

Gauss diagram formulas of Vassiliev invariants of spatial 2-bouquet graphs in view of application

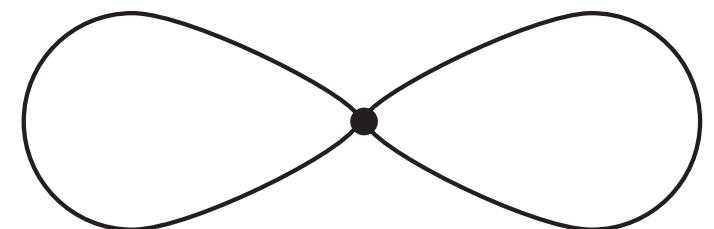
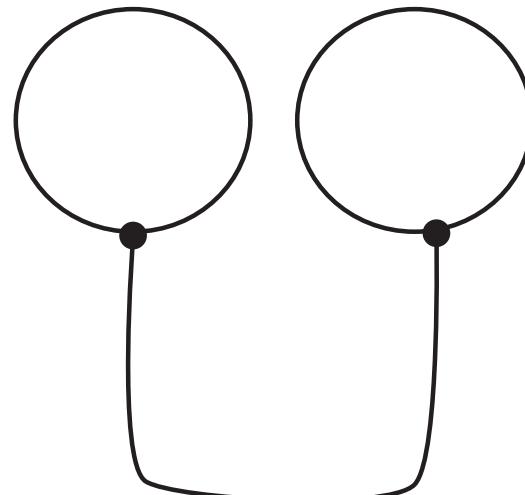
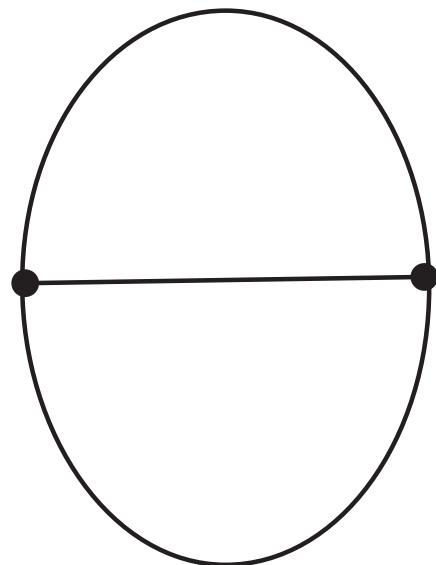
伊藤 昇(茨城高専) & 大山口 菜都美(秀明大学)

東京女子大学トポロジーセミナー

2020年7月11日

空間グラフ

Theta-curve Hand-cuff graph 2-bouquet graph



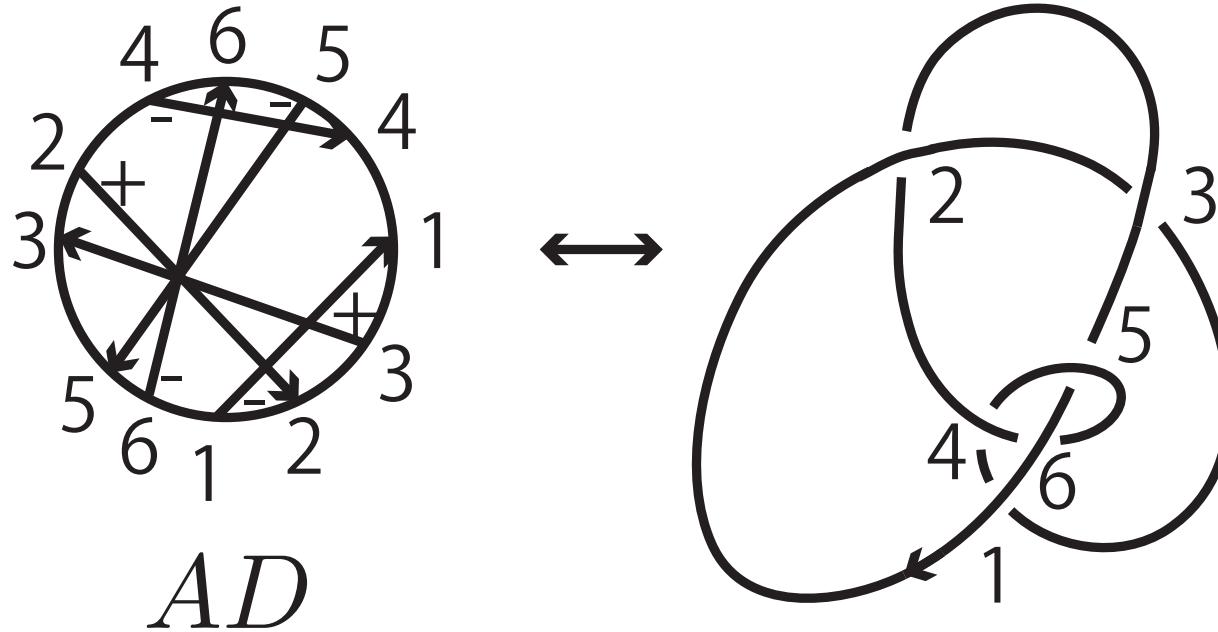
既知の式 (a_2 :Conwayの2次)

	deg 2
knot	a_2
θ (金信)	$-5 + 24 \sum_{i=1}^3 a_2(K_i(T)).$ (半谷千春が正規化 (2008))
H-C	[Kanenobu-Sugita (2012)]
Bouquet	無

文献(敬称略)

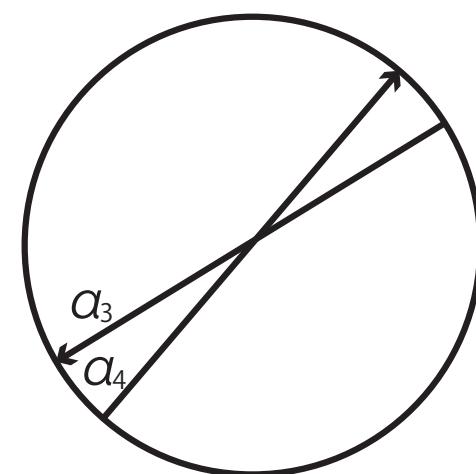
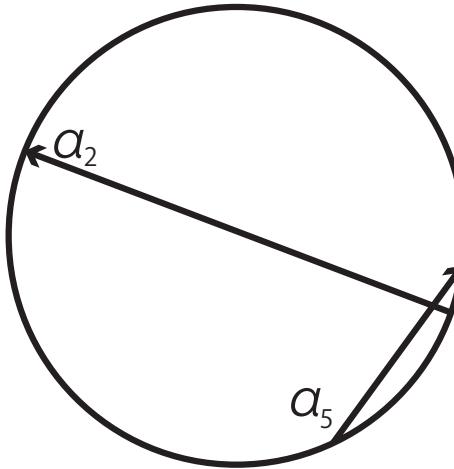
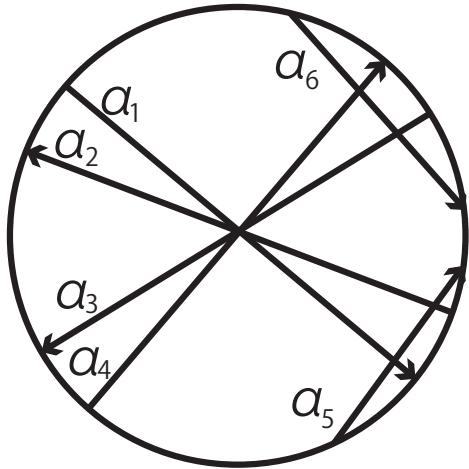
- 山田, 山田多項式の導入 (1989) J. Graph Theory
- 多くの空間グラフの不变量の研究 (新國先生に聞いてください)
- Vassiliev, Vassiliev不变量の導入 (1990) Advanced Soviet Math
- 金信, θ のVassiliev不变量を定義 (1997) JKTR (2重化を使う)
- 小池, 不变量計算(θ , deg 4の基底, 1999) Yokohama Math
- 大山-谷山, 一般の空間グラフのVassiliev不变量の定義 (2001) PJM
- Huh-Jin, 正規化(θ , 2002) JKTR
- 安原, Cn-moveと θ のVassiliev不变量, (2002) TAIA
- 半谷千春, 正規化(deg 2, 東女大修士論文2008)
- 石井, 正規化(general case, 2011) IJM
- 石井-岸本, 不变量計算 (θ , at most deg 4, 2012) TAIA
- 金信-杉田, Handcuff (at most deg 3, 2012) TAIA
- 吉田英生, 正規化 (θ , deg 3, 関西大修士論文2014)

arrow diagram



- arrow diagram $AD \stackrel{\text{def}}{=} \text{oriented knot diagram}$ が
もつ 2 重点 k の逆像 (k, \bar{k}) の配置.
- arrow diagram $\stackrel{\text{def}}{=}$ 円周上 2 個ずつペアとなる点
配置.

arrow diagram



- arrow diagram $\stackrel{\text{def}}{=}$ 円周上 2 個ずつペアとなる点配置.
- ここから sub arrow diagramを数え上げる関数を定義する.

sub diagramの数え上げ関数の定義(院生向け)

- arrow diagram G に対し,

$$\text{Sub}(G) := \{\text{sub diagrams of } G\}$$

- 内積 $(,)$ により

$$\langle x, G \rangle := \sum_{z \in \text{Sub}(G)} (x, z).$$

→線型拡張しGauss diagram formulaと呼ぶ.

→ $\sum_x \lambda_x \langle x, G \rangle$ の λ_x を選び不変量を作ろう！

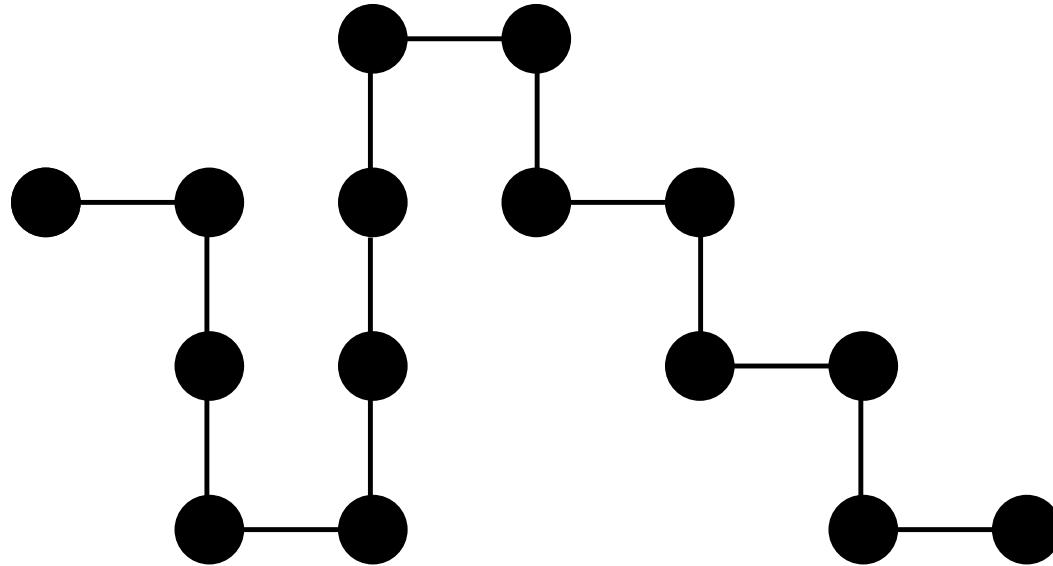
なぜ“Gauss diagram formula?”(院生向け)

- classical knotsの, 任意のVassiliev不変量は Gauss diagram formulaで表示される。
→大山-谷山の式を経由すれば, 空間グラフへも相当な範囲で応用が見込まれる.
- 不変量作成方法の自動化が確立(I. 2019).
- 計算効率から、ポリマーへの応用が期待.
- 量子群の表現論, 配置空間論(積分論)へ連携.
(例：ビオ・サバール“=”Gauss diagram formula)

高分子方面(関連だけ抜粋)

- 出口哲生 [結び目確率(どのくらいの頂点数で結び目が確率的に現れるか)の定式化を90年代にしている]→Shimamura-Deguchiなどを経て、実際の実験でknot-typeの物性があることをつきとめた(Uehara-Deguchi, 2017).
- 手塚研究室 実際に、高分子で空間グラフが合成されている(Heguri-Yamamoto-Tezuka, 2015)
- 環状かそうでないかだけでも物性は大きく異なることがある(東工大プレスリリース, 山本, 2013)

希薄溶液のポリマー

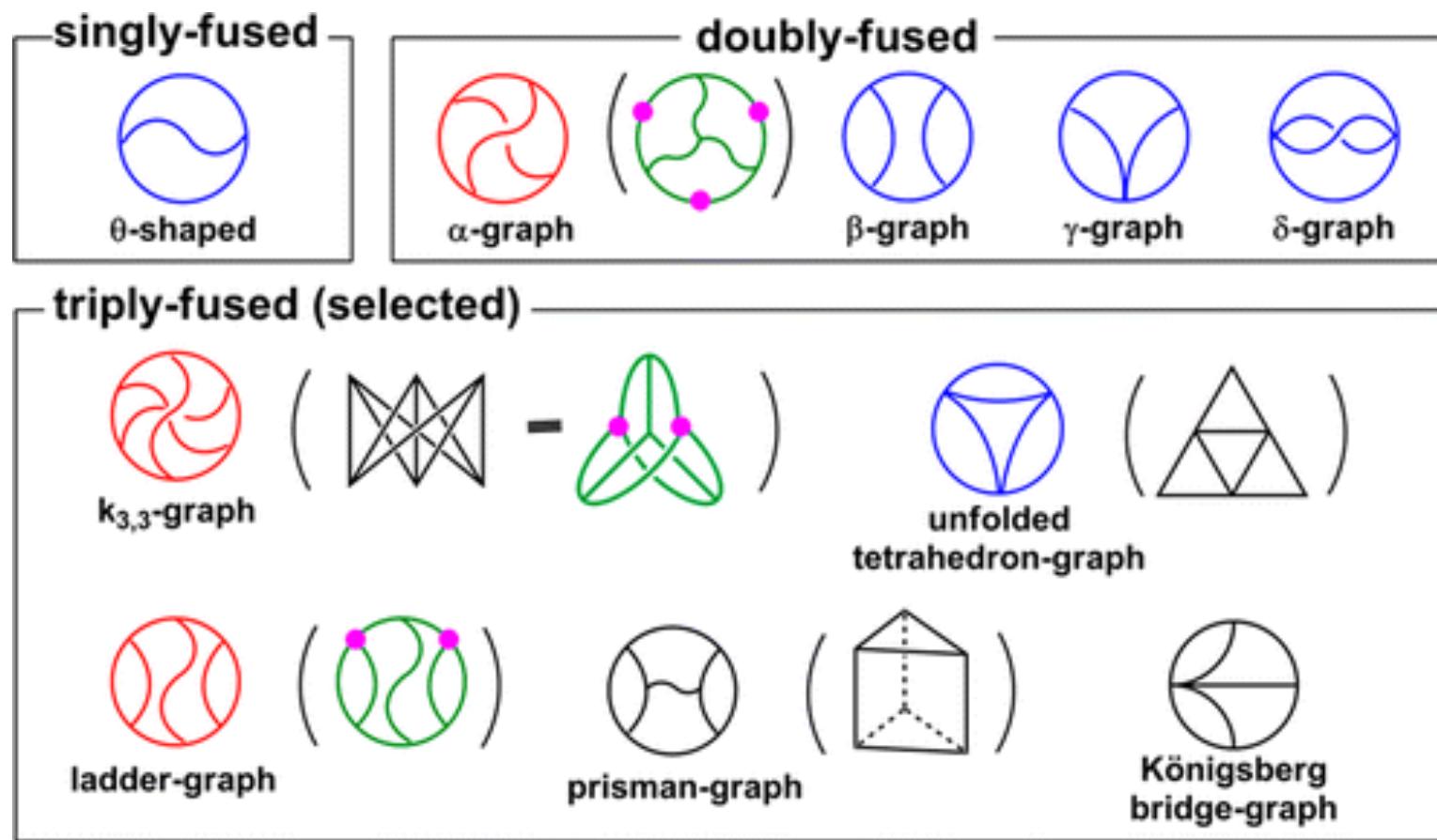


希薄溶液のポリマー[自分自身との重なりを許さない実在鎖(self-avoiding random walk, SAW)]

