On crosscap numbers of knots

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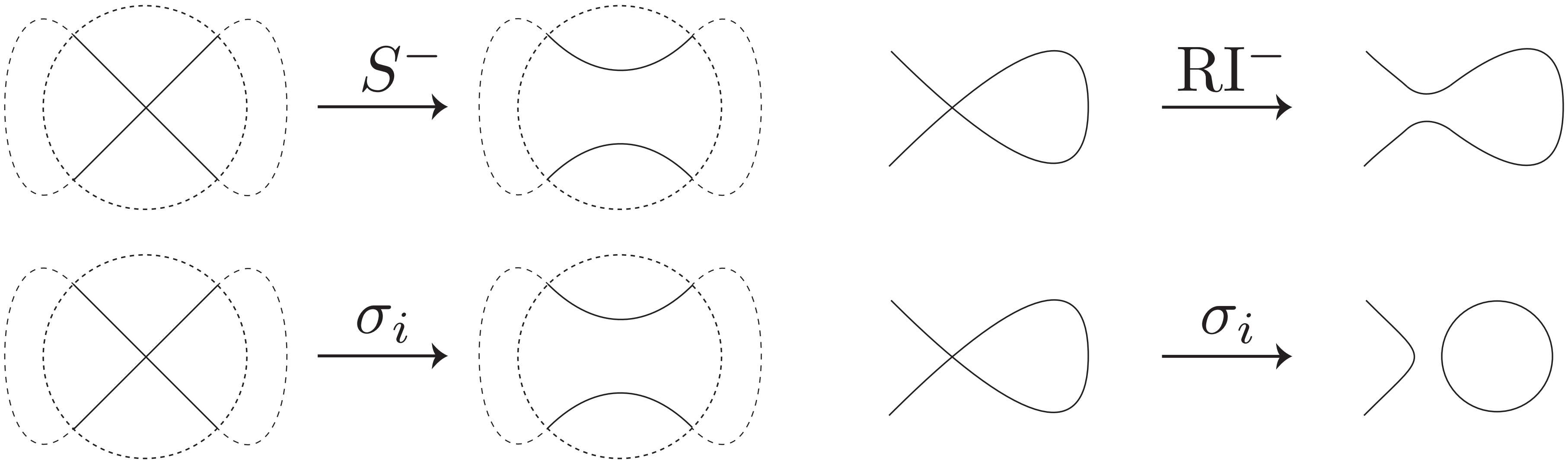
