x}_j)$ (we call it the "correction term") is two-fold: Firstly, the expression $\prod_{j=1}^{n}(u +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^{n}[t](u +_{\mathbb{L}} \overline{x}_j)$ is divisible by u, and therefore $f_2(u)$ becomes indeed a formal power series in u with coefficients in $MU^*(X)$. Secondly, when we substitute x_1 for u, we have $f(x_1) = f_2(x_1)$ by the obvious identity $\prod_{j=1}^{n}[t](x_1 +_{\mathbb{L}} \overline{x}_j) = 0$. Therefore the fundamental Gysin formula (2.2) does apply to $f_2(u)$, and the result is

$$\begin{aligned}
HP^{\mathbb{L}}_{(\lambda_1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) &= \pi^1_*(f_2(x_1)) = [u^{n-1}](f_2(u) \times \mathscr{S}^{\mathbb{L}}(E; 1/u)) \\
&= [u^{n-1}]\Bigg[\frac{[u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}}}{u +_{\mathbb{L}} [t](\overline{u}_1)} \\
&\quad \times \Big\{ \prod_{j=1}^{n}(u +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^{n}[t](u +_{\mathbb{L}} \overline{x}_j) \Big\} \cdot \mathscr{S}^{\mathbb{L}}(E; 1/u) \Bigg] \\
&= [u^{-\lambda_1}]\Bigg[\frac{1}{\mathscr{P}^{\mathbb{L}}(u)} \frac{u}{u +_{\mathbb{L}} [t](\overline{u})} \\
&\quad \times \Big\{ \prod_{j=1}^{n} \frac{u +_{\mathbb{L}} [t](\overline{x}_j)}{u +_{\mathbb{L}} \overline{x}_j} - \prod_{j=1}^{n} \frac{[t](u +_{\mathbb{L}} \overline{x}_j)}{u +_{\mathbb{L}} \overline{x}_j} \Big\} \cdot \prod_{j=1}^{\lambda_1} \frac{u +_{\mathbb{L}} b_j}{u} \Bigg].
\end{aligned}$$

Example 4.1. As a special case of the above formula, the ordinary factorial Hall–Littlewood P-polynomial corresponding to the one-row (λ_1) is given by

$$HP_{(\lambda_1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) = [u^{-\lambda_1}]\Bigg[\frac{1}{1-t} \Big(\prod_{j=1}^{n} \frac{u - tx_j}{u - x_j} - t^n \Big) \times \prod_{j=1}^{\lambda_1} \frac{u + b_j}{u} \Bigg].$$

In particular,

$$\begin{aligned}
HP_{(1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) &= [u^{-1}]\Bigg[\frac{1}{1-t} \Big(\prod_{j=1}^{n} \frac{u - tx_j}{u - x_j} - t^n \Big) \times \frac{u + b_1}{u} \Bigg] \\
&= \frac{1}{1-t} q_1(\boldsymbol{x}_n; t) + \frac{1 - t^n}{1-t} b_1 \\
&= x_1 + x_2 + \cdots + x_n + (1 + t + t^2 + \cdots + t^{n-1}) b_1.
\end{aligned}$$

Here $q_r(\boldsymbol{x}_n;t)$ $(r=0,1,\dots)$ are given by the following generating functions:

$$\prod_{j=1}^{n}\frac{z-tx_j}{z-x_j}\bigg|_{z=u^{-1}}=\prod_{j=1}^{n}\frac{1-tx_ju}{1-x_ju}=\sum_{r=0}^{\infty}q_r(\boldsymbol{x}_n;t)u^r.$$

For a general sequence $\lambda=(\lambda_1,\dots,\lambda_r)$ of positive integers with $r\le n$, we need to compute the image of

$$[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))$$

under the Gysin map

$$(\pi^{r,r-1,\dots,1})_*:MU^*(\mathcal{F}\ell^{r,\dots,1}(E))\to MU^*(X).$$

This image can be computed by applying $\pi^r_*,\pi^{r-1}_*,\dots,\pi^1_*$ successively. In each step, we use the modification such as (4.1), i.e., subtracting the "correction term". This technique enables us to apply the fundamental Gysin formula (2.2), and we are able to show the following result:

Lemma 4.2. *For a sequence $\lambda=(\lambda_1,\dots,\lambda_r)$ of positive integers with $r\le n$, we have*

$$\begin{aligned}(4.2)\qquad &(\pi^{r,r-1,\dots,1})_*\Big([\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\Big)\\ &=\Big[\prod_{i=1}^{r}u_i^{-\lambda_i}\Big]\Big[\prod_{i=1}^{r}\frac{u_i}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\cdot\frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}\\ &\qquad\times\Big\{\prod_{j=1}^{n}\frac{u_i+_{\mathbb{L}}[t](\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}-\prod_{j=1}^{i-1}\frac{u_i+_{\mathbb{L}}[t](\overline{u}_j)}{[t](u_i+_{\mathbb{L}}\overline{u}_j)}\prod_{j=1}^{n}\frac{[t](u_i+_{\mathbb{L}}\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}\Big\}\\ &\qquad\times\prod_{1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}\times\prod_{i=1}^{r}\prod_{j=1}^{\lambda_i}\frac{u_i+_{\mathbb{L}}b_j}{u_i}\Big].\end{aligned}$$

Proof. To find the desired push-forward under the Gysin map

$$(\pi^{r,r-1,\dots,1})_*=\pi^1_*\circ\cdots\circ\pi^{r-1}_*\circ\pi^r_*,$$

as explained above, we proceed inductively. For $a=1,\dots,r-1$ we assume that

$$(4.3)\qquad(\pi^{r-a+1}\circ\cdots\circ\pi^{r-1}\circ\pi^r)_*\Big([\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\Big)$$

$$
\begin{aligned}
&= [u_{r-a+1}^{n-1} \cdots u_{r-1}^{n-1} u_r^{n-1}] \\
&\Bigg[\prod_{i=1}^{r-a} [x_i|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_i} \prod_{j=i+1}^{n} (x_i +_{\mathbb{L}} [t](\overline{x}_j)) \cdot \prod_{i=r-a+1}^{r} \frac{[u_i|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_i}}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \\
&\times \Bigg\{ \prod_{j=r-a+1}^{n} (u_i +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=r-a+1}^{i-1} \frac{u_i +_{\mathbb{L}} [t](\overline{u}_j)}{[t](u_i +_{\mathbb{L}} \overline{u}_j)} \prod_{j=r-a+1}^{n} [t](u_i +_{\mathbb{L}} \overline{x}_j) \Bigg\} \\
&\times \prod_{i=r-a+1}^{r} \prod_{j=1}^{r-a} (u_i +_{\mathbb{L}} \overline{x}_j) \cdot \prod_{r-a+1 \le i < j \le r} \frac{u_j +_{\mathbb{L}} \overline{u}_i}{u_j +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \prod_{i=r-a+1}^{r} \mathscr{S}^{\mathbb{L}}(E; 1/u_i) \Bigg].
\end{aligned}
$$