頂点は体積排除効果を持ち、その効果が大きくなるとトポロジー効果の観察が難しくなる。

<なお、セグメント、ボンドの用語に数学者は注意。>

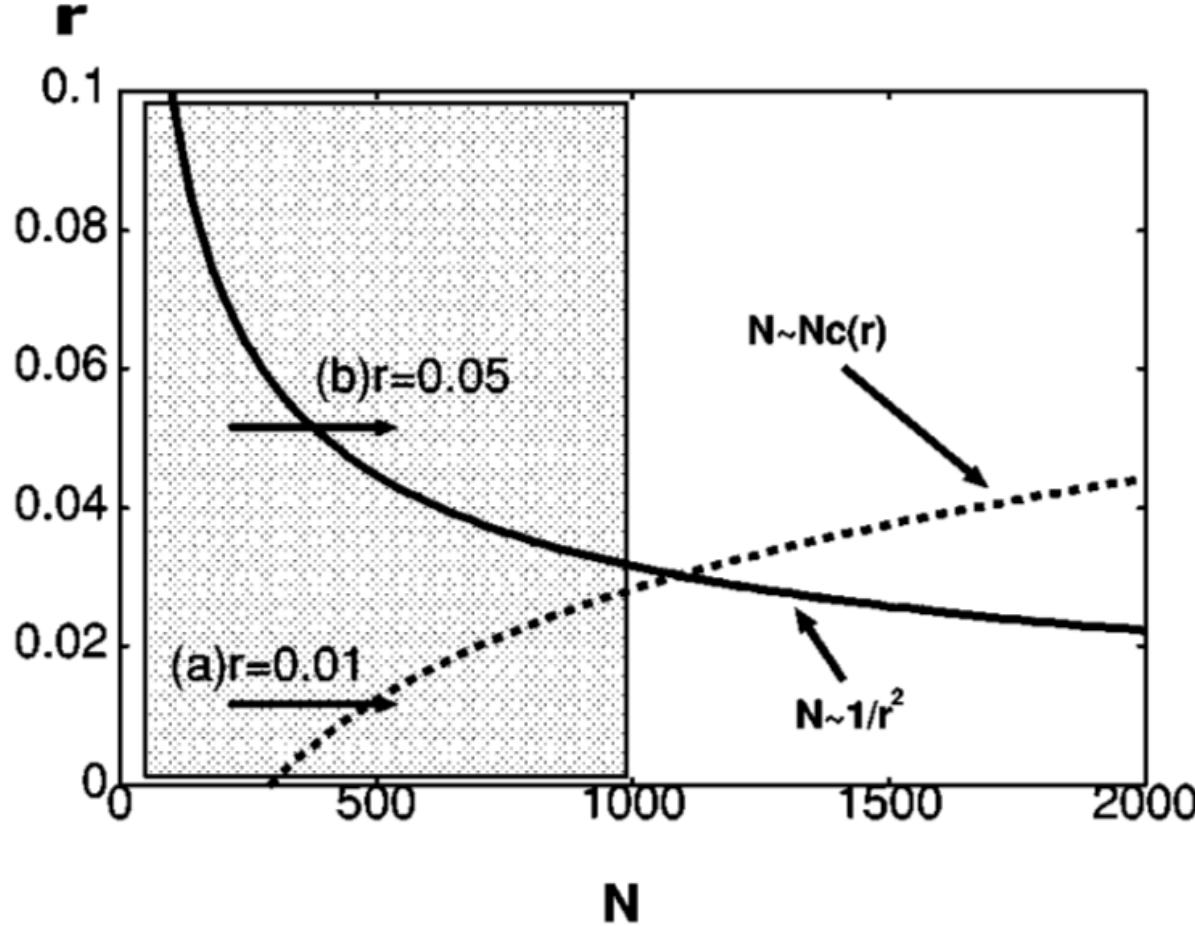
空間グラフの合成(手塚研究室)



T. Suzuki, T. Yamamoto, and Y. Tezuka, *Constructing a Macromolecular K_{3,3} Graph through Electrostatic Self-Assembly and Covalent Fixation with a Dendritic Polymer Precursor*, J. Am. Chem. Soc. 2014, 136, 28, 10148–10155

ポリマーとして空間グラフは実際に合成されており、
その性質は研究対象となっている。

頂点数Nと排除体積効果



M. K. Shimamura and T. Deguchi, *Finite-size and asymptotic behaviors of the gyration radius of knotted cylindrical self-avoiding polygons*, Phys. Rev. E 2002, **65**, 051802

頂点数Nは排除体積効果が小さい範囲でありながら,
knotするくらい大きくあってほしい。

山田多項式

S. Yamada (1989) —————

g : 空間グラフの正則図 (diagram)

山田多項式 $R(g)(A) \in \mathbb{Z}[A, A^{-1}]$ は以下の性質を満たす
ものとして帰納的に定義される

$$(1) \quad R(\text{ \circlearrowleft }) = A + 1 + A^{-1}$$

$$(2) \quad R(\text{ \times }) = AR(\text{ \curvearrowright } \subset) + A^{-1}R(\text{ \curvearrowleft } \supset) + R(\text{ \times with dot })$$

$$(3) \quad R(\text{ \rightarrow with e } \leftarrow) = R(\text{ \rightarrow } \leftarrow) + R(\text{ \times with dot })$$

$$(4) \quad R(g_1 \sqcup g_2) = R(g_1)R(g_2)$$

$$(5) \quad R(g_1 \vee g_2) = -R(g_1)R(g_2)$$

- 山田多項式 $R(g)(A)$ は $(-A)^n$ ($n \in \mathbb{Z}$) をかけることを除いて、flat vertex isotopy 不変量になっている

「正規化」の意味

バシリエフ不变量は定数倍のambiguityがあるので、山田多項式に合わせることを「正規化」と呼ぶことにする。

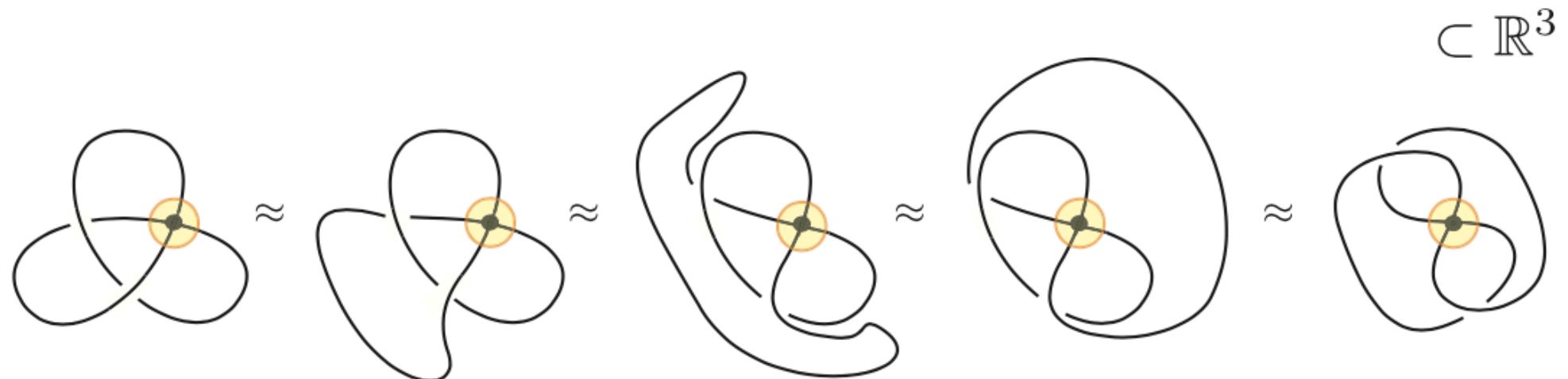
(Ishii 2011)(Yoshida 2014)

flat vertex isotopy

G_1, G_2 : flat vertex graph

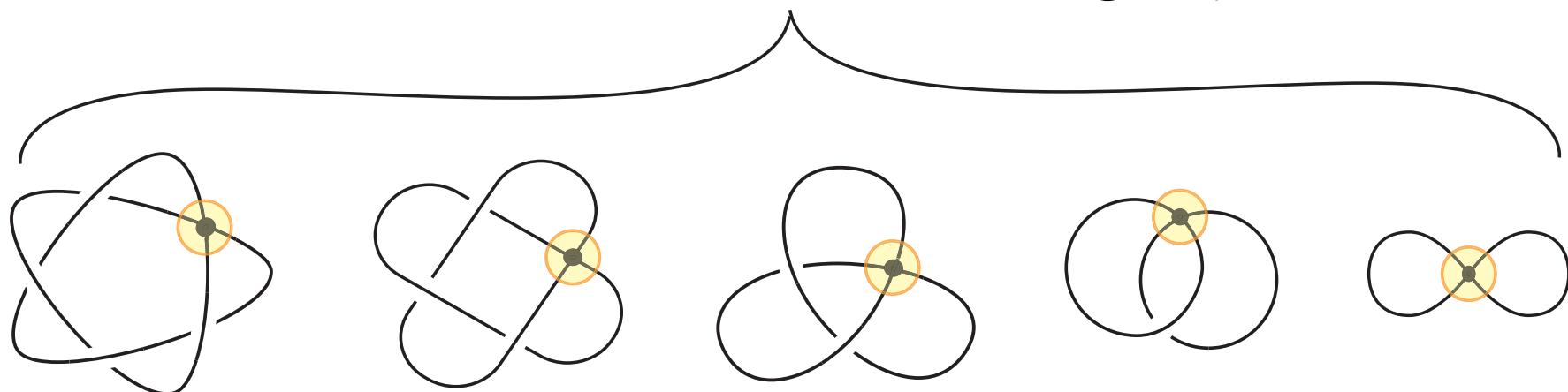
G_1 と G_2 が flat vertex isotopic

$\overset{\text{def}}{\iff} G_1$ から G_2 への ambient isotopy $\Phi : \mathbb{R}^3 \times \mathbb{I} \longrightarrow \mathbb{R}^3 \times \mathbb{I}$ が存在して, $\forall t \in [0, 1]$ に対して $\Phi(G_1, t)$ が flat vertex graph になっている

