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EIGENVALUE PROBLEM ASSOCIATED WITH NONLINEAR BUCKLING OF A COLUMN UNDER LARGE DISPLACEMENT

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Abstract

The classical theory of elastica, which describes the geometrical nonlinearity of a column under the critical axial loading with the supposition of material linearity, provides us some analytical predictions about the post buckling behavior of the column under large displacement. The axial load is supposed to increase as the deformation of the column develops. This is one of those well-known results from the model of elastica. This paper reviews the suppositions of the classical theory of elastica with the purpose of extending it into the material nonlinear model. There is an identical nonlinear eigenvalue problem behind the elastica model and the material nonlinear buckling model, which denotes the inelastic model in the latter part of this paper. Once this identity is accepted, the same nonlinear differential equation describes the tow different physical phenomena: the elastic buckling and the inelastic buckling. Authors start creating a supposition in the beginning and proceed to the theoretical development with the logically rigorous nonlinear buckling model. At the end of this paper, we come to understand that the buckling load becomes smaller as the deformation of the column develops, even if the material property is constant and elastic. A small modification is necessary for creating the compatibility between the elastic and the inelastic models.

Keywords: Elastica, Elliptic function, Eigenvalue, Material nonlinearity, Stability

1. Introduction

Authors start reviewing the model of elastica as is shown in Fig.1 where the deformation curve with several notations is illustrated. There are boundary conditions and geometrical parameters defined in Fig.1. The previous studies [1, 2, 3] are compatible with the following definitions and Fig. 1 and 2 are referred to [1, 2].

$$M(s) = N_0 x(s) + Q_0 y(s) \tag{1}$$

$$M(s) = N_{cr} x(s) \tag{2}$$

$$\frac{dx}{ds} = sin\theta \tag{3}$$

$$\frac{d\theta}{ds} = \varphi(s) \tag{4}$$

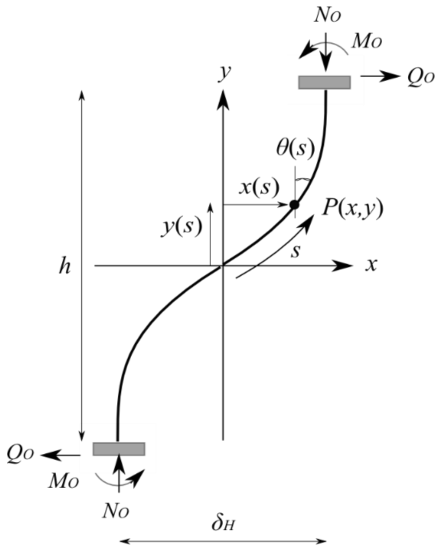
$$M(s) = -EI\varphi(s) \tag{5}$$

The classical theory of elastica sets four suppositions. The first supposition is the definition of bending moment given by Eq. (1), which represents the equilibrium at $P(s)$. The equilibrium of the column with reactions at both ends in Fig.1 gives us Eq. (1), from which we naturally derive Eq. (2) that prescribes the definition of buckling load. As the axial reaction or N_o increases, the horizontal reaction or Q_o approaches to zero. When it comes to zero, the axial reaction reaches the buckling load or N_{cr}. Therefore it is quite natural that we define the buckling load by Eq. (2). The second supposition is the geometrical nonlinearity. The rigorous relation between the coordinate $x(s)$ and the rotarional angle $\theta(s)$ is represented by Eq. (3). The third supposition is the definition of curvature $\varphi(s)$ given by Eq. (4), which describes the curvature of the column curve under large deformation. The last supposition is the material elasticity, which is defined by Eq. (5). The bending stiffness is a constant value *EI*, or equivalently the Young's modulus *E* times by the geometrical moment of inertia or *I*. Supposition of Eq. (2) and Eq. (5) yields Eq. (6). This means that the theory of elastica has nothing to do with function $M(s)$. In summary, there are three equations necessary to derive the elastica differential equation (7), if and only if the bending moment function does exist. There are no constraints imposed on $M(s)$ by the classical theory of elastica. Starting from four suppositions but satisfying only three equations for three variables implies that the solution of elastica might have some room for extension of improvement.