We would like to push forward this formula via the Gysin map

$$\pi_*^{r-a} : MU^*(G^1(U_{n-r+a+1})) \to MU^*(G^1(U_{n-r+a+2})).$$

Taking (4.1) into account, we modify the right-hand side of (4.3) as

$$
\begin{aligned}
&[u_{r-a+1}^{n-1} \cdots u_{r-1}^{n-1} u_r^{n-1}] \Bigg[\prod_{i=1}^{r-a-1} [x_i|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_i} \prod_{j=i+1}^{n} (x_i +_{\mathbb{L}} [t](\overline{x}_j)) \\
&\quad \times \frac{[x_{r-a}|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_{r-a}}}{x_{r-a} +_{\mathbb{L}} [t](\overline{x}_{r-a})} \Bigg\{ \prod_{j=r-a}^{n} (x_{r-a} +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=r-a}^{n} [t](x_{r-a} +_{\mathbb{L}} \overline{x}_j) \Bigg\} \\
&\quad \times \prod_{i=r-a+1}^{r} \frac{[u_i|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_i}}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \Bigg\{ \frac{1}{u_i +_{\mathbb{L}} [t](\overline{x}_{r-a})} \prod_{j=r-a}^{n} (u_i +_{\mathbb{L}} [t](\overline{x}_j)) \\
&\qquad\qquad - \prod_{j=r-a+1}^{i-1} \frac{u_i +_{\mathbb{L}} [t](\overline{u}_j)}{[t](u_i +_{\mathbb{L}} \overline{u}_j)} \cdot \frac{1}{[t](u_i +_{\mathbb{L}} \overline{x}_{r-a})} \prod_{j=r-a}^{n} [t](u_i +_{\mathbb{L}} \overline{x}_j) \Bigg\} \\
&\quad \times \prod_{i=r-a+1}^{r} \prod_{j=1}^{r-a-1} (u_i +_{\mathbb{L}} \overline{x}_j) \times \prod_{i=r-a+1}^{r} (u_i +_{\mathbb{L}} \overline{x}_{r-a}) \\
&\qquad\qquad \times \prod_{r-a+1 \le i < j \le r} \frac{u_j +_{\mathbb{L}} \overline{u}_i}{u_j +_{\mathbb{L}} [t](\overline{u}_i)} \times \prod_{i=r-a+1}^{r} \mathscr{S}^{\mathbb{L}}(E; 1/u_i) \Bigg].
\end{aligned}
$$

Then, we apply (2.2). In the above modification, we divide both the denominator and the numerator of $\frac{1}{u_i +_{\mathbb{L}} [t](\overline{x}_{r-a})}$ by u_i, and consider it as a formal power series in x_{r-a}. We also treat $\frac{1}{[t](u_i +_{\mathbb{L}} \overline{x}_{r-a})}$ in the same manner. With this remark, the result is just replacing x_{r-a} by the formal variable u_{r-a}, and multiplying by $\mathscr{S}^{\mathbb{L}}(U_{n-r+a+1}; 1/u_{r-a})$. Then, we extract the coefficient of u_{r-a}^{n-r+a}. Since we know from (2.1) that

$$\mathscr{S}^{\mathbb{L}}(U_{n-r+a+1}; 1/u_{r-a}) = u_{r-a}^{-(r-a-1)} \prod_{j=1}^{r-a-1} (u_{r-a} +_{\mathbb{L}} \overline{x}_j) \times \mathscr{S}^{\mathbb{L}}(E; 1/u_{r-a}),$$

we see directly that formula (4.3) holds for $a+1$. Therefore, when $a = r$,

$$
\begin{aligned}
&(\pi^{r,r-1,\dots,1})_*\Big([\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i +_{\mathbb{L}} [t](\overline{x}_j))\Big) \\
&= [u_1^{n-1}\dots u_r^{n-1}] \\
&\quad \Bigg[\prod_{i=1}^{r} \frac{[u_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}}}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \Bigg\{ \prod_{j=1}^{n}(u_i +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} [t](\overline{u}_j)}{[t](u_i +_{\mathbb{L}} \overline{u}_j)} \prod_{j=1}^{n} [t](u_i +_{\mathbb{L}} \overline{x}_j)\Bigg\} \\
&\qquad\qquad \times \prod_{1\le i<j\le r} \frac{u_j +_{\mathbb{L}} \overline{u}_i}{u_j +_{\mathbb{L}} [t](\overline{u}_i)} \times \prod_{i=1}^{r} \mathscr{S}^{\mathbb{L}}(E; 1/u_i)\Bigg].
\end{aligned}
$$

Then, using the Segre series (2.1), we obtain the required formula. ■

By (3.5), the left-hand side of (4.2) is $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$, and hence the right-hand side gives a generating function for $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$. Let us simplify this generating function in the following way: First note that

$$
\prod_{1\le i<j\le r} \frac{u_j +_{\mathbb{L}} \overline{u}_i}{u_j +_{\mathbb{L}} [t](\overline{u}_i)} = \prod_{1\le j<i\le r} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)} = \prod_{i=1}^{r}\prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)}.
$$

Therefore if we put

$$
\begin{aligned}
\mathcal{HP}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}) &:= \frac{u_i}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)} \\
&\quad \times \Bigg(\prod_{j=1}^{n} \frac{u_i +_{\mathbb{L}} [t](\overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j} \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)} \prod_{j=1}^{\lambda_i} \frac{u_i +_{\mathbb{L}} b_j}{u_i} \\
&\qquad - \prod_{j=1}^{n} \frac{[t](u_i +_{\mathbb{L}} \overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j} \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{[t](u_i +_{\mathbb{L}} \overline{u}_j)} \prod_{j=1}^{\lambda_i} \frac{u_i +_{\mathbb{L}} b_j}{u_i} \Bigg),
\end{aligned}
$$

$$
\mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b}) := \prod_{i=1}^{r} \mathcal{HP}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}),
$$

then

$$
HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}](\mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})). \tag{4.4}
$$

Moreover, observe that:

- $\frac{u_i}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}$ is a formal power series in u_i.
- $\prod_{j=1}^{n} \frac{[t](u_i +_{\mathbb{L}} \overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j} \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{[t](u_i +_{\mathbb{L}} \overline{u}_j)}$ is regarded as a formal power series in u_i with constant term t^{n-i+1}.
- $\prod_{j=1}^{\lambda_i} \frac{u_i +_{\mathbb{L}} b_j}{u_i}$ is a formal Laurent series in u_i whose lowest degree term is $u_i^{-\lambda_i}$ with coefficient $\prod_{j=1}^{\lambda_i} b_j$.