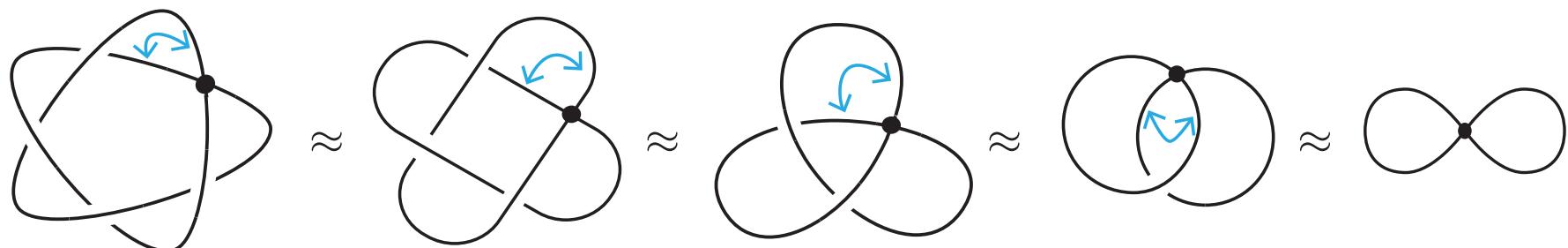


flat vertex isotopy と ambient isotopy の差

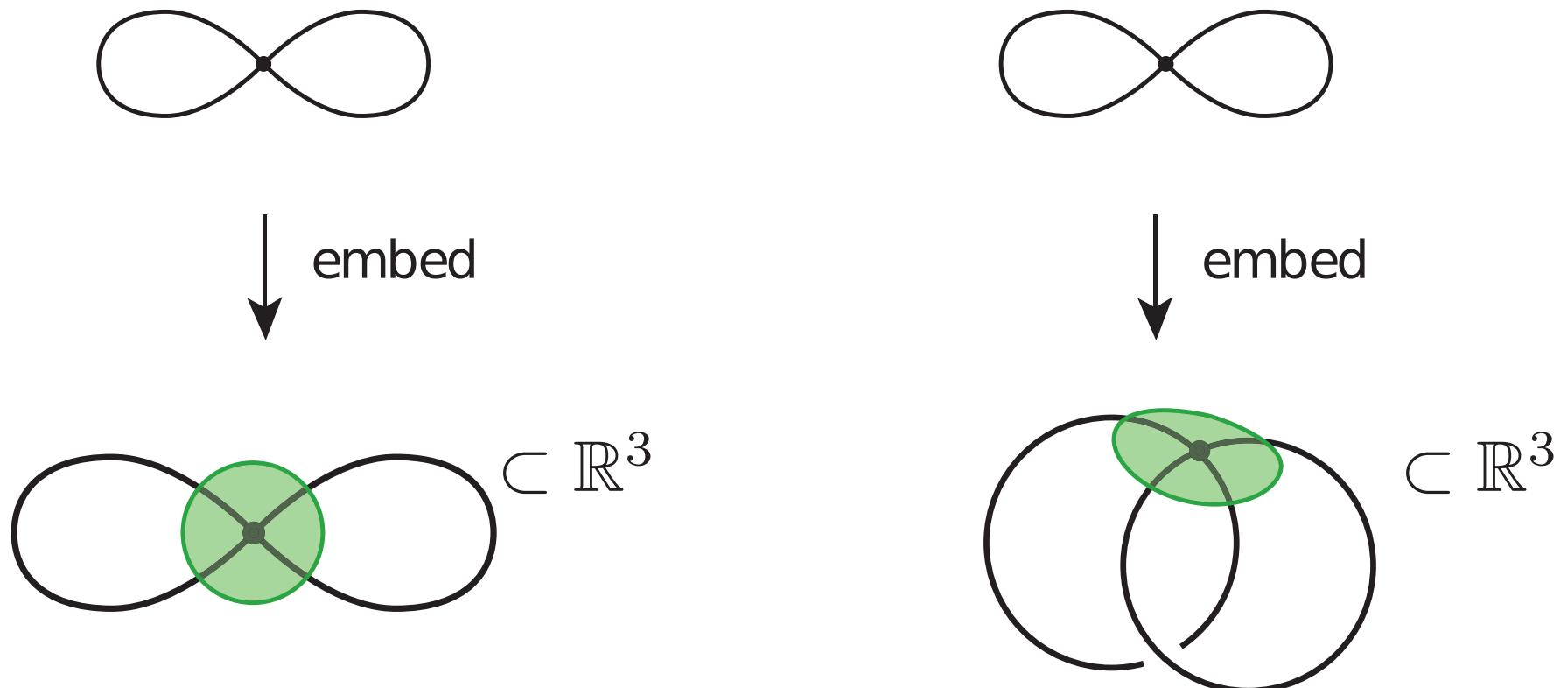
すべて異なるflat vertex graph



up to ambient isotopy



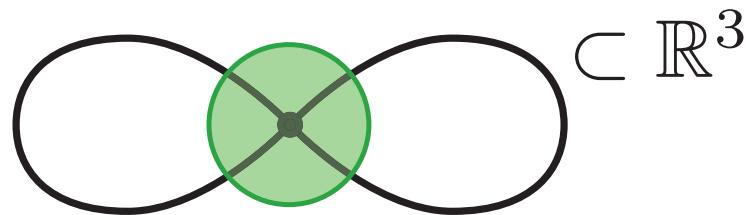
2-bouquet graph



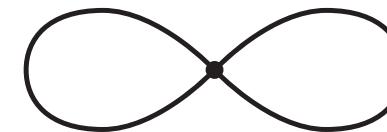
2-bouquet graph



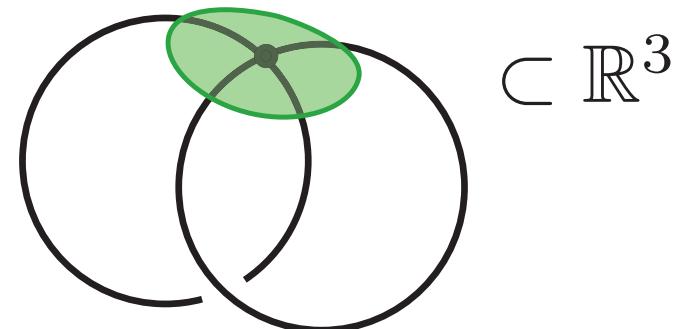
↓ embed



K型 (knot型)

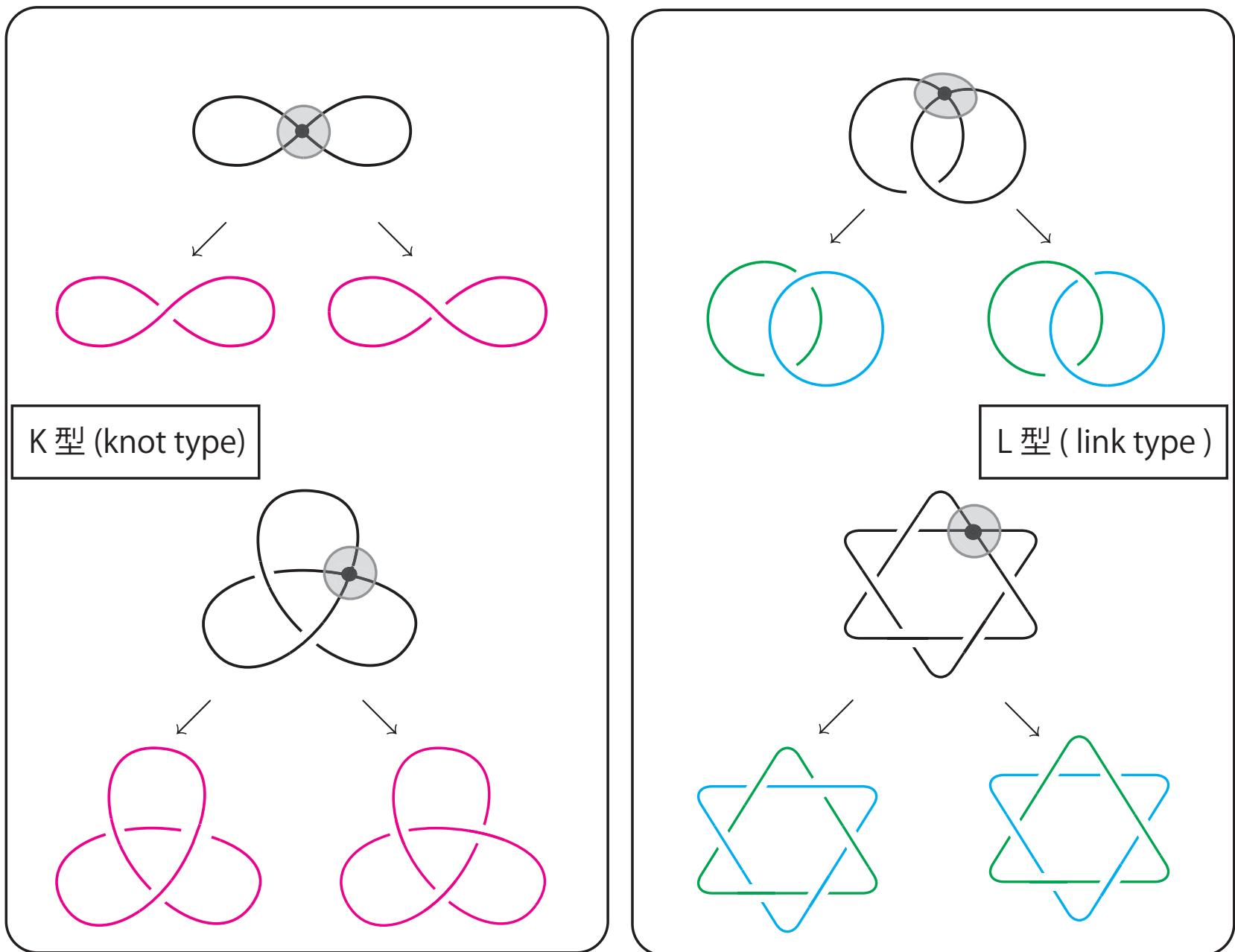


↓ embed

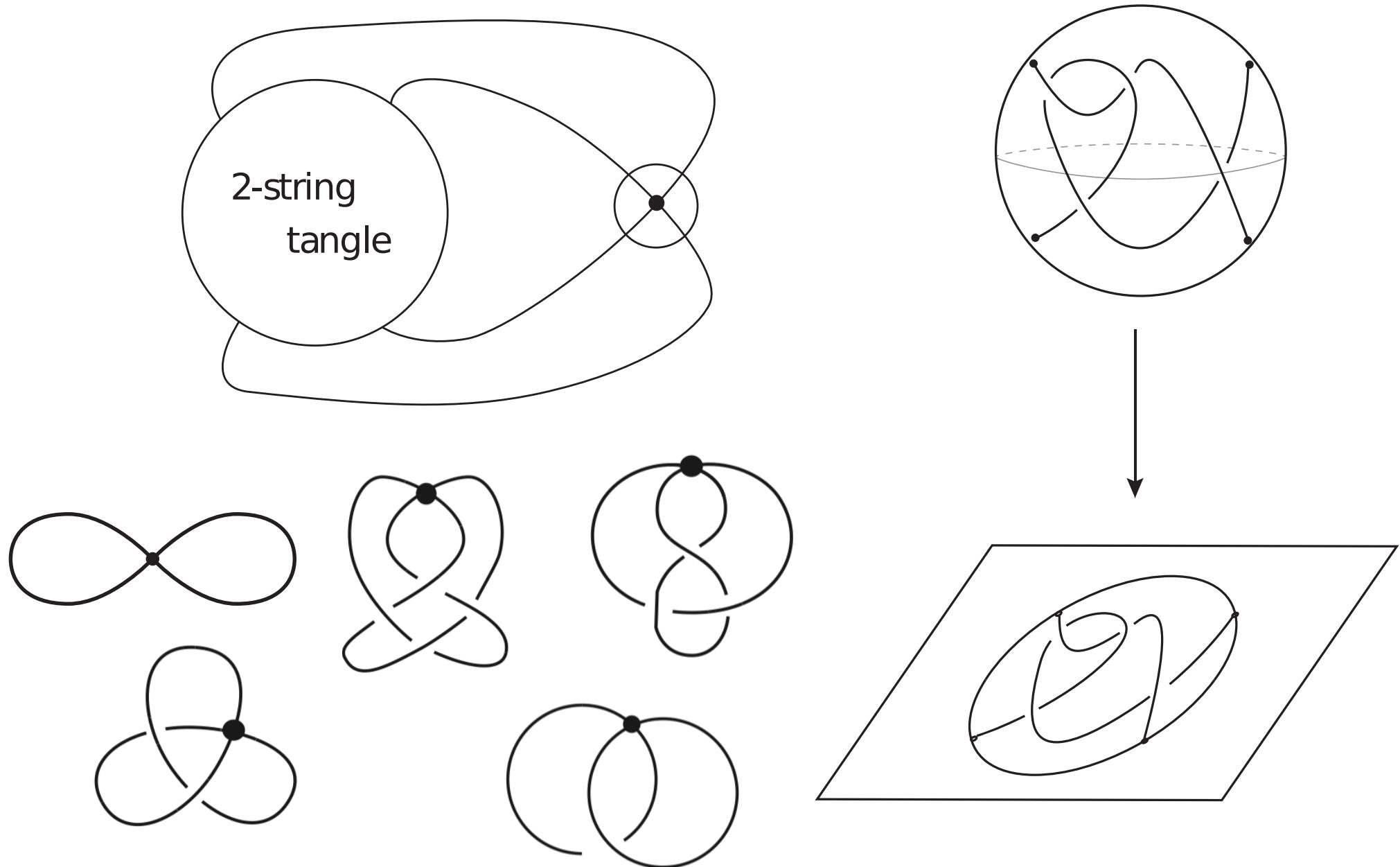


L型 (link型)

K型とL型

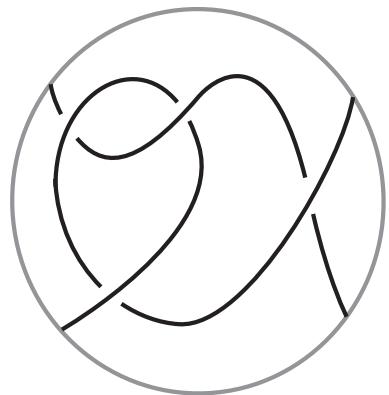


2-string tangleと2-bouquetの関係



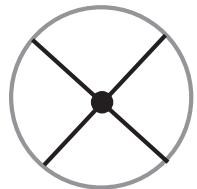
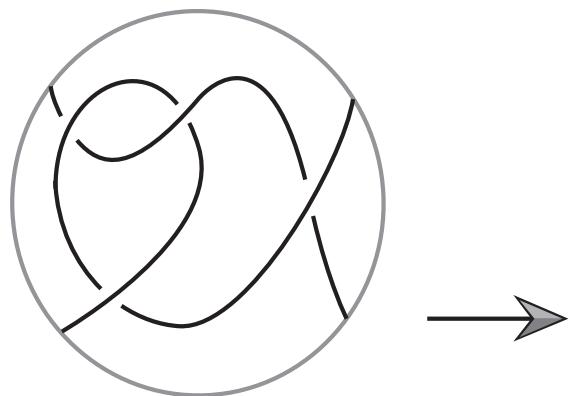
2-string tangleから2-bouquetを構成する