$$-EI\varphi(s) = N_{cr}x(s) \tag{6}$$

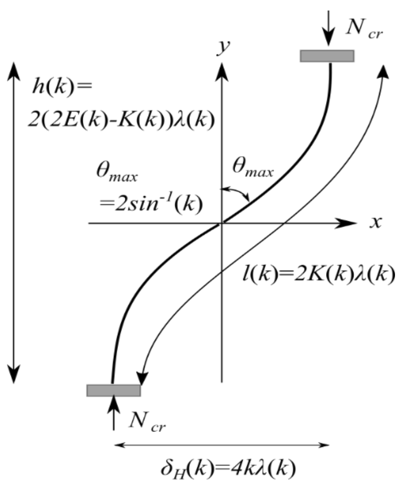
$$\frac{d^2\theta}{ds^2} + \frac{1}{\lambda^2}sin\theta = 0 \tag{7}$$

$$where \quad \frac{1}{\lambda^2} = \frac{N_{cr}}{EI} \tag{8}$$



S : Distance parameter from the origin to $P(x, y)$
$\theta(s)$: Rotational angle caused by bending moment
$x(s)$: coordinate of $P(x, y)$
$y(s)$: coordinate of $P(x, y)$
$M(s)$: Bending moment at $P(x, y)$
Q_o : Horizontal reaction at the end
N_o : Axial reaction at the end
M_o : Bending moment reaction at the end
δ_H : Lateral deformation of the column
h : Height of the column after loading
l : Length of the column after loading
l_o : Initial length of the column before loading

Fig. 1 Deformation of the column under axial loading and boundary conditions



$K(k)$: Complete elliptic integral of the first kind
$E(k)$: Complete elliptic integral of the second kind
$l(k)$: Length of the column under buckling load
$h(k)$: Height of the column under buckling load
$\lambda(k)$: Unit length of the column
k : modulus of the elliptic integral
θ_{max} : rotational angle at the origin

Fig. 2 Deformation of the column under buckling load

The well-known nonlinear differential equation (7) has the solution given by Eq. (9) and (10) that satisfy the boundary condition or equivalently Eq. (11) in common, where $K(k)$ is the complete elliptic integral of the first kind.

$$sin\left(\frac{\theta}{2}\right) = k\ sn\left(\frac{s}{\lambda} + K(k)\right) \tag{9}$$

$$cos\left(\frac{\theta}{2}\right) = dn\left(\frac{s}{\lambda} + K(k)\right) \tag{10}$$

$$2\lambda(k)\ K(k) = l(k) \tag{11}$$

We can verify the above solution by substituting Eq. (9) and (10) into Eq. (7). In fact, we obtain Eq. (12) from Eq. (7). Integrating Eq. (12), we obtain Eq. (13).

$$\frac{d^2\theta}{ds^2} + \frac{2k}{\lambda^2} sn\left(\frac{s}{\lambda} + K(k)\right) dn\left(\frac{s}{\lambda} + K(k)\right) = 0 \tag{12}$$

$$\varphi(s) = \frac{d\theta}{ds} = \frac{2k}{\lambda} cn\left(\frac{s}{\lambda} + K(k)\right) \tag{13}$$

The modulus k of the elliptic function is determined by Eq. (14) shown in Fig.2. Considering the rotational angle at the origin, we obtain Eq. (14) that determines the parameter k for Eq. (9), (10), and (11).

$$sin\left(\frac{\theta_{max}}{2}\right) = k \tag{14}$$

Substitution of Eq. (11) into Eq. (8) yields Eq. (15) from which we would predict how the buckling load influenced by the modulus k.

$$N_{cr}(k) = \frac{EI}{\lambda^2} = \frac{4EI}{l^2} K^2(k) \tag{15}$$

$$where\quad K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \cdots\right) \tag{16}$$