Taking the above observation into account, we put

$$\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}) := \frac{u_i}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}$$
$$\times \left(\prod_{j=1}^{n} \frac{u_i +_{\mathbb{L}} [t](\overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j} \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)} \prod_{j=1}^{\lambda_i} \frac{u_i +_{\mathbb{L}} b_j}{u_i} - t^{n-i+1} \prod_{j=1}^{\lambda_i} \frac{b_j}{u_i} \right),$$

$$\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b}) := \prod_{i=1}^{r} \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}).$$

Then we can reduce $\mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})$ to $\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})$, and obtain from (4.4) the following result:

THEOREM 4.3 (Generating function for $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$). *For a sequence $\lambda = (\lambda_1,\dots,\lambda_r)$ of positive integers with $r \le n$, the universal factorial Hall–Littlewood P-function $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1}\cdots u_r^{-\lambda_r}$ in $\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b})$:*

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}](\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})).$$

If we specialize the universal formal group law $F_{\mathbb{L}}(u,v) = u +_{\mathbb{L}} v$ to $F_a(u,v) = u+v$, the above generating function takes a relatively simple form:

$$\widetilde{\mathcal{HP}}^{(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b})$$
$$= \frac{1}{1-t}\left(\prod_{j=1}^{n} \frac{u_i - tx_j}{u_i - x_j} \prod_{j=1}^{i-1} \frac{u_i - u_j}{u_i - tu_j} \prod_{j=1}^{\lambda_i} \frac{u_i + b_j}{u_i} - t^{n-i+1} \prod_{j=1}^{\lambda_i} \frac{b_j}{u_i} \right),$$

$$\widetilde{\mathcal{HP}}^{(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \widetilde{\mathcal{HP}}^{(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b}) = \prod_{i=1}^{r} \widetilde{\mathcal{HP}}^{(n)}_{i,\lambda_i}(u_1,\dots,u_i|\boldsymbol{b}).$$

Thus we have the following corollary:

COROLLARY 4.4 (Generating function for $HP_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$). *For a sequence $\lambda = (\lambda_1,\dots,\lambda_r)$ of positive integers with $r \le n$, the factorial Hall–Littlewood P-polynomial $HP_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1}\cdots u_r^{-\lambda_r}$ in $\widetilde{\mathcal{HP}}^{(n)}_{\lambda}(u_1,\dots,u_r|\boldsymbol{b})$:*

$$HP_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}](\widetilde{\mathcal{HP}}^{(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})).$$

4.2. Generating function for $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$. Next we shall derive the generating function for $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$. In the one-row case $\lambda = (\lambda_1)$ of (3.6), we push forward the expression

$$[[x_1;t|\boldsymbol{b}]]_{\mathbb{L}}^{\lambda_1}\prod_{j=2}^{n}(x_1+_{\mathbb{L}}[t](\overline{x}_j)) = \frac{(x_1+_{\mathbb{L}}[t](\overline{x}_1))[x_1|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_1-1}}{x_1+_{\mathbb{L}}[t](\overline{x}_1)}\prod_{j=1}^{n}(x_1+_{\mathbb{L}}[t](\overline{x}_j))$$
$$= [x_1|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_1-1}\prod_{j=1}^{n}(x_1+_{\mathbb{L}}[t](\overline{x}_j)),$$

which is a formal power series in x_1, and so we can apply (2.2) to obtain

$$HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [u_1^{-\lambda_1}\cdots u_r^{-\lambda_r}]$$
$$\left[\prod_{i=1}^{r}\frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}\prod_{j=1}^{n}\frac{u_i+_{\mathbb{L}}[t](\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}\prod_{1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}\cdot\prod_{i=1}^{r}\prod_{j=1}^{\lambda_i-1}\frac{u_i+_{\mathbb{L}}b_j}{u_i}\right].$$

For each non-negative integer k, we set

$$\mathcal{HQ}_k^{\mathbb{L},(n)}(u|\boldsymbol{b}) := \frac{1}{\mathscr{P}^{\mathbb{L}}(u)}\prod_{j=1}^{n}\frac{u+_{\mathbb{L}}[t](\overline{x}_j)}{u+_{\mathbb{L}}\overline{x}_j}\times\prod_{j=1}^{k}\frac{u+_{\mathbb{L}}b_j}{u}.$$

For a sequence $\lambda=(\lambda_1,\ldots,\lambda_r)$ of positive integers with $r\le n$, we set

$$\mathcal{HQ}_\lambda^{\mathbb{L},(n)}(\boldsymbol{u}_r|\boldsymbol{b}) = \mathcal{HQ}_\lambda^{\mathbb{L},(n)}(u_1,\ldots,u_r|\boldsymbol{b})$$
$$:= \prod_{i=1}^{r}\mathcal{HQ}_{\lambda_i-1}^{\mathbb{L},(n)}(u_i|\boldsymbol{b})\prod_{1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}.$$

Thus we have the following result:

Theorem 4.5 (Generating function for $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b})$). *For a sequence $\lambda=(\lambda_1,\ldots,\lambda_r)$ of positive integers with $r\le n$, the universal factorial Hall–Littlewood Q-function $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda}=u_1^{-\lambda_1}\cdots u_r^{-\lambda_r}$ in $\mathcal{HQ}_\lambda^{\mathbb{L},(n)}(u_1,\ldots,u_r|\boldsymbol{b})$:*

$$HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}](\mathcal{HQ}_\lambda^{\mathbb{L},(n)}(\boldsymbol{u}_r|\boldsymbol{b})).$$