2-string tangle



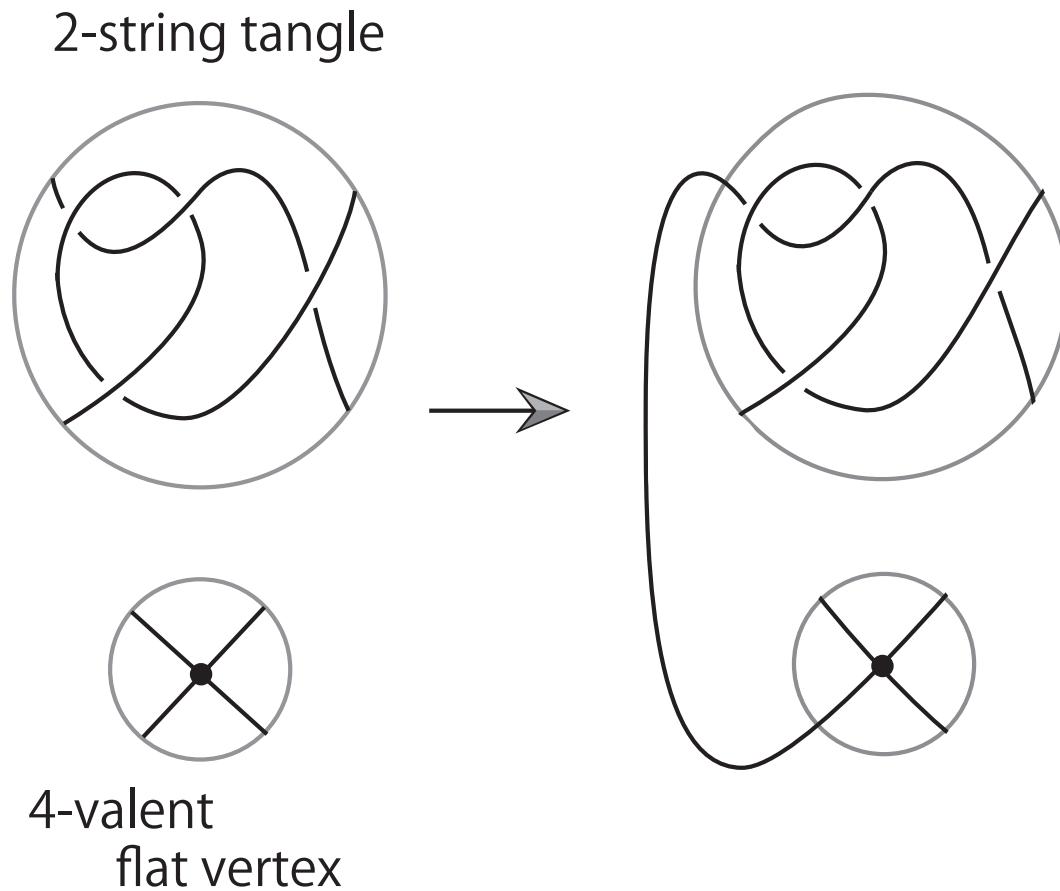
2-string tangleから2-bouquetを構成する

2-string tangle

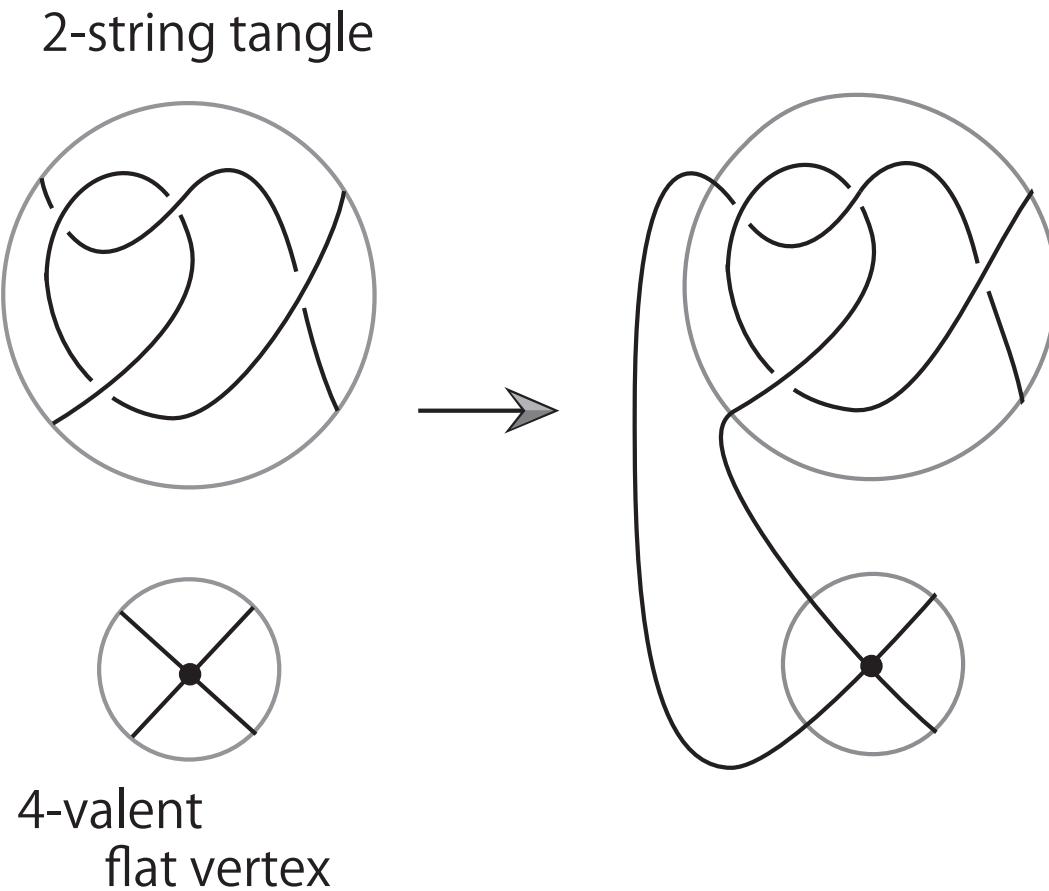


4-valent
flat vertex

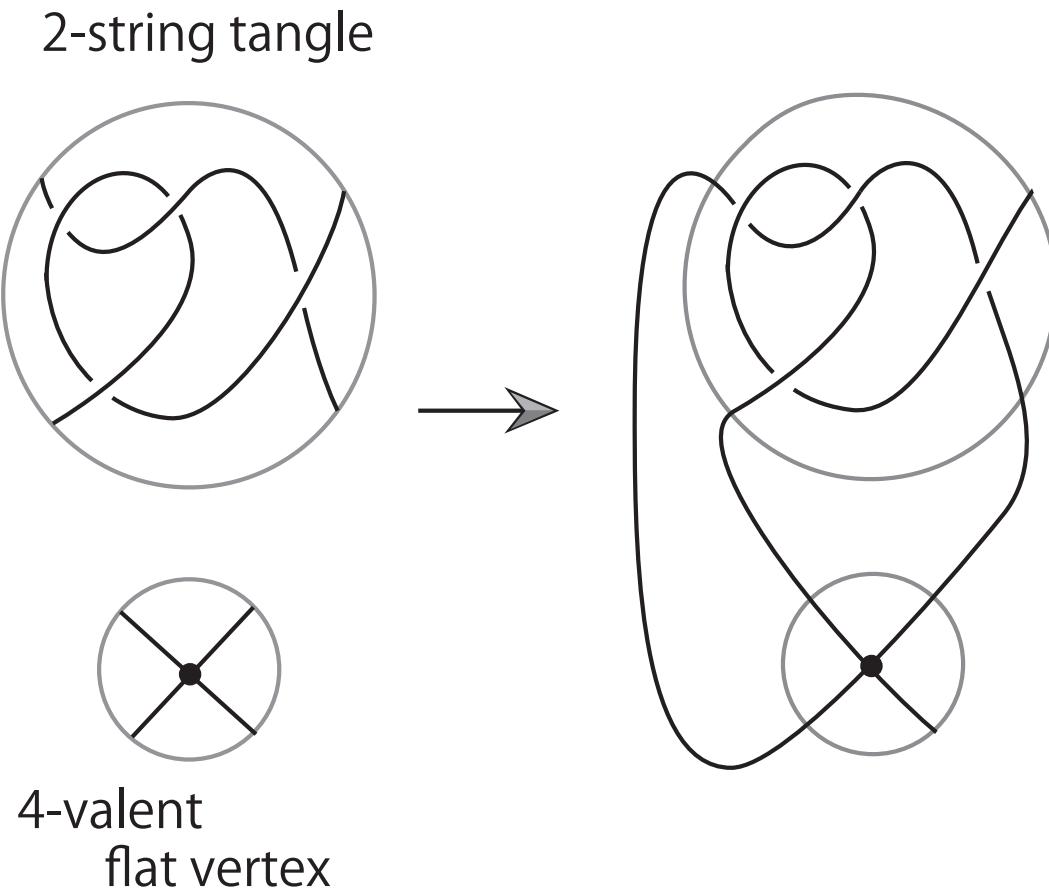
2-string tangleから2-bouquetを構成する



2-string tangleから2-bouquetを構成する

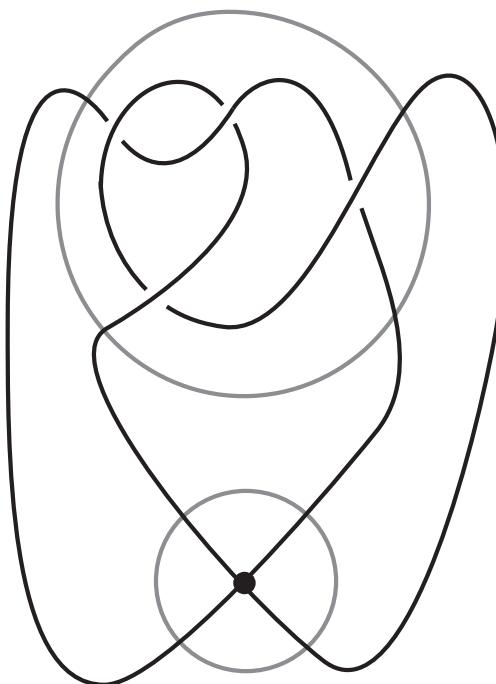
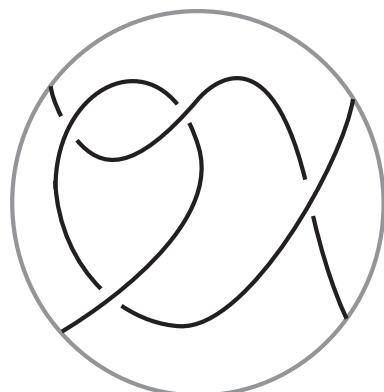