Under the condition that the length of the column is supposed to be constant or l_o, the buckling load N_{cr} increases as the deformation of the column develops. We could numerically estimate N_{cr}, if the complete elliptic integral of the first kind is approximated by Eq. (16).

$$N_{cr}(k) = \frac{\pi^2 EI}{{l_o}^2}\left(1 + \frac{1}{2}k^2 + \frac{11}{32}k^4 + \cdots\right) \tag{17}$$

As far as the author knows, the past study paid little attention to the rest of the variables such as $x(s)$, $M(s)$, and $\varphi(s)$. Once we obtained $\theta(s)$ from the nonlinear differential equation (7), we can determine $\varphi(s)$ from Eq. (13). As far as material is linear according to Eq. (5), $M(s)$ is the same function as $\varphi(s)$. In other words, $M(s)$ should be Eq. (18). We apply the same logic to $x(s)$ for modeling it as Eq. (19). However, we have not yet proved that there is compatibility between Eq. (17) and Eq. (18), because the theory of elastica has nothing to do with $M(s)$. We would continue the discussion whether $M(s)$ does exist to be consistent with Eq. (17) in the following chapters.

$$M(s) = -M_o(k)\, cn\left(\frac{s}{\lambda} + K(k)\right) \tag{18}$$

$$x(s) = -\frac{1}{2}\delta_H(k)\, cn\left(\frac{s}{\lambda} + K(k)\right) \tag{19}$$

2. Eigenvalue problem required for the elastica theory

In the previous chapter, we reviewed the supposition of elastica and the theoretical predictions about the critical load or equivalently the buckling load under large deformation. The theory tells us that the buckling load gradually increases as the deformation develops. There is one required condition: the bending moment $M(s)$ does exist. In this chapter, we would consider the required condition for the existence of bending moment $M(s)$. We have reached the exact solution for $x(s)$ as Eq. (19) from which we can obtain the derivative of $x(s)$ with respect to s.

$$\frac{dx}{ds} = \frac{1}{2\lambda}\delta_H(k)\, sn\left(\frac{s}{\lambda} + K(k)\right) dn\left(\frac{s}{\lambda} + K(k)\right) \tag{20}$$

Substitution of Eq. (9) and (10) into Eq. (20), we obtain another geometrical boundary condition.

$$\frac{1}{2\lambda}\delta_H(k) = 2k \qquad or \qquad \delta_H(k) = 4k\lambda(k) \tag{21}$$

Taking the derivative of both sides of Eq. (20), we obtain Eq. (22).

$$\frac{d^2x}{ds^2} + \frac{1}{\lambda^2}\big(1 - V(s)\big)x(s) = 0 \tag{22}$$

$$\text{where} \qquad V(s) = 2k^2 sn^2\left(\frac{s}{\lambda} + K(k)\right) \tag{23}$$

We could recognize Eq. (22) as a time independent Schrödinger equation with the potential function of $V(s)$. It is quite natural that we should consider the elastica as an eigenvalue problem with the potential function of Eq. (23). Indeed, Eq. (19) is the solution of Eq. (22), where we have the freedom to select the amplitude of $x(s)$ because this is the solution of eigenvalue problem. Once $x(s)$ is determined, $M(s)$ should satisfy Eq. (18), which means that $M(s)$ should satisfy the same eigenvalue problem of Eq. (22) and (23).

The theory of elastica requires the existence of $M(s)$. If it does exist, it should be the solution of nonlinear eigenvalue problem specified by Eq. (22) and (23). In other words, $M(s)$ should be Eq. (18). No constrains are imposed on $M(s)$ for the theory of elastica. Because it required three suppositions that impose no constrains on the selection of $M(s)$, which means $M_o(k)$ can be any number. The variable $x(s)$ and $M(s)$ should satisfy the identical eigenvalue problem and expressed as the same elliptic function. The amplitude of $x(s)$, however, should be Eq. (21) that is one of those boundary conditions. At the same time, the amplitude of $x(s)$ should be any number, because it is the solution of eigenvalue function. We still have the freedom to select the constant value k, which is the modulus of elliptic integral and determines the amplitude of $x(s)$ as well.