5. Application of generating functions

5.1. e-Cancellation property. A symmetric polynomial $f(x_1,\ldots,x_n)$ with coefficients in $\mathbb{Z}$ has the *Q-cancellation property* if whenever the substitution $x_1=a$, $x_2=-a$, a an indeterminate, is made in f, the resulting polynomial is independent of a (Pragacz [31, §2]). It is known that the Schur P- and Q-polynomials satisfy this cancellation property. The notion of the Q-cancellation property is generalized in the following way: Let $e\ge 2$ be a fixed integer, and $\zeta=\zeta_e$ be a primitive eth root of unity. We define $\boldsymbol{a}^e(\zeta)$ to be the sequence $(a,\zeta a,\zeta^2 a,\ldots,\zeta^{e-1}a)$. Suppose that $e\le n$. Then a symmetric polynomial $f(x_1,\ldots,x_n)$ with coefficients in $\mathbb{Z}[\zeta]$ has the *e-cancellation property* if $f(\boldsymbol{a}^e(\zeta), x_{e+1}, \ldots, x_n) = f(a, \zeta a, \zeta^2 a, \ldots, \zeta^{e-1}a, x_{e+1}, \ldots, x_n)$ does not depend on a. In the case $e = 2$, this property is nothing but the Q-cancellation property. By specializing t to be ζ, the factorial Hall–Littlewood polynomials $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ are symmetric polynomials with coefficients in $\mathbb{Z}[\zeta] \otimes \mathbb{Z}[\boldsymbol{b}] = \mathbb{Z}[\zeta] \otimes \mathbb{Z}[b_1, b_2, \ldots]$. Thus one can ask if these symmetric polynomials have the e-cancellation property or not. In this subsection, as the first application of our generating functions, we shall establish the e-cancellation property of the factorial Hall–Littlewood P- and Q-polynomials.

Proposition 5.1 (e-Cancellation property). *Assume that $e \leq n$. The factorial Hall–Littlewood polynomials $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ have the e-cancellation property.*

Proof. Let r be the length of λ. By Corollary 4.4, $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda}$ in the generating function

$$\frac{1}{(1-\zeta)^r} \prod_{i=1}^{r} \left(\prod_{j=1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j} \prod_{j=1}^{i-1} \frac{u_i - u_j}{u_i - \zeta u_j} \prod_{j=1}^{\lambda_i} \frac{u_i + b_j}{u_i} - \zeta^{n-i+1} \prod_{j=1}^{\lambda_i} \frac{b_j}{u_i} \right). \tag{5.1}$$

Substituting $(x_1, \ldots, x_e)$ with $\boldsymbol{a}^e(\zeta)$ in each factor $\prod_{j=1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j}$, we have

$$\prod_{j=1}^{e} \frac{u_i - \zeta^j a}{u_i - \zeta^{j-1} a} \times \prod_{j=e+1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j} = \prod_{j=e+1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j},$$

since $\zeta^e = 1$. Therefore, (5.1) depends neither on a nor on $x_1, \ldots, x_e$ after substitution. From this, the e-cancellation property of $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ follows. By Theorem 4.5, $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ is given as the coefficient of $\boldsymbol{u}^{-\lambda}$ in the generating function

$$\prod_{i=1}^{r} \prod_{j=1}^{n} \frac{u - \zeta x_j}{u - x_j} \times \prod_{j=1}^{\lambda_i - 1} \frac{u + b_j}{u} \times \prod_{1 \leq i < j \leq r} \frac{u_j - u_i}{u_j - \zeta u_i}.$$

From this, the e-cancellation property of $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ follows just as for $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$. ∎

Remark 5.2. For the universal factorial Hall–Littlewood P- and Q-functions, we substitute $(x_1, \ldots, x_e)$ with

$$\boldsymbol{a}^e[\zeta] := (a, [\zeta](a), [\zeta^2](a), \ldots, [\zeta^{e-1}](a)).$$

Using Theorems 4.3 and 4.5, one can then verify easily that $HP_\lambda^{\mathbb{L}}(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ and $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ satisfy the e-cancellation property too.

5.2. Pfaffian formula for $GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b})$. As another application of our generating functions, we shall derive the *Pfaffian formulas* for the K-theoretic factorial Q-polynomial $GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b})$, which seems to be new. In what follows, we assume that the length $\ell(\nu)$ of a strict partition ν is $2m$ (even). We consider the specialization from $F_{\mathbb{L}}(u,v) = u +_{\mathbb{L}} v$ to $F_m(u,v) = u \oplus v$ with $t = -1$. Then $HQ^{\mathbb{L}}_\nu(\boldsymbol{x}_n; t|\boldsymbol{b})$ specializes to $GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b})$, and the generating function $\mathcal{HQ}^{\mathbb{L},(n)}_\nu(\boldsymbol{u}_{2m}|\boldsymbol{b})$ reduces to

$$\mathcal{GQ}^{(n)}_\nu(\boldsymbol{u}_{2m}|\boldsymbol{b}) = \prod_{i=1}^{2m} \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b}) \prod_{1\le i<j\le 2m} \frac{u_j \ominus u_i}{u_j \oplus u_i}, \tag{5.2}$$

where, for each non-negative integer k, we define

$$\begin{aligned}\mathcal{GQ}^{(n)}_k(u|\boldsymbol{b}) &:= \frac{1}{1+\beta u}\prod_{j=1}^{n}\frac{u\oplus x_j}{u\ominus x_j} \times \prod_{j=1}^{k}\frac{u\oplus b_j}{u} \\ &= \frac{1}{1+\beta u}\prod_{j=1}^{n}\frac{1+(u^{-1}+\beta)x_j}{1+(u^{-1}+\beta)\overline{x}_j} \times \prod_{j=1}^{k}\{1+(u^{-1}+\beta)b_j\}.\end{aligned}$$

This is a generating function for the factorial K-theoretic Q-polynomials $GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b})$. Here we recall from Ikeda–Naruse [14, Lemma 2.4] the formula

$$\mathrm{Pf}\left(\frac{x_j - x_i}{x_j \oplus x_i}\right)_{1\le i<j\le 2m} = \prod_{1\le i<j\le 2m}\frac{x_j - x_i}{x_j \oplus x_i}.$$