2-string tangleから2-bouquetを構成する



2-string tangleから2-bouquetを構成する

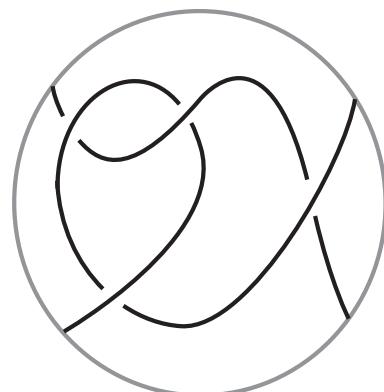
2-string tangle



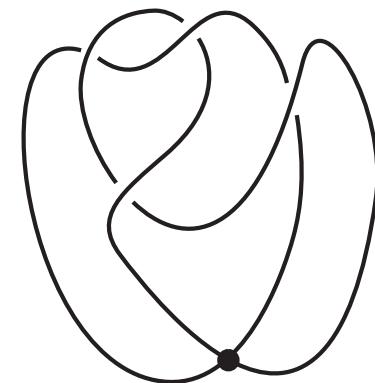
4-valent
flat vertex

2-string tangleから2-bouquetを構成する

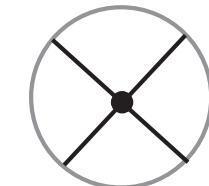
2-string tangle



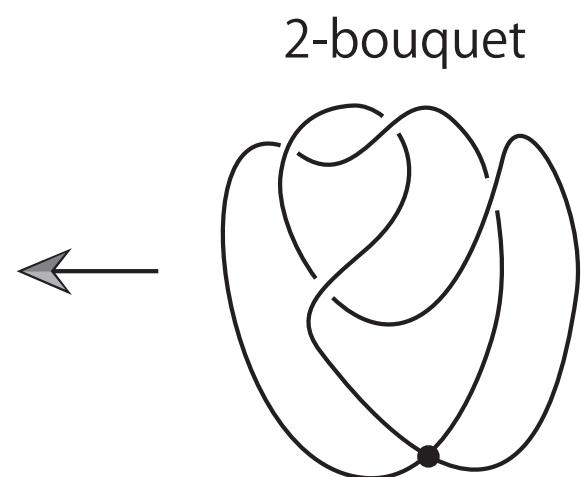
2-bouquet



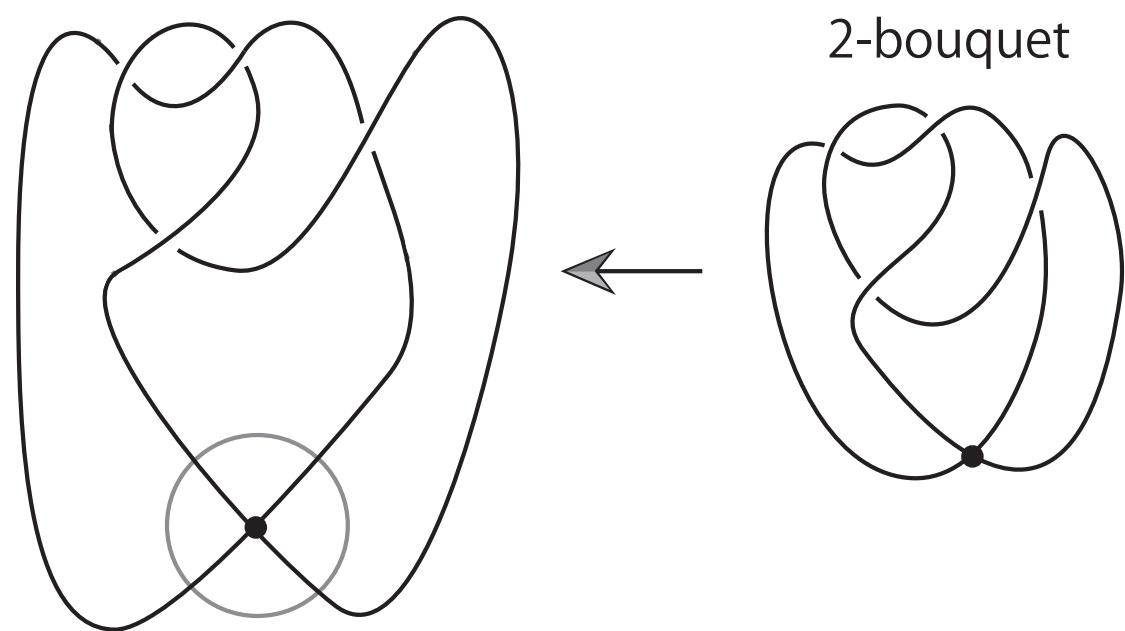
4-valent
flat vertex



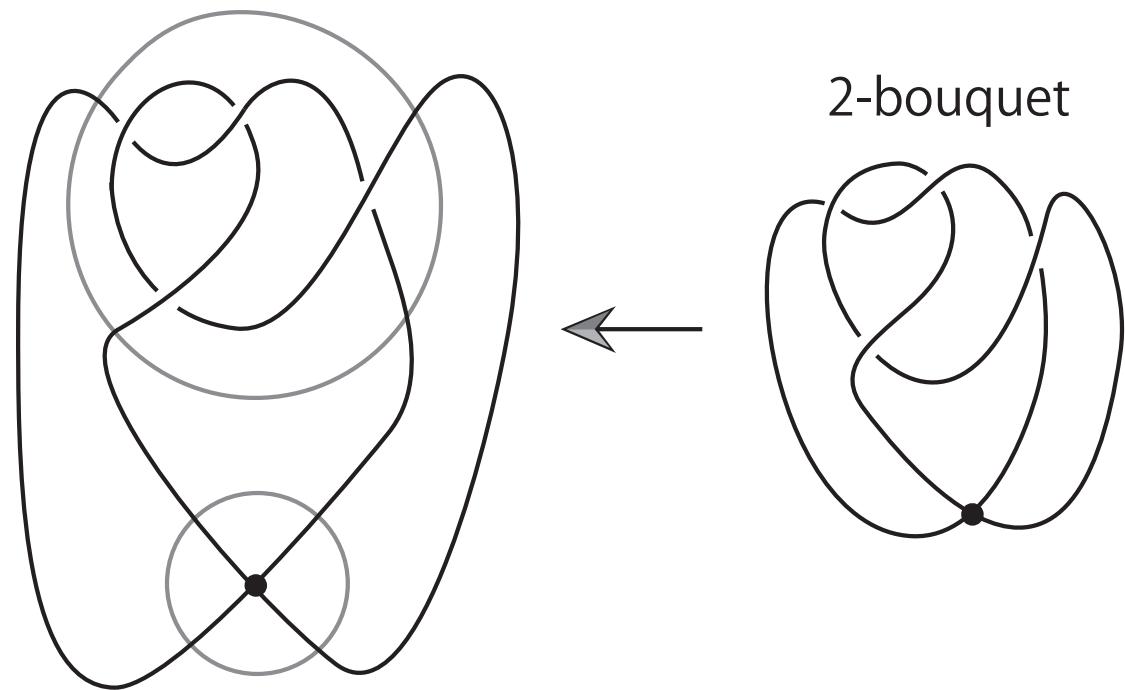
2-bouquetが与えられたとき



2-bouquetが与えられたとき

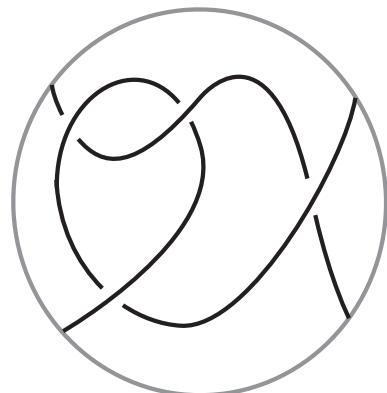


2-bouquetが与えられたとき

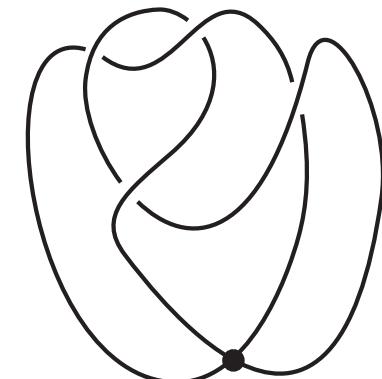


2-bouquetが与えられたとき

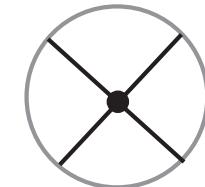
2-string tangle



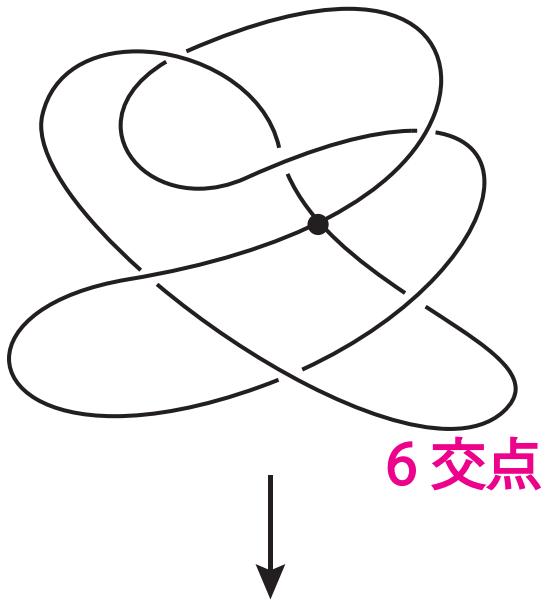
2-bouquet