As far as the geometrical variable $x(s)$ and $\varphi(s)$ are concerned, we have the freedom to select their amplitude by means of k. It, however, has no influence on $M(s)$, because it is not a geometrical variable but a stress variable. Here arises a problem, because we suppose that material linearity is compatible with Eq. (5). As a result, $M_o(k)$ is one of those boundary conditions that should be geometrical parameters such as $\delta_H(k)$ should be consistent with Eq. (21). This means that we have no more guarantee to select an appropriate $M_o(k)$ to be consistent with the rest of the boundary conditions including Eq. (17).

3. Modified theory of elastica

There should be one more required condition for the existence of bending moment of $M(s)$. At least we should be more careful to select the bending moment $M(s)$, even if the material is elastic. This is the motivation of the author to select four fundamental equations to solve the elastica problem instead of three. Instead of solving the set of three equations in the theory of elastica, we would set four equations for material nonlinear buckling model. The geometrical nonlinearity Eq. (3), the definition of buckling state Eq. (2), the definition of curvature of the column Eq. (4) are identical to those in the previous chapter. We would replace material linearity Eq. (5) by Eq. (24) and (25). The illustration in Fig.3 describes the physical meaning of Eq. (25). Additional equation provides us one more freedom to determine the boundary condition compatible with the four fundamental equations. We introduce one parameter α, which represents the nonlinearity of the bending stiffness $K_B(\alpha)$. Even if the bending stiffness depends on $M_o(k)$, it does not depend on the variable s. This situation is similar to the buckling load $N_{cr}(k)$ on which the modulus k has a significant effect.

$$M(s) = -K_B(\alpha)\varphi(s) \tag{24}$$

$$-EI\frac{d^2x}{ds^2} = M(s) + \left(\frac{\alpha}{M_o}\right)^2 M(s)^3 \tag{25}$$

$$\frac{d^2M}{ds^2} + \frac{N_{cr}}{EI}\left(1 + \left(\frac{\alpha}{M_o}\right)^2 M(s)^2\right)M(s) = 0 \tag{26}$$

The nonlinear differential equation of (26) has the rigorous solution as Eq. (27), which is consistent with Eq. (18). We can prover it by substituting Eq. (27) into Eq. (26). The derivative of Eq. (27) is Eq. (28) followed by the second derivative of Eq. (29). It is quite interesting that the nonlinear eigenvalue problem of Eq. (29) is identical to Eq. (22) in the previous chapter. We have reached the same equation, even if we began from two different starting points. One is geometrical nonlinearity while the other one is material nonlinearity.

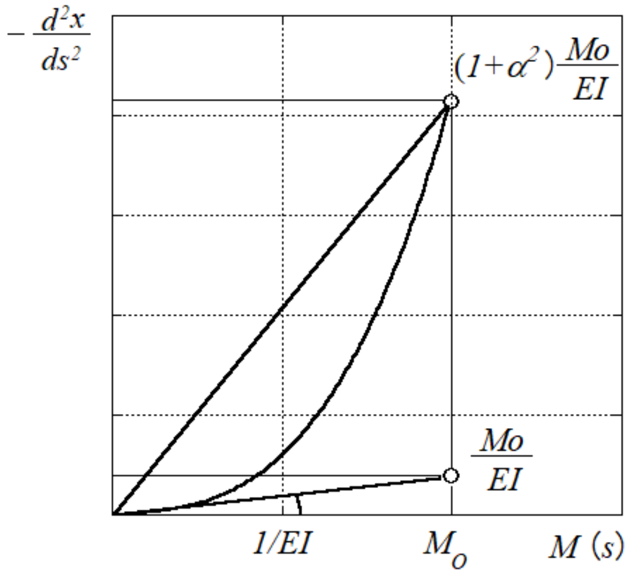


Fig.3 Material property between $M(s)$ and $x(s)$

$$M(s) = -M_o(\alpha)\, cn\left(\frac{s}{\lambda} + K(k)\right) \tag{27}$$

$$\frac{dM}{ds} = \frac{1}{\lambda} M_o(\alpha)\, sn\left(\frac{s}{\lambda} + K(k)\right) dn\left(\frac{s}{\lambda} + K(k)\right) \tag{28}$$