Thus we can compute ([6])

$$\begin{aligned}\mathcal{GQ}^{(n)}_\nu(\boldsymbol{u}_{2m}|\boldsymbol{b}) &= \prod_{i=1}^{2m} \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b}) \prod_{1\le i<j\le 2m} \frac{u_j \ominus u_i}{u_j \oplus u_i} \\ &= \prod_{i=1}^{2m} \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b}) \prod_{i=1}^{2m}\frac{1}{(1+\beta u_i)^{2m-i}} \cdot \mathrm{Pf}\left(\frac{u_j - u_i}{u_j \oplus u_i}\right)_{1\le i<j\le 2m} \\ &= \mathrm{Pf}_{2m}\Bigg(\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b}) \\ &\qquad\qquad \times \frac{1}{(1+\beta u_i)^{2m-i}}\,\frac{1}{(1+\beta u_j)^{2m-j}}\cdot\frac{u_j-u_i}{u_j\oplus u_i}\Bigg) \\ &= \mathrm{Pf}_{2m}\Bigg((1+\beta u_i)^{i+1-2m}(1+\beta u_j)^{j-2m} \\ &\qquad\qquad \times \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b})\cdot\frac{u_j\ominus u_i}{u_j\oplus u_i}\Bigg).\end{aligned}$$

([6]) Below we use $\mathrm{Pf}_{2m}\,(a_{i,j})$ as an abbreviation of $\mathrm{Pf}\,(a_{i,j})_{1\le i<j\le 2m}$ when the expression is too long.

For non-negative integers $p, q \geq 0$ and positive integers $k, l \geq 1$, we define polynomials $GQ^{(p,q)}_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b})$ to be

$$GQ^{(p,q)}_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b}) := [u_1^{-k}u_2^{-l}]\left(\mathcal{GQ}^{(n)}_{p-1}(u_1|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{q-1}(u_2|\boldsymbol{b}) \cdot \frac{u_2 \ominus u_1}{u_2 \oplus u_1}\right).$$

Note that, by Theorem 4.5, we have $GQ_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b}) = GQ^{(k,l)}_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b})$ for positive integers $k > l > 0$. Then, by Theorem 4.5, one obtains

$$\begin{aligned}
GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b}) &= \Big[\prod_{i=1}^{2m} u_i^{-\nu_i}\Big]\Bigg(\mathrm{Pf}\,\Big((1+\beta u_i)^{i+1-2m}(1+\beta u_j)^{j-2m} \\
&\qquad \times \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b}) \times \frac{u_j \ominus u_i}{u_j \oplus u_i}\Big)_{1\leq i<j\leq 2m}\Bigg) \\
&= \mathrm{Pf}\,\Bigg([u_i^{-\nu_i}u_j^{-\nu_j}]\Big((1+\beta u_i)^{i+1-2m}(1+\beta u_j)^{j-2m} \\
&\qquad \times \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b}) \times \frac{u_j \ominus u_i}{u_j \oplus u_i}\Big)\Bigg)_{1\leq i<j\leq 2m} \\
&= \mathrm{Pf}\,\Bigg([u_i^{-\nu_i}u_j^{-\nu_j}]\Big(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+1-2m}{k}\binom{j-2m}{l}\beta^{k+l}u_i^k u_j^l \\
&\qquad \times \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b}) \cdot \frac{u_j \ominus u_i}{u_j \oplus u_i}\Big)\Bigg)_{1\leq i<j\leq 2m} \\
&= \mathrm{Pf}\,\Bigg(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+1-2m}{k}\binom{j-2m}{l}\beta^{k+l}GQ^{(\nu_i,\nu_j)}_{(\nu_i+k,\nu_j+l)}(\boldsymbol{x}_n|\boldsymbol{b})\Bigg)_{1\leq i<j\leq 2m}.
\end{aligned}$$

Thus we have obtained the following:

Theorem 5.3 (Pfaffian formula for $GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b})$). *For a strict partition ν of length $2m$, we have*

$$GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b}) \\ = \mathrm{Pf}_{2m}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+1-2m}{k}\binom{j-2m}{l}\beta^{k+l}GQ^{(\nu_i,\nu_j)}_{(\nu_i+k,\nu_j+l)}(\boldsymbol{x}_n|\boldsymbol{b})\right).$$

Remark 5.4.

(1) Putting $\boldsymbol{b} = \boldsymbol{0}$ in (5.2), we obtain a generating function for the (non-factorial) K-theoretic Q-polynomials $GQ_\nu(\boldsymbol{x}_n)$. On the other hand, dual K-theoretic P- and Q-polynomials were introduced in our previous papers [23, §5], [25]. We have a conjecture on a generating function for the dual K-theoretic Q-polynomials, and their Pfaffian formula (see §6.2).

(2) The generating function technique can also be applied to derive the determinantal formula for factorial Grothendieck polynomials $G_\lambda(\boldsymbol{x}_n|\boldsymbol{b})$. On the other hand, a generating function for the dual Grothendieck polynomials $g_\lambda(\boldsymbol{x}_n)$ (for their definition, see Lascoux–Naruse [18]) can be obtained in a purely algebraic manner. We shall give the details in §6.1.

6. Appendix

6.1. Generating function for the dual Grothendieck polynomials. As mentioned in Remark 5.4, we give a generating function for the dual Grothendieck polynomials. Following Lascoux–Naruse [18], let us introduce the dual Grothendieck polynomials $g_\lambda(\boldsymbol{y}_n)$, where $\boldsymbol{y}_n = (y_1, \ldots, y_n)$ is a set of independent variables and $\lambda \in \mathcal{P}_n$. First we need some notation: Given two sets $\mathbf{A}$, $\mathbf{B}$ of variables (called *alphabets* as usual), the *complete functions* $s_k(\mathbf{A} - \mathbf{B})$ $(k = 0, 1, \ldots)$ are given by the following generating function:

$$\sum_{k=0}^{\infty} s_k(\mathbf{A} - \mathbf{B}) z^k = \prod_{a \in \mathbf{A}} \frac{1}{1 - az} \prod_{b \in \mathbf{B}} (1 - bz).$$

In particular, when we add r letters specialized to 1, that is, the sequence $\{1, \ldots, 1\}$ (r times), to one of the alphabets $\mathbf{A}$ or $\mathbf{B}$, we have

$$\sum_{k=0}^{\infty} s_k(\mathbf{A} - \mathbf{B} \pm r) z^k = (1 - z)^{\mp r} \prod_{a \in \mathbf{A}} \frac{1}{1 - az} \prod_{b \in \mathbf{B}} (1 - bz).$$