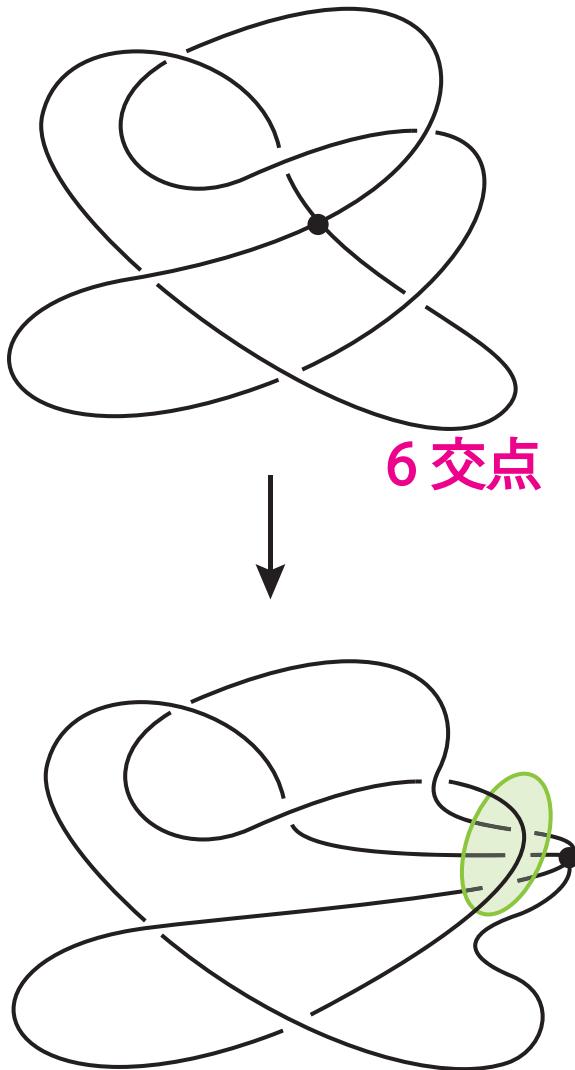
4-valent
flat vertex



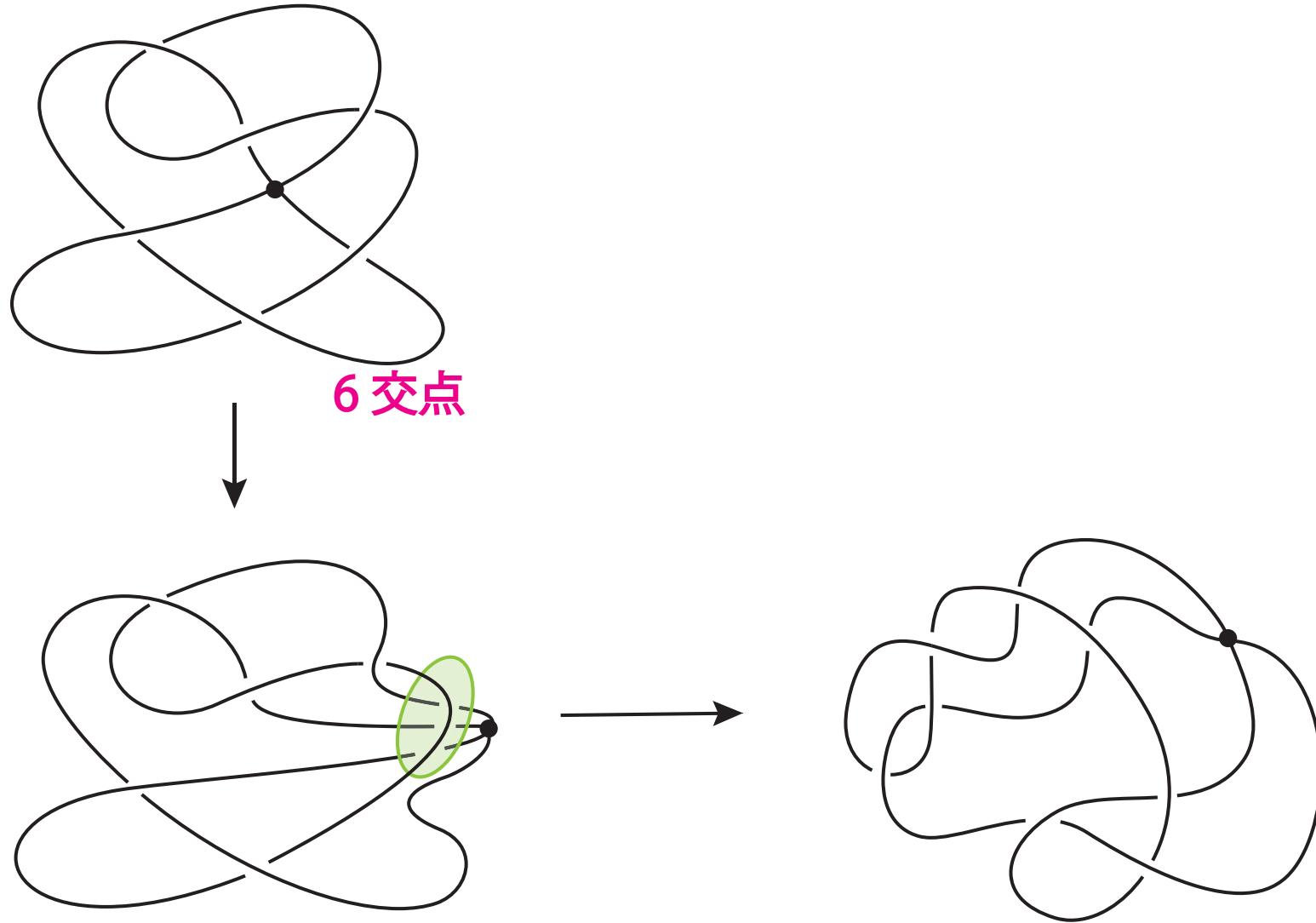
頂点が内部にある場合



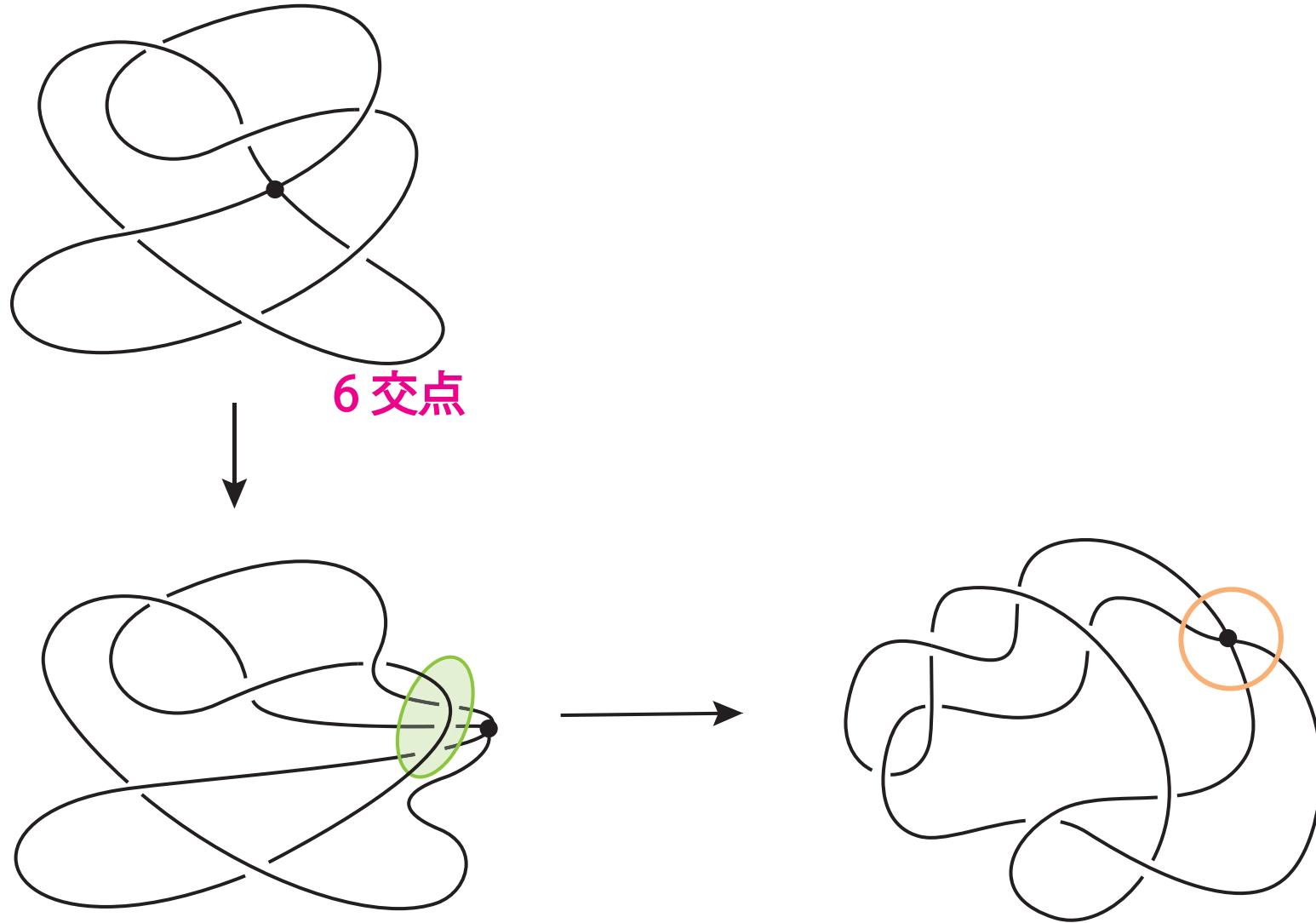
頂点が内部にある場合



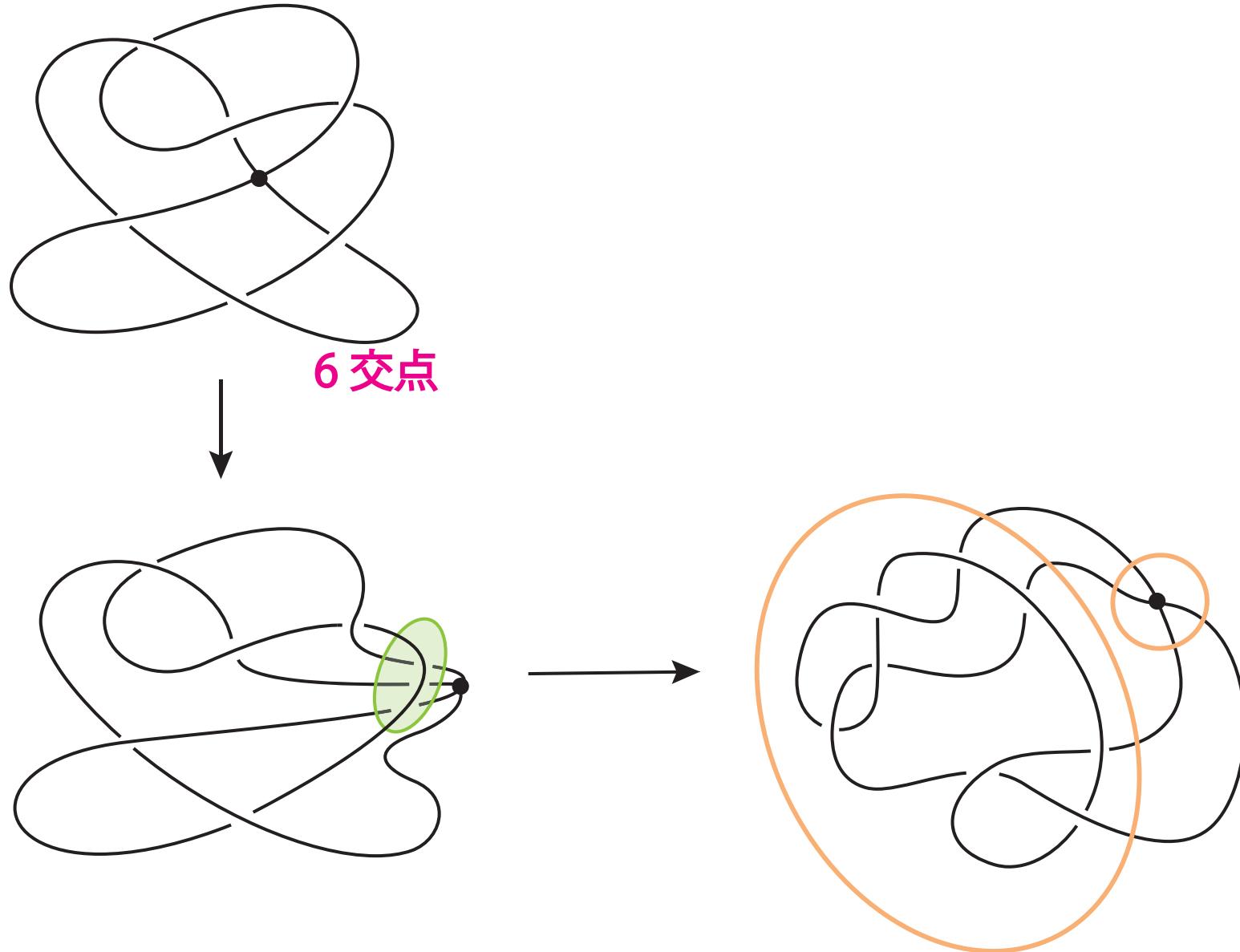
頂点が内部にある場合



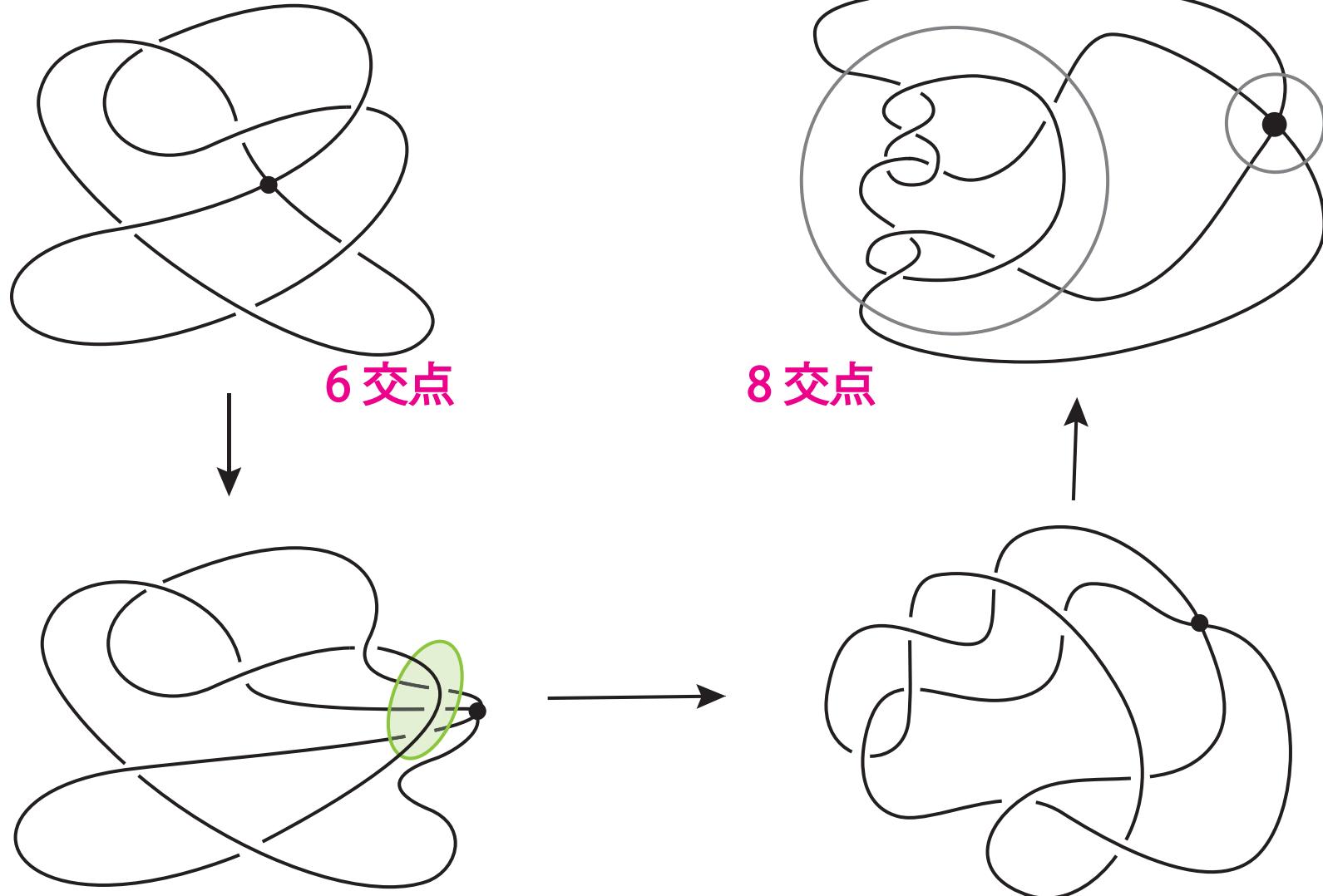
頂点が内部にある場合



頂点が内部にある場合

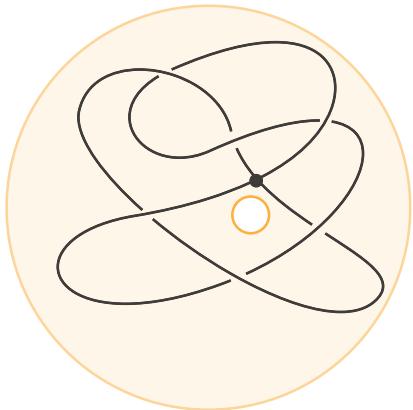


頂点が内部にある場合



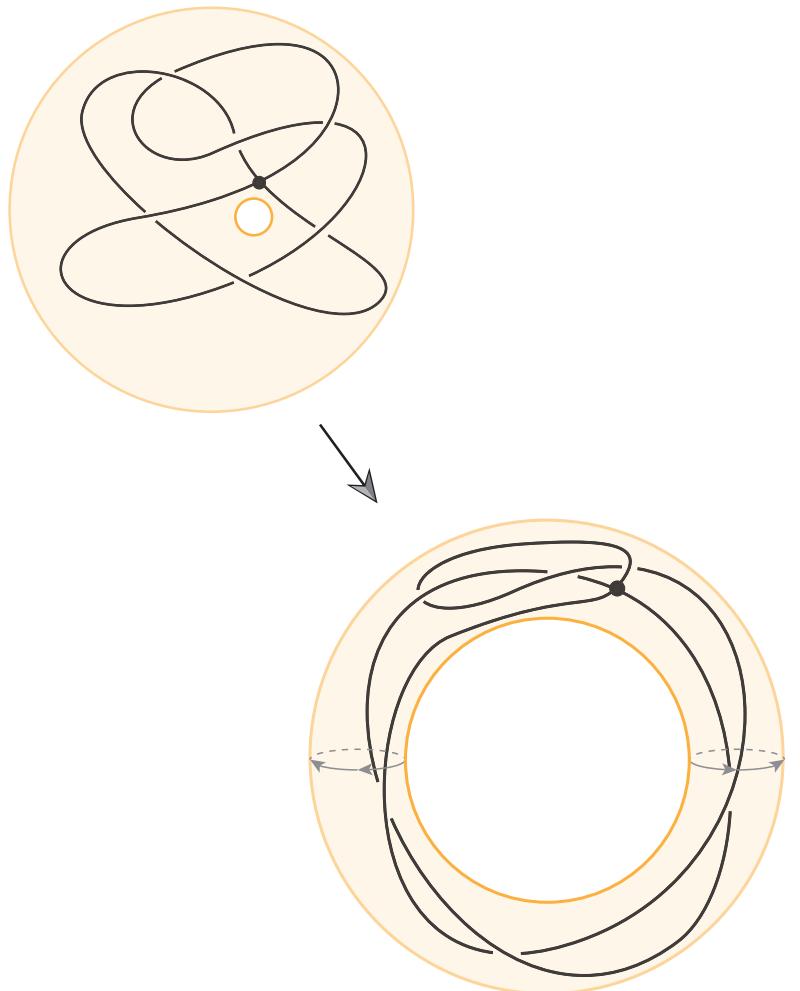
頂点が内部にある場合

solid torusに入れて



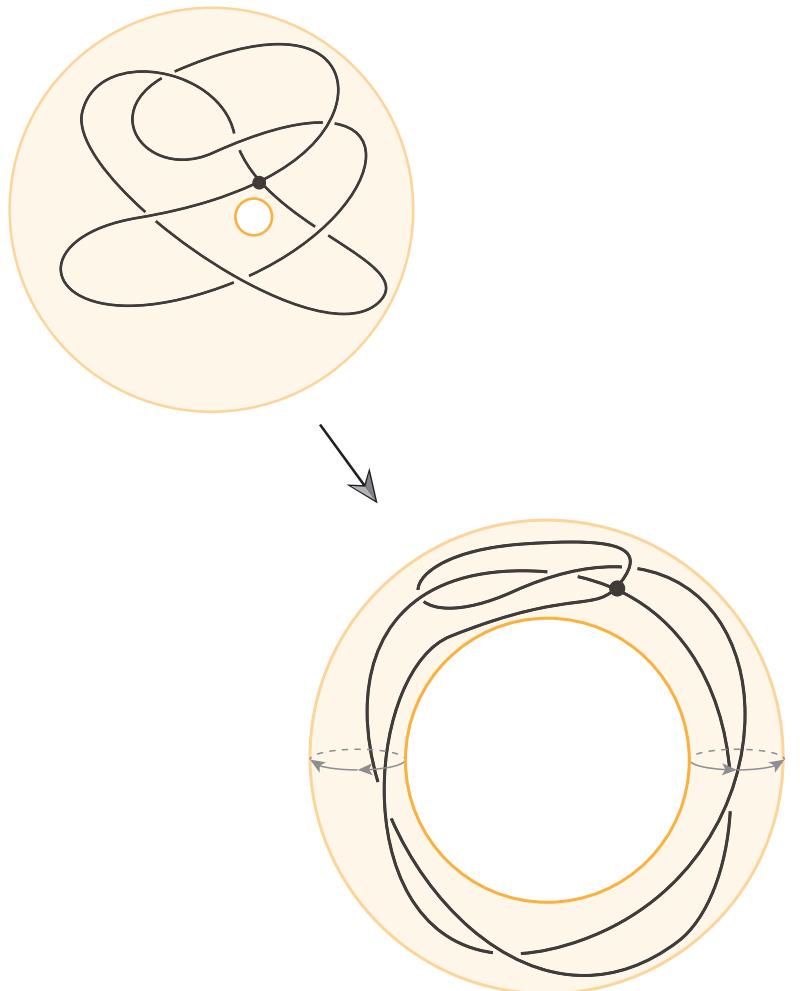
頂点が内部にある場合

solid torusに入れて



頂点が内部にある場合

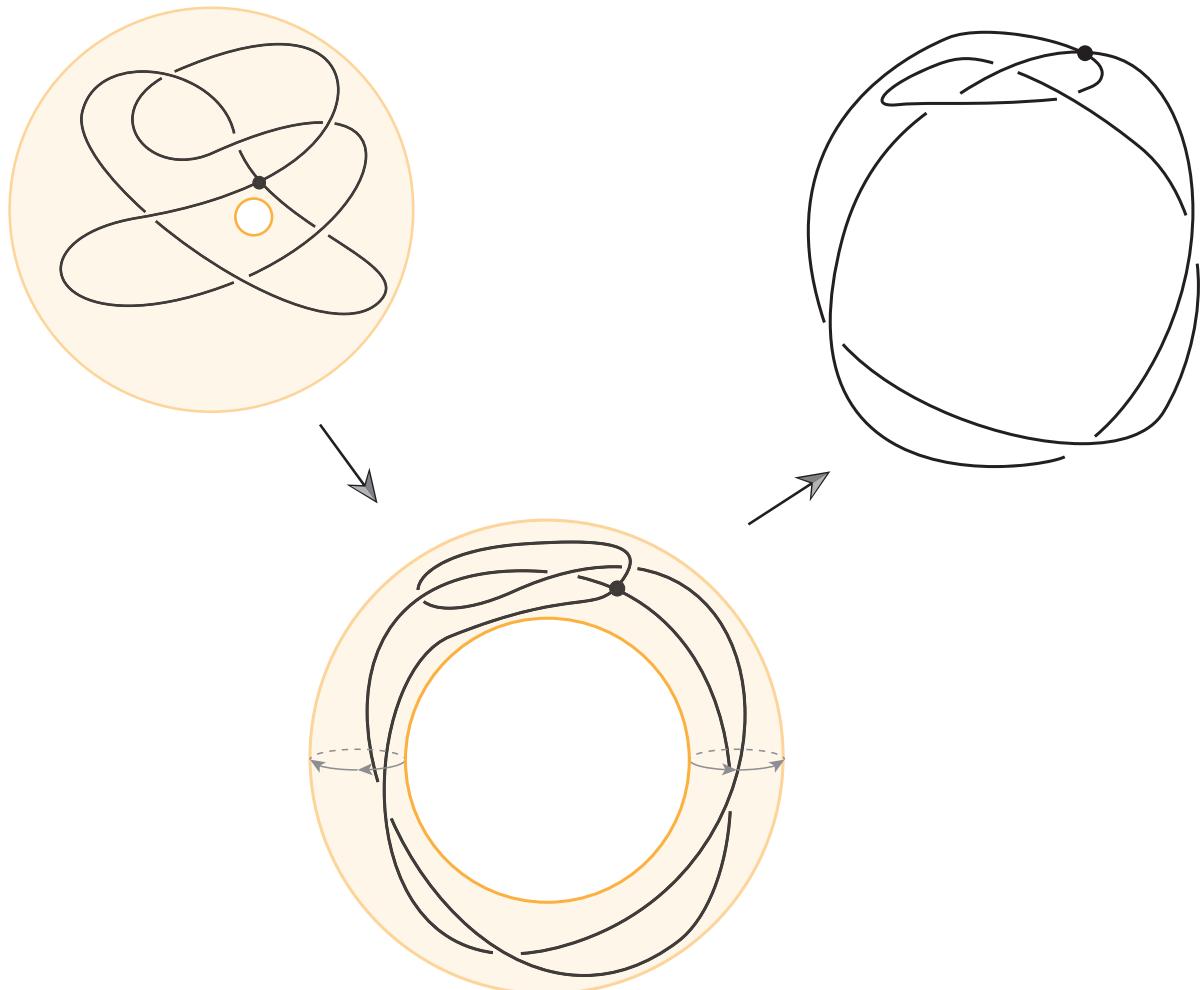
solid torusに入れて



180度回転させる

頂点が内部にある場合

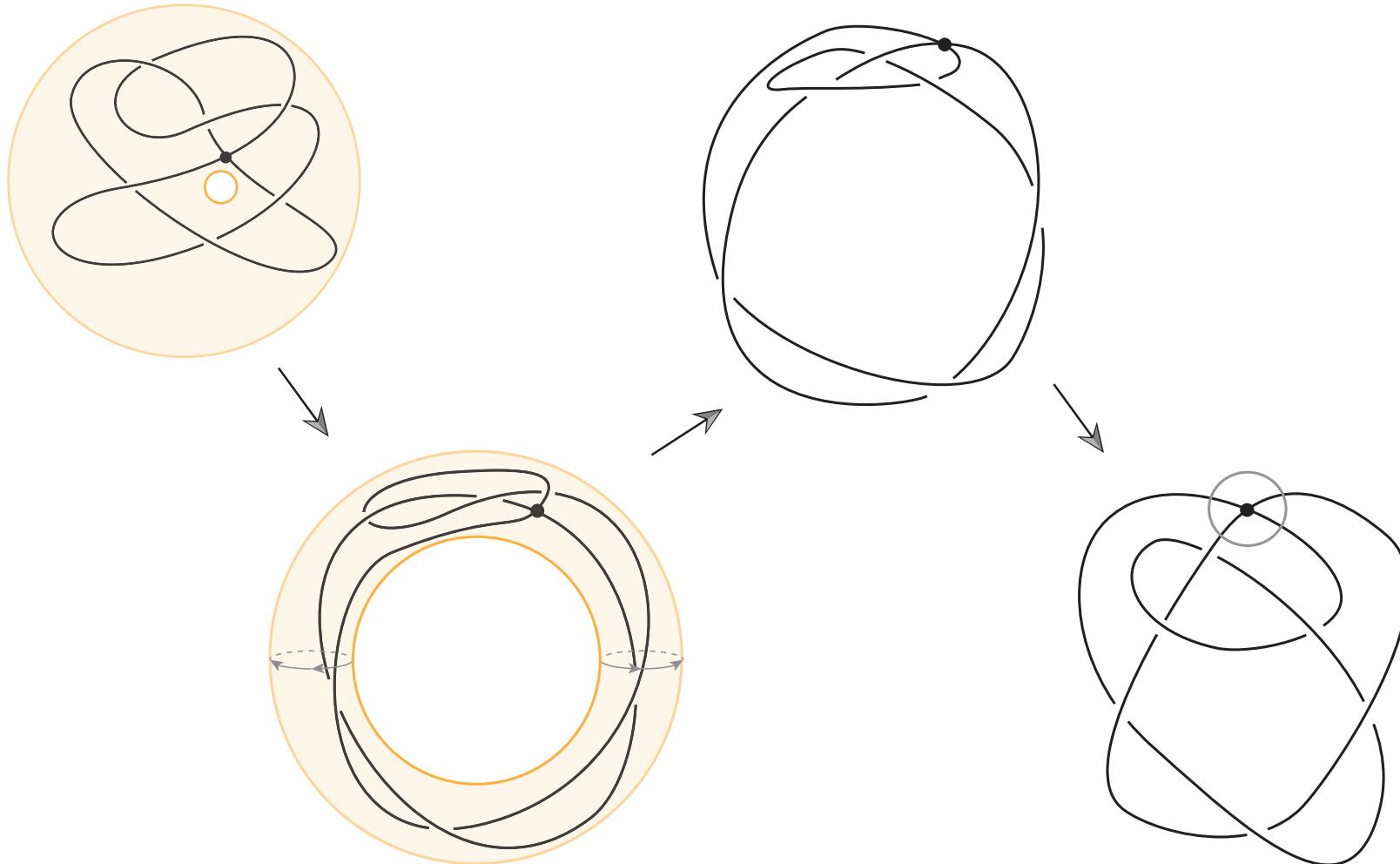
solid torusに入れて



180度回転させる

頂点が内部にある場合

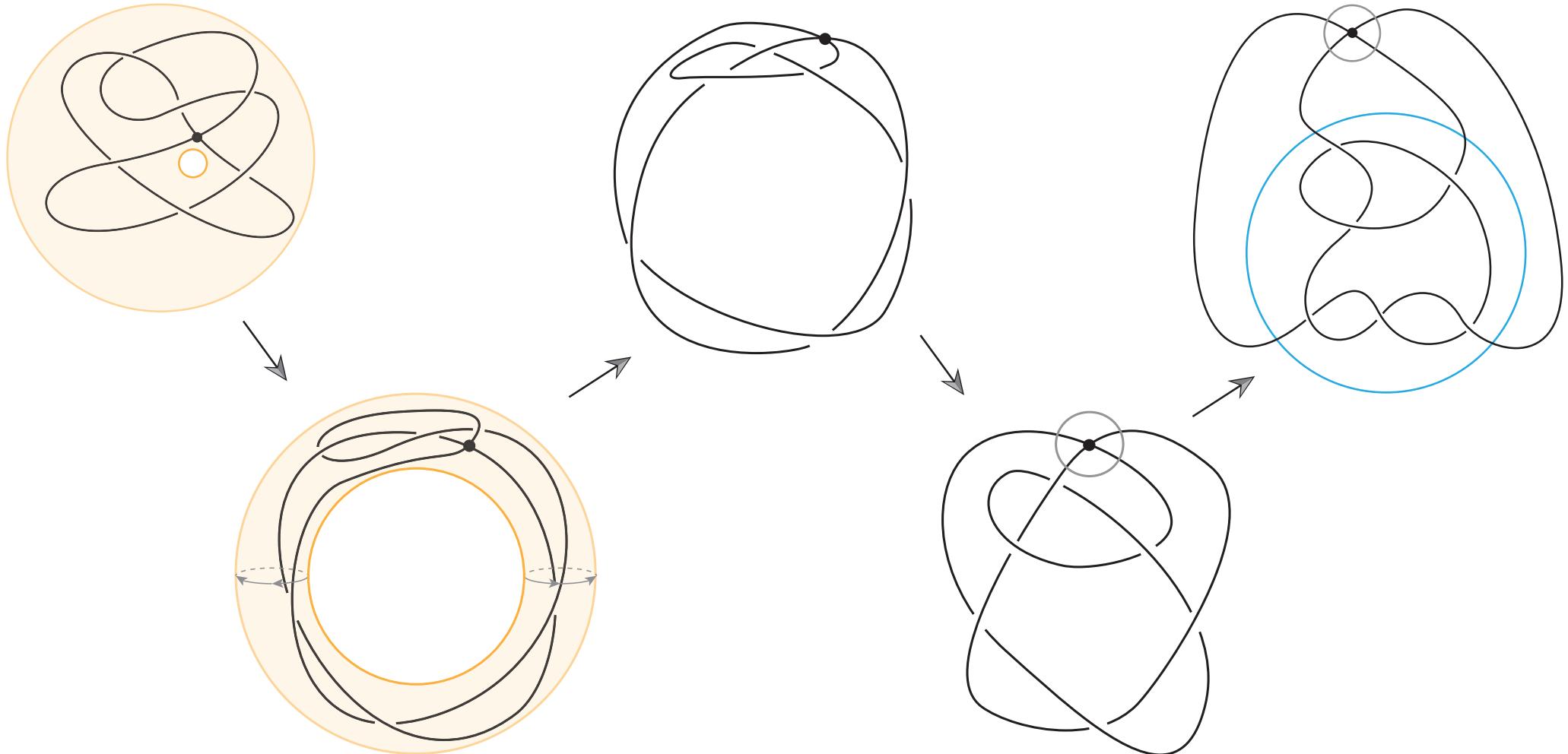
solid torusに入れて



180度回転させる