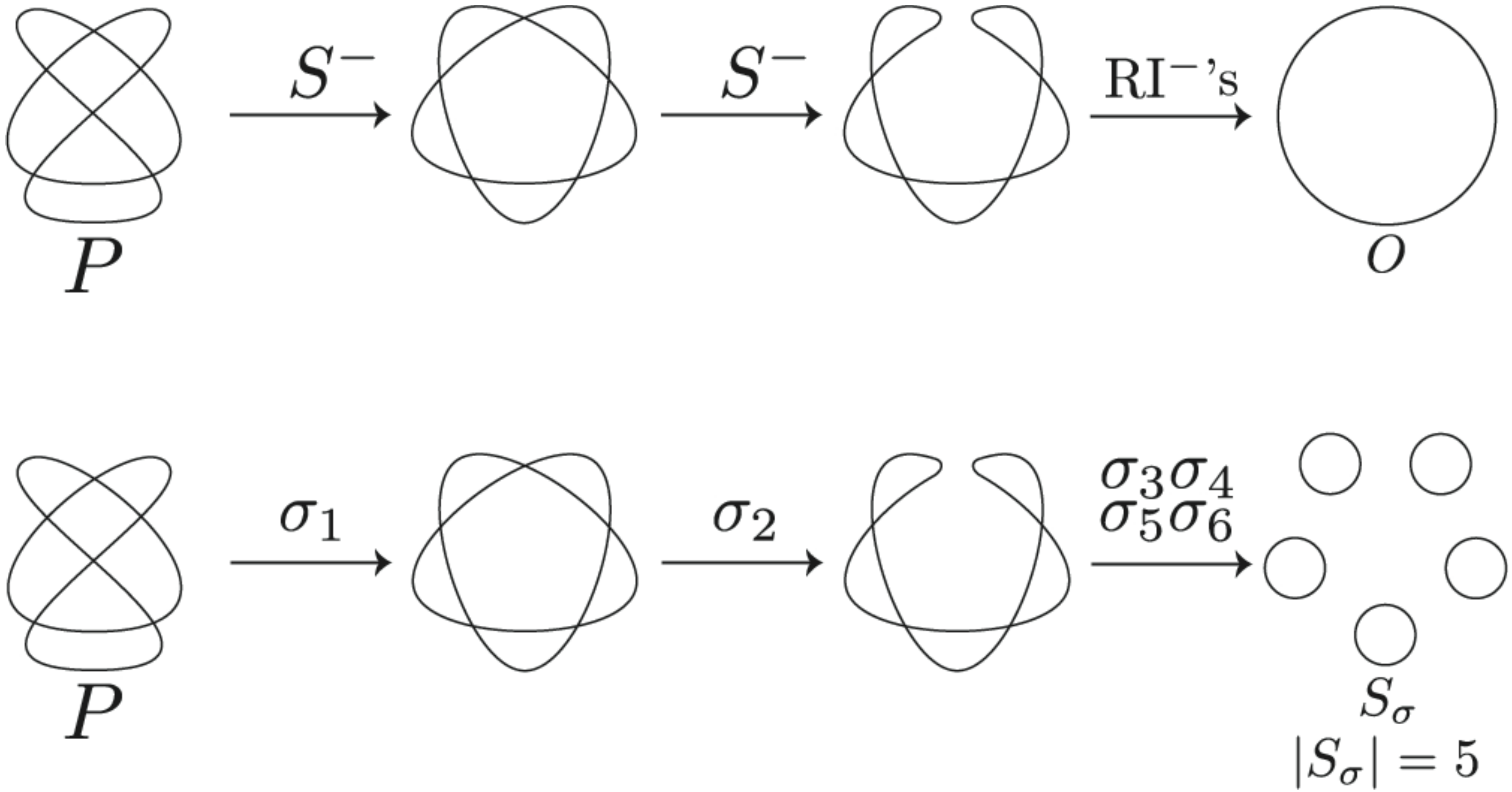
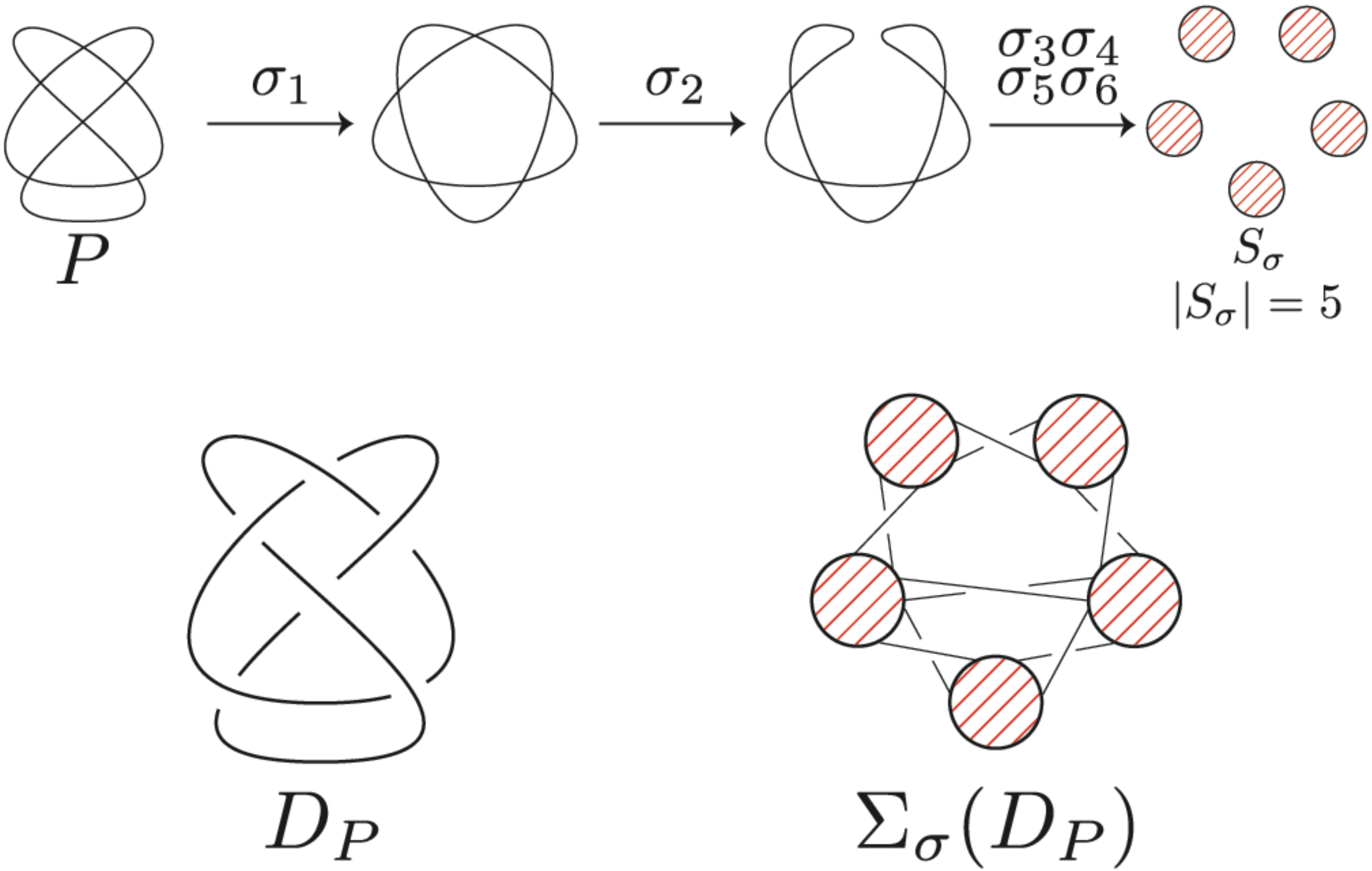