Then, for the variables $\boldsymbol{y}_n = (y_1, \ldots, y_n)$ and any integer r, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} s_k(\boldsymbol{y}_n + r) z^k &= (1 - z)^{-r} \prod_{i=1}^{n} \frac{1}{1 - y_i z} \\
&= \left(\sum_{i=0}^{\infty} \binom{-r}{i} (-z)^i \right) \left(\sum_{j=0}^{\infty} h_j(\boldsymbol{y}_n) z^j \right) \\
&= \left(\sum_{i=0}^{\infty} (-1)^i \binom{r + i - 1}{i} (-z)^i \right) \left(\sum_{j=0}^{\infty} h_j(\boldsymbol{y}_n) z^j \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{r + i - 1}{i} h_{k-i}(\boldsymbol{y}_n) \right) z^k,
\end{aligned}$$

and hence

$$s_k(\boldsymbol{y}_n + r) = \sum_{i=0}^{k} \binom{r + i - 1}{i} h_{k-i}(\boldsymbol{y}_n) \quad (k = 0, 1, \ldots). \tag{6.1}$$

Using (6.1), the dual Grothendieck polynomial $g_\lambda(\boldsymbol{y}_n)$ for $\lambda \in \mathcal{P}_n$ of length r is given by (see Lascoux–Naruse [18, (3)])

$$
\begin{aligned}
g_\lambda(\boldsymbol{y}_n) &= s_\lambda(\boldsymbol{y}_n, \boldsymbol{y}_n + 1, \ldots, \boldsymbol{y}_n + n - 1) \\
&= \det\left(s_{\lambda_i - i + j}(\boldsymbol{y}_n + i - 1)\right)_{1 \le i,j \le r} \\
&= \det\left(\sum_{k=0}^{\lambda_i - i + j} \binom{(i-1)+k-1}{k} h_{\lambda_i - i + j - k}(\boldsymbol{y}_n)\right)_{1 \le i,j \le r} \\
&= \det\left(\sum_{k=0}^{\infty} \binom{i+k-2}{k} h_{\lambda_i - i + j - k}(\boldsymbol{y}_n)\right)_{1 \le i,j \le r}.
\end{aligned} \tag{6.2}
$$

We set

$$
\begin{aligned}
H(z) = H^{(n)}(z) &:= \prod_{j=1}^{n} \frac{1}{1 - y_j z} = \sum_{k=0}^{\infty} h_k(\boldsymbol{y}_n) z^k, \\
g(\boldsymbol{z}_r) = g(z_1, \ldots, z_r) &:= \prod_{i=1}^{r} H(z_i) \prod_{1 \le i < j \le r} \frac{z_i \ominus z_j}{z_i}.
\end{aligned}
$$

We shall show that $g(\boldsymbol{z}_r)$ is the generating function for the dual Grothendieck polynomials:

THEOREM 6.1 (Generating function for $g_\lambda(\boldsymbol{y}_n)$). *For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of length $\ell(\lambda) = r \le n$, the dual Grothendieck polynomial $g_\lambda(\boldsymbol{y}_n)$ is the coefficient of $\boldsymbol{z}^\lambda = z_1^{\lambda_1} \cdots z_r^{\lambda_r}$ in $g(z_1, \ldots, z_r)$:*

$$g_\lambda(\boldsymbol{y}_n) = [\boldsymbol{z}^\lambda](g(\boldsymbol{z}_r)).$$

Proof. By the Vandermonde determinant formula, we have

$$
\begin{aligned}
\prod_{1 \le i < j \le r} \frac{z_i \ominus z_j}{z_i} &= \prod_{i=1}^{r-1} \prod_{j=i+1}^{r} \frac{1}{1 + \beta z_j} \cdot \frac{z_i - z_j}{z_i} \\
&= \prod_{i=1}^{r-1} \frac{1}{(1 + \beta z_{i+1}) \cdots (1 + \beta z_r)} \cdot \prod_{i=1}^{r-1} \frac{1}{z_i^{r-i}} \cdot \prod_{1 \le i < j \le r} (z_i - z_j) \\
&= \prod_{i=1}^{r} \frac{1}{(1 + \beta z_i)^{i-1}} \cdot \prod_{i=1}^{r} \frac{1}{z_i^{r-i}} \cdot \det\left(z_i^{r-j}\right)_{1 \le i,j \le r} \\
&= \det\left((1 + \beta z_i)^{1-i} z_i^{i-j}\right)_{1 \le i,j \le r}.
\end{aligned}
$$

Therefore

$$g(z_1, \ldots, z_r) = \prod_{i=1}^{r} H(z_i) \prod_{1 \le i < j \le r} \frac{z_i \ominus z_j}{z_i} = \det\left((1+\beta z_i)^{1-i} z_i^{i-j} H(z_i)\right)_{1 \le i,j \le r}.$$

Extracting the coefficient of the monomial $\boldsymbol{z}^\lambda = \prod_{i=1}^r z_i^{\lambda_i}$, we obtain

$$
\begin{aligned}
[\boldsymbol{z}^\lambda](g(z_1,\ldots,z_r)) &= \Big[\prod_{i=1}^r z_i^{\lambda_i}\Big]\Big(\det\left((1+\beta z_i)^{1-i} z_i^{i-j} H(z_i)\right)_{1\le i,j\le r}\Big) \\
&= \det\Big([z_i^{\lambda_i}]((1+\beta z_i)^{1-i} z_i^{i-j} H(z_i)\Big)_{1\le i,j\le r} \\
&= \det\Big([z_i^{\lambda_i}]\Big(\sum_{k=0}^\infty \binom{1-i}{k}\beta^k z_i^k \cdot z_i^{i-j} H(z_i)\Big)\Big)_{1\le i,j\le r} \\
&= \det\Big(\sum_{k=0}^\infty \binom{1-i}{k}\beta^k h_{\lambda_i-i+j-k}(\boldsymbol{y}_n)\Big)_{1\le i,j\le r} \\
&= \det\Big(\sum_{k=0}^\infty \binom{i+k-2}{k}(-\beta)^k h_{\lambda_i-i+j-k}(\boldsymbol{y}_n)\Big)_{1\le i,j\le r}.
\end{aligned}
$$