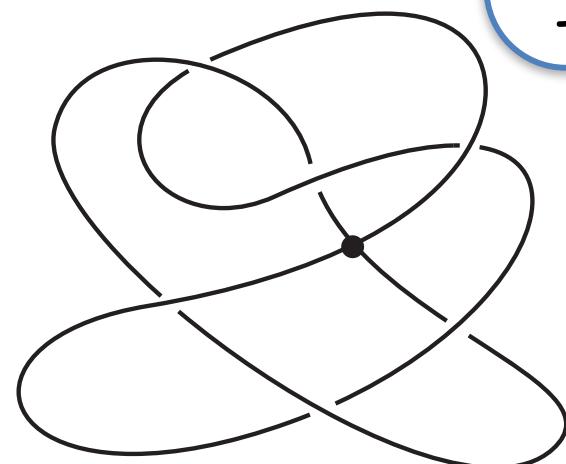
頂点が内部にある場合

solid torusに入れて

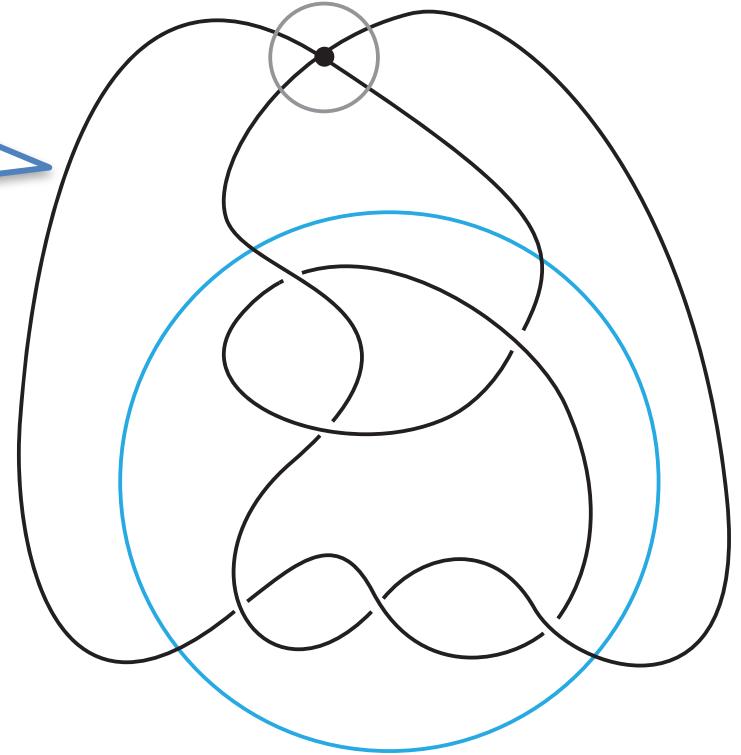


180度回転させる

2-bouquetと2-string tangleの関係

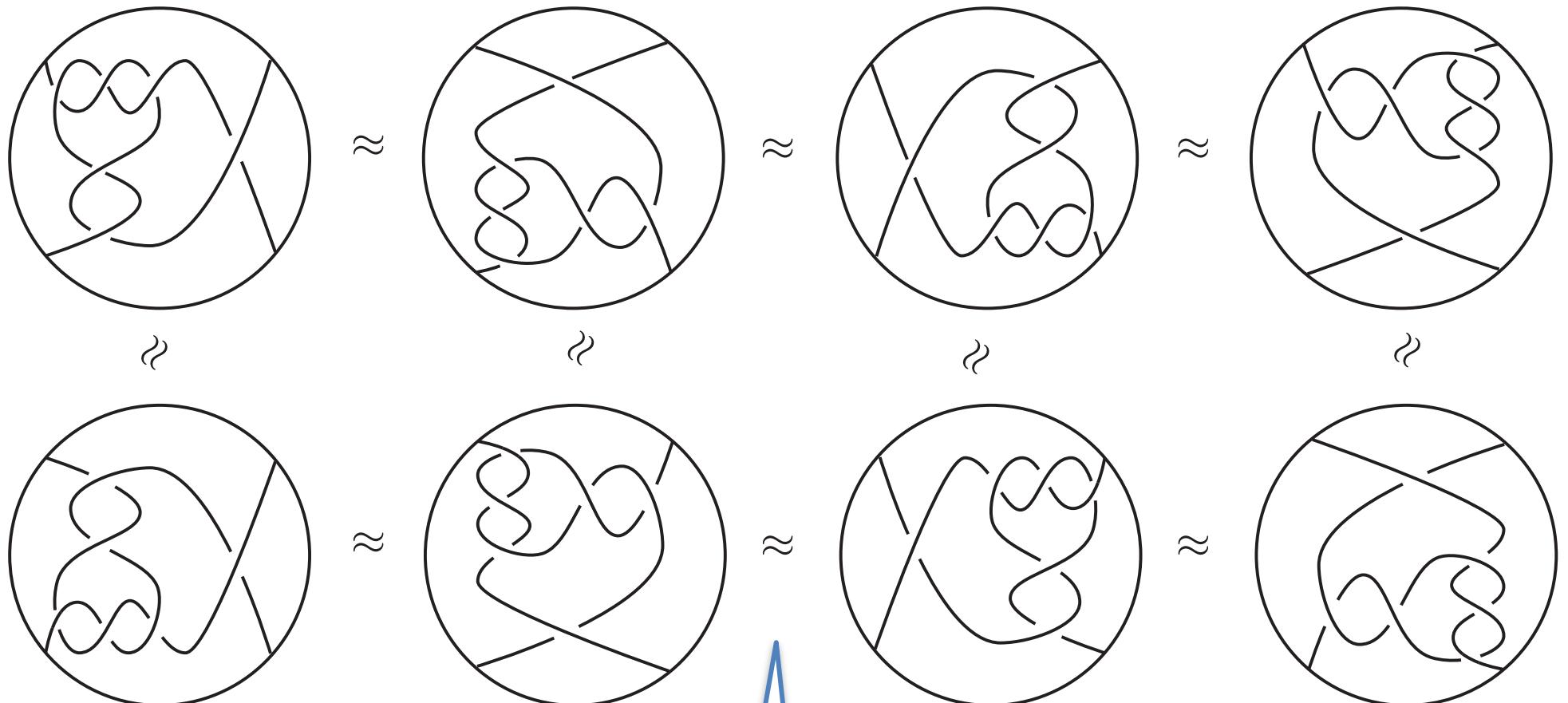


2-string tangleに
4価のflat vertexを
つないだもの



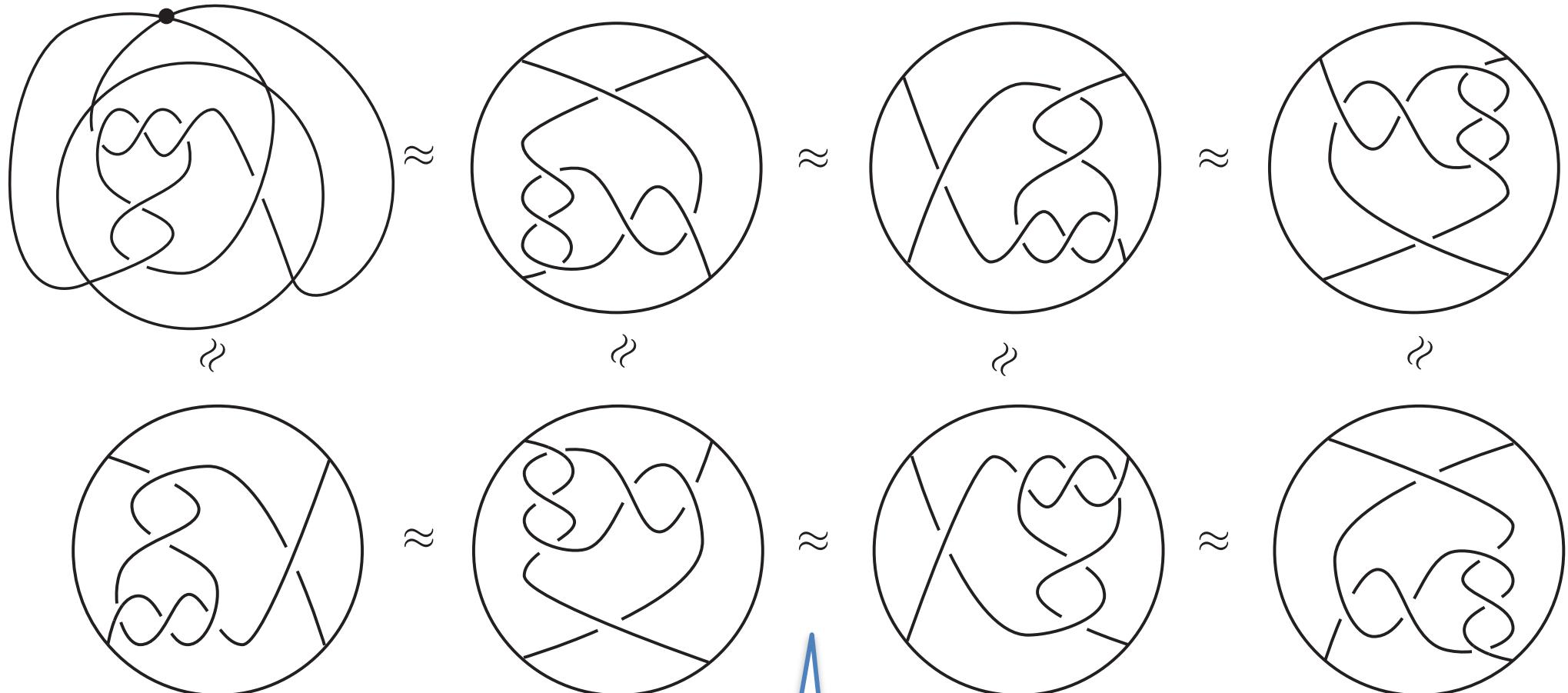
- 2-bouquetは2-string tangleと1対1に対応する

2-string tangleの同値関係



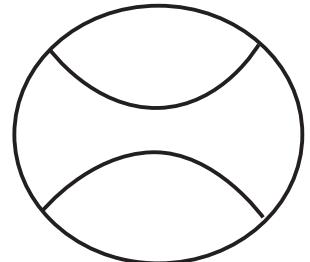
これらは同じ2-string tangleみなす

2-string tangleの同値関係

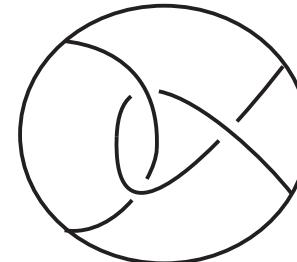
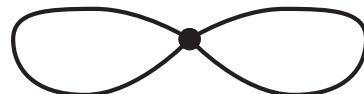


これらは同じ2-string tangleとみなす

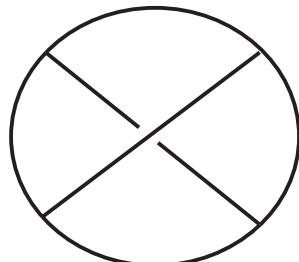
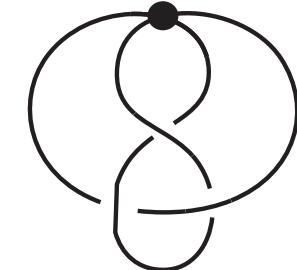
2-string tangleと2-bouquetの組



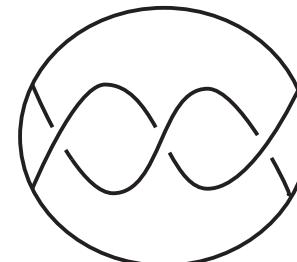
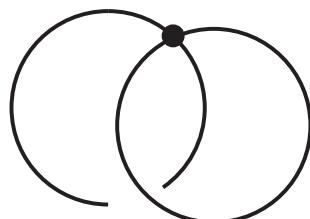
...