$$\frac{d^2M}{ds^2} + \frac{1}{\lambda^2}\left(1 - 2k^2 sn^2\left(\frac{s}{\lambda} + K(k)\right)\right) M(s) = 0 \tag{29}$$

The solution of Eq. (26) should be Eq. (27), which means that Eq. (29) is identical to Eq. (26). This requirement is equivalent to the following condition.

$$\alpha^2 = \frac{2k^2}{1 - 2k^2} \tag{30}$$

$$\frac{1}{\lambda^2} = \frac{N_{cr}}{EI(1 - 2k^2)} = \frac{N_{cr}}{K_B(\alpha)} \tag{31}$$

Final goal of this chapter is to prove that the four solutions of $M(s)$, $x(s)$, $\theta(s)$, and $\varphi(s)$ are consistent with the four suppositions along with the required boundary conditions. The solution of $x(s)$ is consistent with Eq. (2). This requirement forces us to select $x(s)$ equivalent to Eq. (19). Derivative of Eq. (19) is equivalent to Eq. (20) along with the boundary conditions of Eq. (21). Supposition of Eq. (3) requires that the solution of $\theta(s)$ should be Eq. (9) and (10). Finally, the curvature of $\varphi(s)$ should be Eq. (13).

$$\varphi(s) = \frac{d\theta}{ds} = \frac{2k}{\lambda} cn\left(\frac{s}{\lambda} + K(k)\right) \tag{13}$$

$$x(s) = -\frac{1}{2}\delta_H(k)\, cn\left(\frac{s}{\lambda} + K(k)\right) \tag{19}$$

We set a new supposition that is different from the classical elastica model. If the new supposition of Eq. (24) is consistent with the rest of the boundary conditions, the introduced modification of the theory of elastica leads us to a different conclusive result. As the modulus of k increases, the bending stiffness decreases as is expressed in Eq. (32). The complete elliptic integral of the first kind $K(k)$ is substituted by Eq. (16) and we would expect the buckling load $N_{cr}(k)$ comes down as the modulus k increases in Eq. (33).

$$N_{cr}(k) = \frac{K_B(\alpha)}{\lambda^2} = \frac{EI(1 - 2k^2)}{\lambda^2} = \frac{4EI(1 - 2k^2)}{l^2} K^2(k) \tag{32}$$

$$N_{cr}(k) = \frac{4EI(1 - 2k^2)}{l^2} K^2(k) = \frac{\pi^2 EI}{{l_o}^2}\left(1 - \frac{7}{4}k^2 - \frac{23}{64}k^4 - \cdots\right) \tag{33}$$

It is surprising to know that the buckling load comes down as the modulus k increases. This conclusion totally contradicts Eq. (17). Here arises a very interesting question about the validity of the classical theory of elastic buckling of a column under axial loading.