Here we have used the identity

$$
\binom{1-i}{k} = \binom{-(i-1)}{k} = (-1)^k\binom{i+k-2}{k}
$$

for integers $i \ge 1$, $k \ge 0$. This last determinant is the dual Grothendieck polynomial $g_\lambda(\boldsymbol{y}_n)$ introduced in (6.2) with $\beta = -1$. ∎

6.2. Conjecture on a generating function for $gq_\nu(\boldsymbol{y}_n)$. In [14, §3.4], Ikeda–Naruse introduced the K-theoretic P- and Q-functions $GP_\nu(\boldsymbol{x})$ and $GQ_\nu(\boldsymbol{x})$ in countably many variables $\boldsymbol{x} = (x_1, x_2, \ldots)$. Let $G\mathit{\Gamma}'(\boldsymbol{x})$ denote the ring of symmetric functions satisfying the K-theoretic Q-cancellation property (see [14, Definition 1.1]). Similarly, let $G\mathit{\Gamma}(\boldsymbol{x})$ denote the subring of $G\mathit{\Gamma}'(\boldsymbol{x})$ consisting of all functions f such that $f(t, x_2, \ldots) - f(0, x_2, \ldots)$ is divisible by $t \oplus t$ (7). Ikeda–Naruse showed that $GP_\nu(\boldsymbol{x})$'s and $GQ_\nu(\boldsymbol{x})$'s (ν strict) form a formal $\mathbb{Z}[\beta]$-basis of $G\mathit{\Gamma}'(\boldsymbol{x})$ and $G\mathit{\Gamma}(\boldsymbol{x})$ respectively. Using this "basis theorem" and the "Cauchy kernel"

$$
\mathit{\Delta}(\boldsymbol{x};\boldsymbol{y}) = \prod_{i=1}^\infty\prod_{j=1}^\infty \frac{1-\overline{x}_i y_j}{1-x_i y_j},
$$

where $\boldsymbol{y} = (y_1, y_2, \ldots)$ is another set of independent variables, we can define the *dual K-theoretic P- and Q-functions*, denoted by $gp_\nu(\boldsymbol{y})$ and $gq_\nu(\boldsymbol{y})$, as follows (see also Nakagawa–Naruse [23, Definition 5.3, Remark 5.4]):

Definition 6.2 (Dual K-theoretic Schur P- and Q-functions). Let $\mathcal{SP}$ denote the set of all strict partitions. We define $gp_\nu(\boldsymbol{y})$ and $gq_\nu(\boldsymbol{y})$ for a

(7) We have slightly changed the notation of [14], where $G\mathit{\Gamma}'(\boldsymbol{x})$ and $G\mathit{\Gamma}(\boldsymbol{x})$ are written as $G\mathit{\Gamma}$ and $G\mathit{\Gamma}_+$ respectively.

strict partition $\nu \in \mathcal{SP}$ by the identities

(6.3)
$$\Delta(\boldsymbol{x};\boldsymbol{y}) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\nu\in\mathcal{SP}} GP_\nu(\boldsymbol{x}) gq_\nu(\boldsymbol{y}) = \sum_{\nu\in\mathcal{SP}} GQ_\nu(\boldsymbol{x}) gp_\nu(\boldsymbol{y}).$$

One can check that $gp_\nu(\boldsymbol{y})$ and $gq_\nu(\boldsymbol{y})$ are actually symmetric functions, i.e., they are elements of $\Lambda(\boldsymbol{y}) \otimes \mathbb{Z}[\beta]$, where $\Lambda(\boldsymbol{y})$ is the ring of symmetric functions in the variables $\boldsymbol{y} = (y_1, y_2, \ldots)$ over $\mathbb{Z}$. For each positive integer n, one can define a surjective ring homomorphism $\rho^{(n)} : \Lambda(\boldsymbol{y}) \twoheadrightarrow \Lambda(\boldsymbol{y}_n)$ by putting $y_{n+1} = y_{n+2} = \cdots = 0$. Here $\Lambda(\boldsymbol{y}_n) = \mathbb{Z}[y_1, \ldots, y_n]^{S_n}$ is the ring of symmetric polynomials in $\boldsymbol{y}_n = (y_1, \ldots, y_n)$ under the usual action of the symmetric group S_n. We also denote by $\rho^{(n)}$ its extension over $\mathbb{Z}[\beta]$. Then we define the *dual K-theoretic Schur P- and Q-polynomials*, denoted by $gp_\nu(\boldsymbol{y}_n)$ and $gq_\nu(\boldsymbol{y}_n)$ for a strict partition ν of length $\leq n$, by $gp_\nu(\boldsymbol{y}_n) = \rho^{(n)}(gp_\nu(\boldsymbol{y}))$ and $gq_\nu(\boldsymbol{y}_n) = \rho^{(n)}(gq_\nu(\boldsymbol{y}))$ respectively.

Next we set

$$gq(z) = \prod_{j=1}^{n} \frac{1-y_j\overline{z}}{1-y_j z} = \sum_{k=0}^{\infty} gq_k(\boldsymbol{y}_n) z^k,$$
$$gq(\boldsymbol{z}_r) = gq(z_1, \ldots, z_r) := \prod_{i=1}^{r} gq(z_i) \prod_{1\leq i<j\leq r} \frac{z_i \ominus z_j}{z_i \oplus z_j}.$$

We make the following conjectures:

Conjecture 6.3 (Generating function for $gq_\nu(\boldsymbol{y}_n)$). For a strict partition $\nu = (\nu_1, \ldots, \nu_r)$ of length $\ell(\nu) = r \leq n$, the dual K-theoretic Q-polynomial $gq_\nu(\boldsymbol{y}_n)$ is the coefficient of $\boldsymbol{z}_r = z_1^{\nu_1} z_2^{\nu_2} \cdots z_r^{\nu_r}$ in $gq(z_1, \ldots, z_r)$:

$$gq_\nu(\boldsymbol{y}_n) = [\boldsymbol{z}^\nu](gq(\boldsymbol{z}_r)).$$

We have checked that the above conjecture holds for $r \leq 2$. As a corollary to the above conjecture, we immediately obtain the following formula:

Corollary 6.4 (Pfaffian formula for $gq_\nu(\boldsymbol{y}_n)$). *For a strict partition ν of length $2m$, we have*

$$gq_\nu(\boldsymbol{y}_n) = \mathrm{Pf}\left(\sum_{k=0}^{i-1}\sum_{l=0}^{j} \beta^{k+l} \binom{i-1}{k}\binom{j}{l} gq_{(\nu_i-k, \nu_j-l)}(\boldsymbol{y}_n) \right)_{1\leq i<j\leq 2m}.$$

Acknowledgements. Thanks are due to the anonymous referee whose careful reading and useful suggestions improved the readability of our manuscript. The authors dedicate this work to the memory of Professor Piotr Pragacz who sadly passed away on October 25th, 2022. Our work is greatly inspired by his foundational works [31, 32].