For every knot projection P,

$$u^-(P) := \min_{\text{seq.}}\{S^- \text{ in seq. from } P \text{ to } O\ \}.$$

$$u^-(K) = \min_{P} u^-(P).$$


P
S^-
S^-
RI^-'s
O
P
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
S_σ
$|S_\sigma| = 5$


P
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
S_σ
$|S_\sigma| = 5$
D_P
$\Sigma_\sigma(D_P)$

Theorem 1. *Let K be a knot, $C(K)$ a crosscap number of K, $n(K)$ the minimum crossing number, and $u^-(K)$ defined as the above. If K is an prime alternating knot, then*

$$C(K) = u^-(K).$$

If K is a prime (alternating or non-alternating) knot K,

$$C(K) \leq u^-(K) \leq \left\lfloor \frac{n(K)}{2} \right\rfloor$$

(the left inequality holds even if K is non-prime).

Corollary 1. *Let* $V_K(q) = a_n q^n + a_{n+1} q^{n+1} + \cdots + a_{m-1} q^{m-1} + a_m q^m$ *be the Jones polynomial of a knot* K*. If* K *is a prime alternating knot, then (using two results of Murasugi and Dasbach-Lin)*

$$C(K) = u^-(K) \leq \begin{cases} \min\{|a_{n+1}| + |a_{m-1}|, \left\lfloor \frac{m-n}{2} \right\rfloor\} \\ \text{ if } C(K) \neq 2g(K)+1, \\ \min\{|a_{n+1}| + |a_{m-1}| + 1, \left\lfloor \frac{m-n}{2} \right\rfloor\} \\ \text{ otherwise.} \end{cases}$$

Rmk. $C(K) \leq \min\{\lfloor n(K)/2 \rfloor, t+1\}$ by Murakami-Yasuhara, Kalfagianni-Lee. We use *state surface*, introduced by Ozawa.