4. Summary of the inelastic nonlinear buckling model

We started creating a new buckling model from material nonlinearity and reached a nonlinear differential equation to identify the bending moment function $M(s)$ along with other geometrical variables $x(s)$, $\theta(s)$, and $\varphi(s)$. There are four different variables that are $x(s)$, $\theta(s)$, $\varphi(s)$, and $M(s)$ so that we need four equations to specify each one of them. Three variables share the same function but they have different amplitudes: $x(s)$, $\varphi(s)$, and $M(s)$. In other words, there are two constant necessary to specify the ratios between any two functions. This requirement forces us additional two equations as Eq. (2) and Eq. (24). The two constants are bending stiffness $K_B(\alpha)$ and buckling load $N_{cr}(k)$, which means that there are two parameters such as k and α instead of K_B and N_{cr}. We need two more equations for explicitly identify four variables. There should be two differential equations necessary for this purpose, because there are two different physical properties. Three variables are geometrical functions while the rest one is a stress function: $x(s)$, $\theta(s)$, $\varphi(s)$ are geometrical functions while $M(s)$ is a stress function. We adopted two nonlinear equations for this purpose, which are Eq. (3) and (25). Eq. (3) needs one parameter or k for expressing geometrical nonlinearity, while Eq. (25) needs another parameter or α for expressing material nonlinearity. It is rather surprising that these two equations came to one identical nonlinear eigenvalue problem, which clarified the relation between k and α. There are four equations necessary for identifying four functions, but two equations are nonlinear so that there are two more equations necessary for identifying two parameters. One equation is derived from the fact that two nonlinear equations are identical each other. There is the last condition necessary for the set of four functions along with two parameters. Question left over is whether four variables and two parameters are compatible for each other or not. We would answer this question in the rest of this chapter.

The first task is to check the range of modulus k that guarantees the existence of $M(s)$. The modulus k satisfies Eq. (30), which means that it also satisfies the following inequality.

$$0 < k^2 < \frac{1}{2} \quad \Leftrightarrow \quad 0 < k < 0.707 \tag{34}$$

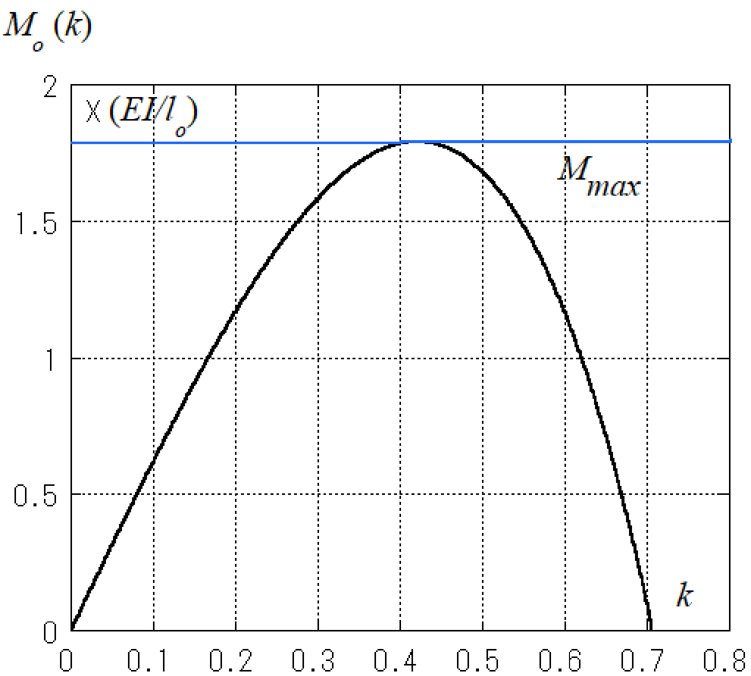


Fig.4 Moment reaction $M_o(k)$ with respect to k

The geometrical boundary conditions for the functions of $x(s)$, $\theta(s)$, $\varphi(s)$ are Eq. (11), (14), and (21), which come from the previous chapters as below.

$$2\lambda(k)K(k) = l(k) = l_o \tag{11}$$

$$sin\left(\frac{\theta_{max}}{2}\right) = k \tag{14}$$

$$\delta_H(k) = 4k\lambda(k) \tag{21}$$

$$\varphi_o(k) = \frac{1}{R_o(k)} = \frac{2k}{\lambda(k)} \qquad where\ \ R_o(k)\ is\ radius\ at\ the\ end \tag{35}$$

The bending moment function $M(s)$ is a stress function but has a boundary condition.