The first author is partially supported by the Grant-in-Aid for Scientific Research (C) 18K03303, Japan Society for the Promotion of Science. The second author is partially supported by the Grant-in-Aid for Scientific Research (B) 16H03921, Japan Society for the Promotion of Science.

References

[1] K. Bremke and G. Malle, *Reduced words and a length function for $G(e,1,n)$*, Indag. Math. (N.S.) 8 (1997), 453–469.

[2] K. Bremke and G. Malle, *Root systems and length functions*, Geom. Dedicata 72 (1998), 83–97.

[3] M. Brion, *The push-forward and Todd class of flag bundles*, in: Parameter Spaces, P. Pragacz (ed.), Banach Center Publ. 36, Inst. Math., Polish Acad. Sci., Warszawa, 1996, 45–50.

[4] L. Darondeau and P. Pragacz, *Universal Gysin formulas for flag bundles*, Internat. J. Math. 28 (2017), no. 11, art. 1750077, 23 pp.

[5] C. De Concini and C. Procesi, *Symmetric functions, conjugacy classes, and the flag variety*, Invent. Math. 64 (1981), 203–219.

[6] J. S. Frame, G. de B. Robinson, and R. W. Thrall, *The hook graphs of the symmetric group*, Canad. J. Math. 6 (1954), 316–325.

[7] W. Fulton, *Young Tableaux*, London Math. Soc. Student Texts 35, Cambridge Univ. Press, 1997.

[8] W. Fulton and P. Pragacz, *Schubert Varieties and Degeneracy Loci*, Lecture Notes in Math. 1689, Springer, Berlin, 1998.

[9] A. M. Garsia and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. Math. 94 (1992), 82–138.

[10] H. Hiller, *Combinatorics and intersection of Schubert varieties*, Comment. Math. Helv. 57 (1982), 41–59.

[11] T. Ikeda, *Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian*, Adv. Math. 215 (2007), 1–23.

[12] T. Ikeda, L. C. Mihalcea, and H. Naruse, *Double Schubert polynomials for the classical groups*, Adv. Math. 226 (2011), 840–866.

[13] T. Ikeda and H. Naruse, *Excited Young diagrams and equivariant Schubert calculus*, Trans. Amer. Math. Soc. 361 (2009), 5193–5221.

[14] T. Ikeda and H. Naruse, *K-theoretic analogues of factorial Schur P- and Q-functions*, Adv. Math. 243 (2013), 22–66.

[15] V. N. Ivanov, *Interpolation analogs of Schur Q-functions*, Zap. Nauchn. Sem. POMI 307 (2004), 99–119 (in Russian); English transl.: J. Math. Sci. 131 (2005), 5495–5507.

[16] T. Józefiak, *Schur Q-functions and cohomology of isotropic Grassmannians*, Math. Proc. Cambridge Philos. Soc. 109 (1991), 471–478.

[17] A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. 119 (2003), 221–260.

[18] A. Lascoux and H. Naruse, *Finite sum Cauchy identity for dual Grothendieck polynomials*, Proc. Japan Acad. Ser. A Math. Sci. 90 (2014), 87–91.

[19] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford Univ. Press, Oxford, 1995.

[20] C. McDaniel, *A GKM description of the equivariant coinvariant ring of a pseudo-reflection group*, arXiv:1609.00849 (2016).

[21] A. I. Molev and B. E. Sagan, *A Littlewood–Richardson rule for factorial Schur functions*, Trans. Amer. Math. Soc. 351 (1999), 4429–4443.
[22] K. Nakada, *Colored hook formula for a generalized Young diagram*, Osaka J. Math. 45 (2008), 1085–1120.
[23] M. Nakagawa and H. Naruse, *Generalized (co)homology of the loop spaces of classical groups and the universal factorial Schur P-and Q-functions*, in: Schubert Calculus – Osaka 2012, Adv. Stud. Pure Math. 71, Math. Soc. Japan, Tokyo, 2016, 337–417.
[24] M. Nakagawa and H. Naruse, *Universal Gysin formulas for the universal Hall–Littlewood functions*, in: Contemp. Math. 708, Amer. Math. Soc., 2018, 201–244.
[25] M. Nakagawa and H. Naruse, *Universal factorial Schur P, Q-functions and their duals*, arXiv:1812.03328 (2018).
[26] M. Nakagawa and H. Naruse, *Darondeau–Pragacz formulas in complex cobordism*, Math. Ann. 381 (2021), 335–361.
[27] M. Nakagawa and H. Naruse, *Equivariant Schubert calculus for unitary reflection groups*, in preparation.
[28] H. Naruse, *Elementary proof and application of the generating function for generalized Hall–Littlewood functions*, J. Algebra 516 (2018), 197–209.
[29] H. Naruse and S. Okada, *Skew hook formula for d-complete posets via equivariant K-theory*, Algebr. Combin. 2 (2019), 541–571.
[30] O. Ortiz, *GKM theory for p-compact groups*, J. Algebra 427 (2015), 426–454.
[31] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials*, in: Topics in Invariant Theory (Paris, 1989/1990), Lecture Notes in Math. 1478, Springer, Berlin, 1991, 130–191.
[32] P. Pragacz, *A Gysin formula for Hall–Littlewood polynomials*, Proc. Amer. Math. Soc. 143 (2015), 4705–4711.
[33] D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. 75 (1969), 1293–1298.
[34] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. Math. 7 (1971), 29–56.
[35] B. Totaro, *Towards a Schubert calculus for complex reflection groups*, Math. Proc. Cambridge Philos. Soc. 134 (2003), 83–93.

Masaki Nakagawa
Graduate School of Education
Okayama University
Okayama, 700-0808 Okayama, Japan
E-mail: nakagawa@okayama-u.ac.jp

Hiroshi Naruse
Graduate School of Education
University of Yamanashi
Kofu, 400-8510 Yamanashi, Japan
E-mail: hnaruse@yamanashi.ac.jp