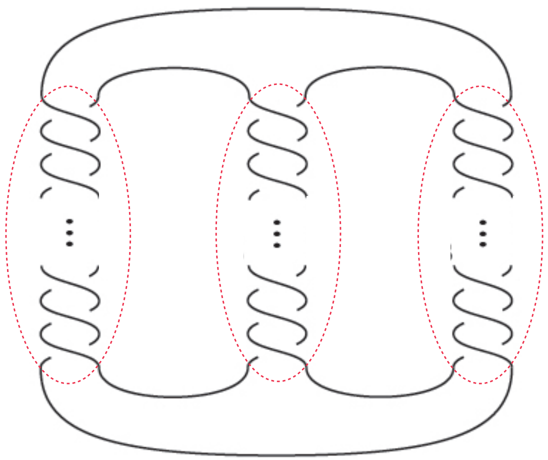
Notation 1. • D *be a knot diagram of a hyperbolic link* K.

• $t(D)$ *: the twist number of* D *(Lackenby, 2004),*

• v_3 *the volume of a regular hyperbolic ideal tetrahedron,*

• v_8 *the volume of a regular hyperbolic ideal octahedron.*

• $\mathrm{vol}(S^3 \setminus K)$*: the volume of knot compliment.*



Corollary 2. *Let K be a prime alternating hyperbolic knot.*

$$\frac{v_8}{2}(u^-(K)-3) \leq \mathrm{vol}(S^3 \setminus K) \leq 10v_3(3u^-(K)-4).$$

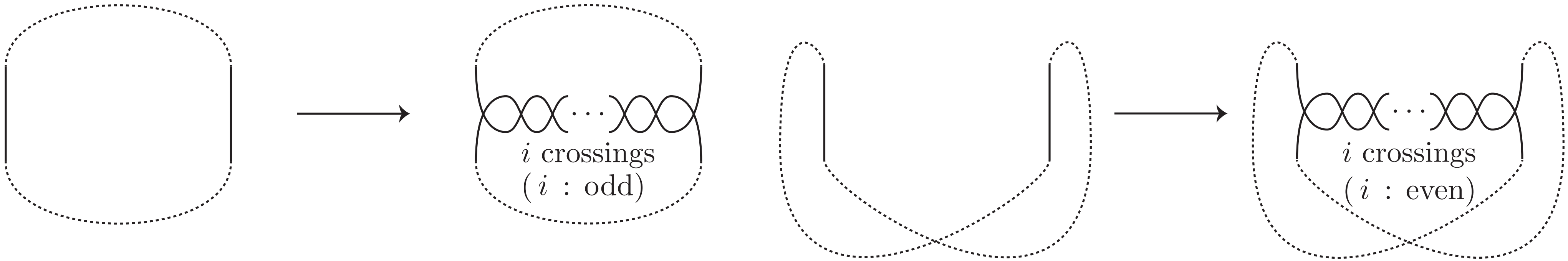
Corollary 3. *Let K be a hyperbolic knot that is the closure of a positive braid with at least three crossings in each twist region.*

$$\mathrm{vol}(S^3 \setminus K) \leq 10v_3(3u^-(K)-4).$$

Corollary 4. *Let K be a prime Montesinos hyperbolic knot.*

$$\mathrm{vol}(S^3 \setminus K) \leq 6v_8(u^-(K)-1).$$

Move 1. *For any pair of simple arcs lying on the boundary of a common region, each of the two local replacements as in Figure is obtained by applying operations of type* RI^+ $i-1$ *times followed by a single operation of type* S^+.



The computation of $u^-(D)$ of D is systematically discussed by using Move 1 to give $u^-(D) = n$ for any positive integer n.

Proposition 1. *Let* RI *be a splice of type* RI^- *or its inverse. Let* D *be a knot diagram and* O *the knot diagram with no crossings. The following conditions are equivalent.*

(A) D *is a knot diagram with* $u^-(D) = n$.

(B) D *is obtained from* O *by applying Move 1 successively* n *times and some* $\mathrm{RI}'s$.

Thank you for your listening.

1978, Clark, Fixed K s.t. $C(K) = 1$.

1996, H. Murakami-Yasuhara, Fixed connected sum.

2004, Teragaito, Fixed torus knots

2006, Hirasawa-Teragaito, Fixed 2-bridge knots

2006, Ichihara-Mizushima, Fixed most of pretzel knots

2013, Adams-Kindred, Algorithm for alternating knots
(using state surface, introduced by Ozawa)

2016, Kalfagianni-Lee, Lower bounds and Jones polynomial
(via Adams-Kindred algorithm)

Thank you for your listening.