$$M_o(k) = \frac{2EIk(1-2k^2)}{\lambda(k)} = \frac{4EIk(1-2k^2)K(k)}{l_o} \tag{36}$$

The modulus k varies from 0.0 to 0.707 as is shown in Eq. (34) and the bending moment at the end satisfies Eq. (36) in Fig. 4. If the strength of the material is more than M_{max} that is the peak of $M_o(k)$ in Fig.4, the material is supposed to be elastic. Young's modulus E and geometrical moment of inertia I are constant, while bending stiffness K_B varies according to the geometrical deformation. Whether the material is elastic or inelastic, the geometrical deformation satisfies the same function as Eq. (19). If the bending stiffness $K_B(\alpha)$ increases as the modulus k increases, there is no solution for $M(s)$. On the other hand, $K_B(\alpha)$ decreases as the modulus k increases, there is a solution for $M(s)$ that is identical to Eq. (18). Consequently, we must admit that the buckling load $N_{cr}(k)$ should be Eq. (33) rather than Eq. (17). There are several other predictions that we could derive from the modified theory of elastica.

5. Predictions from the modified theory of elastica

As the author pointed out so far, the classical thoery of elastica is based on the three suppositions instead of four. There are four variables such as $x(s)$, $\theta(s)$, $\varphi(s)$, and $M(s)$, which means that there should be four equations to specify each one of them. Three variables $x(s)$, $\theta(s)$, $\varphi(s)$ are geometrical numbers, while $M(s)$ expresses inner forces. This situation makes it necessary to set two physically different equations. This is the reason why the author supposed two physical models such as Eq. (3) or Fig. 2 and Eq. (26) or Fig. 3. They are geometrical nonlinearity and stiffness nonlinearity, respectively. Two different physical models came to the same nonlinear eigenvalue problem, which means two parameters are dependent and have a relation with each other. As far as the author checked the modification of the elastica theory, there is no defect in the process of logic. If the supposition of the physical model is appropriate enough to follow what really takes place in the real world, the predictions we could derive from the theoretical results are reliable, visible and tangible. Experimental investigation and observation are the only method to check the validity of the theory. The author wishes the readers to carry out validation tests and check the theoretical predictions.

According to the modified theory of elastica, the buckling load decreases as the modulus k develops. This prediction is valid not only for linear materials but for inelastic materials. This prediction is totally diffrent from the classical model. The Euler load is the upper boundary of the buckling phenomena for any materials. This is the most contraversial prediction as a result of the modification of the elastica model.

Theory tells us another prediction. The shape of the column under buckling loads is not influenced by the fact that the material is either elastic or inelastic. This means that the theory and the physical model in this paper could be extened into the plastic buckling for steel as well as other materials.
The theory has many merits but weak points as well. The solution of $M(s)$ only does exist as far as the modulus k is between 0 and 0.707 according to Eq. (34). In other words, the supposition of Eq. (25) is only consistent with other boundary conditions and suppositions as long as the modulus k satisfies Eq. (34). If the deformation of the column is excessively large and the modulus k is bigger than 0.707, we have to make a different supposision instead of Eq. (34). Otherwise, we could not complete the modification of the theory of elastica. Unfortunately, the author has not yet discovered the appropriate nonlinear inelastic model to comensate for this weak point.

6. Conclusive remarks

The author believes that there should be a modification necessary for the classical theory of elastica. The bending stiffness is no longer a constant value EI but influenced by the modulus k of the elliptic function. The buckling load N_{cr} is less than Euler load and it comes down as the deformation develops, even if the material property is constant and elastic. We could not tell whether the material is elastic or inelastic just by observing the shape of the post buckling state, because the horizontal deformation of the column reduces the bending stiffness and the buckling load as well.
There are several predictions derived from the modified theory of elastica, some of them are contraversial and require further discussion and verification. Experimental studies are also necessary for proving the validity of the modified theory of elastica.

7. References

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