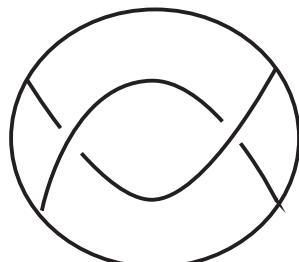
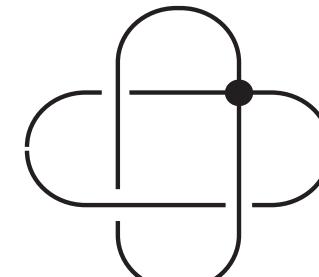
...



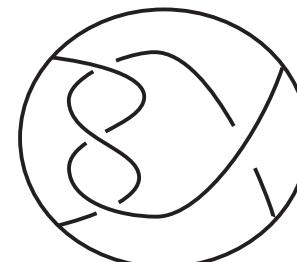
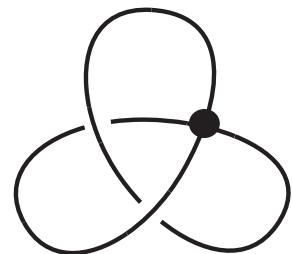
...



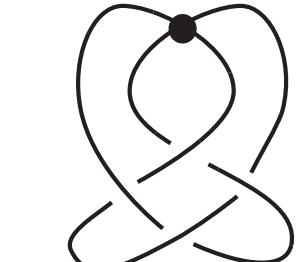
...



...

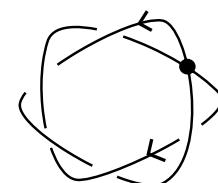
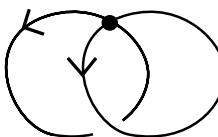
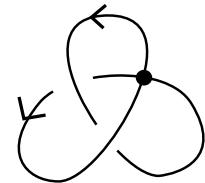
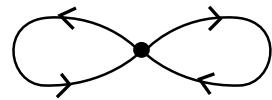


...

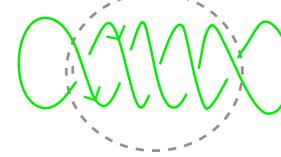
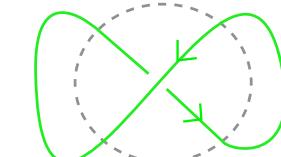
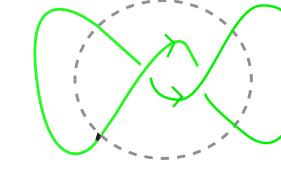
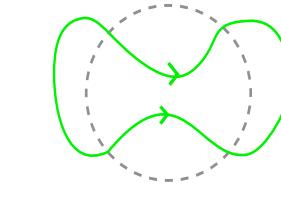
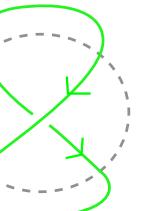
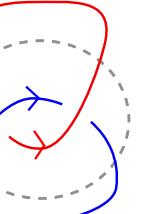
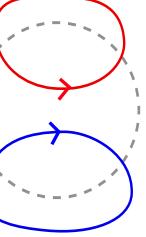
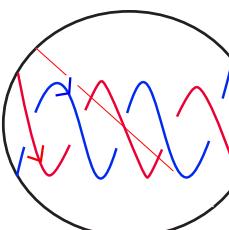
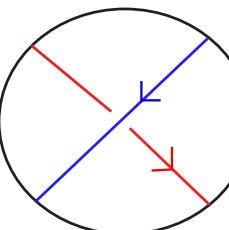
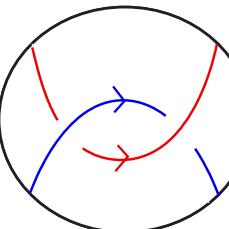
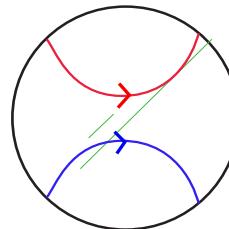
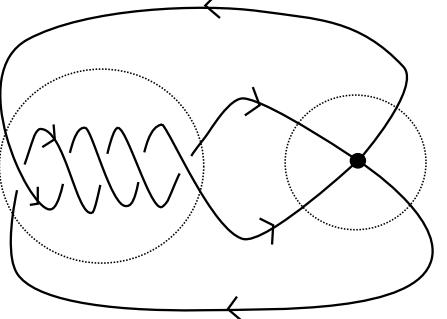
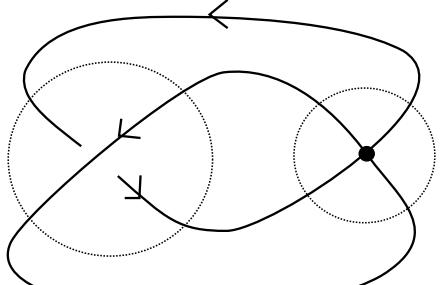
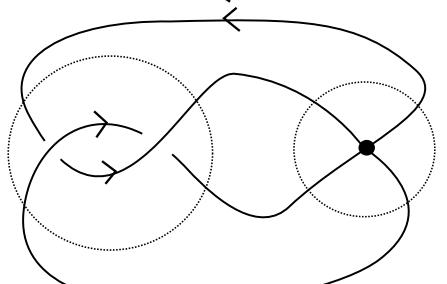
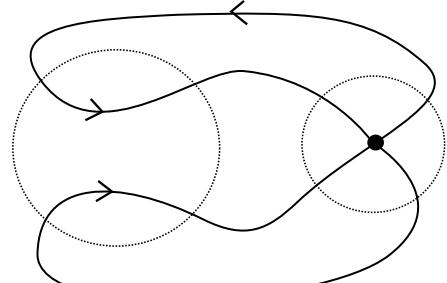


2-bouquet から得られる基点付き2-component link

2-bouquet



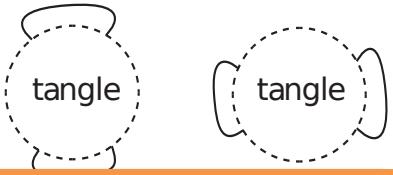
2-string tangle



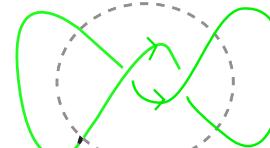
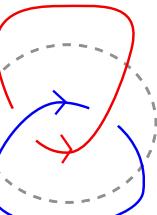
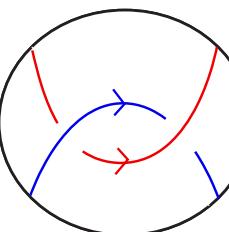
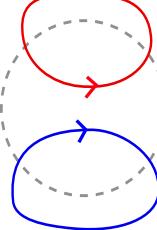
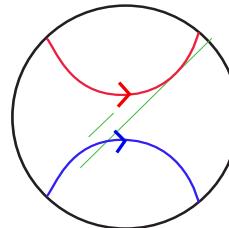
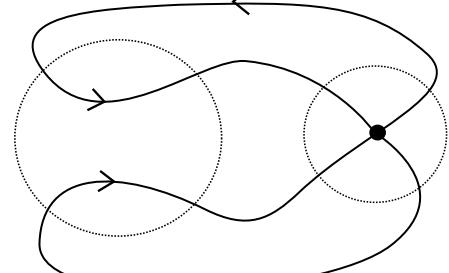
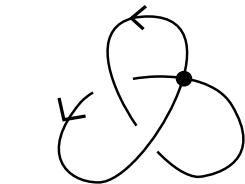
2-bouquet から得られる基点付き2-component link

2-bouquet

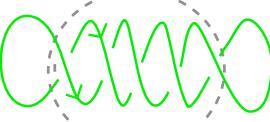
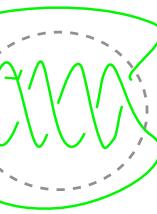
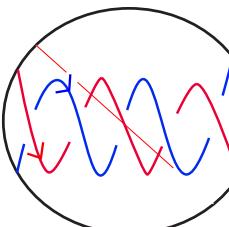
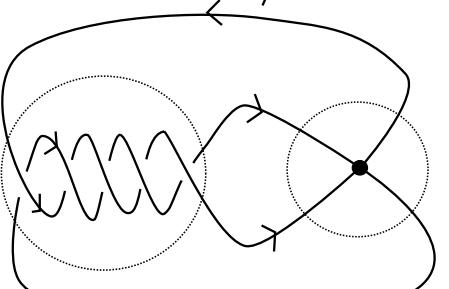
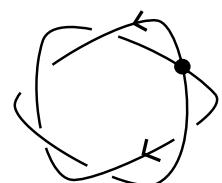
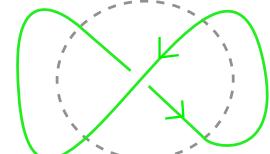
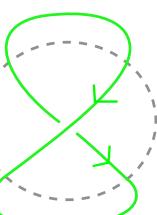
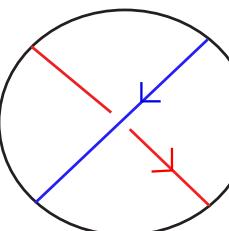
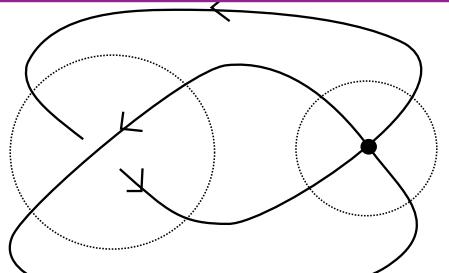
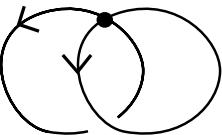
2-string tangle



Knot 型



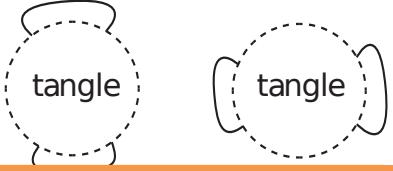
Link 型



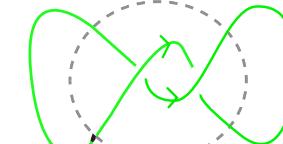
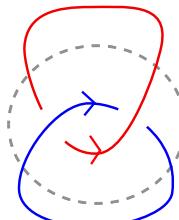
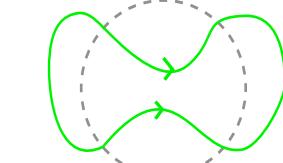
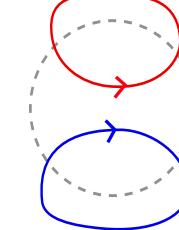
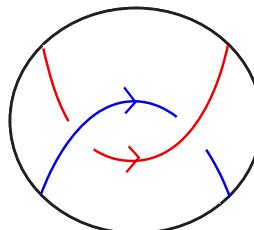
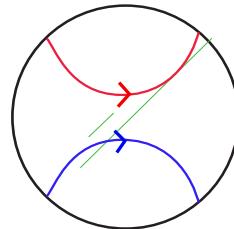
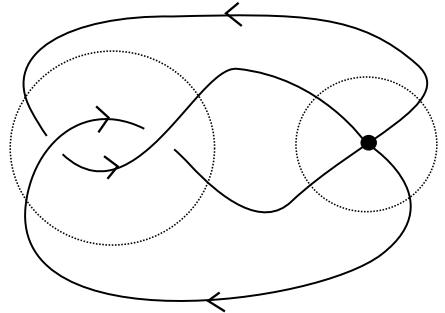
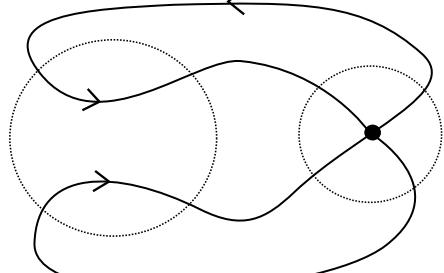
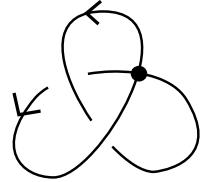
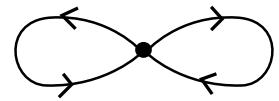
2-bouquet から得られる基点付き2-component link

2-bouquet

2-string tangle



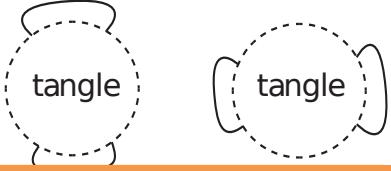
Knot 型



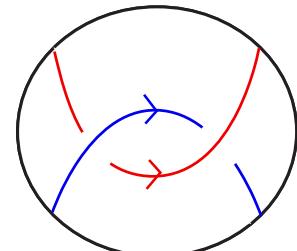
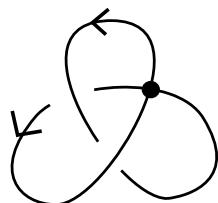
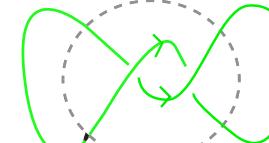
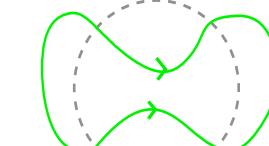
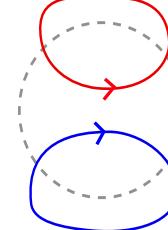
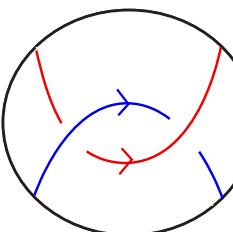
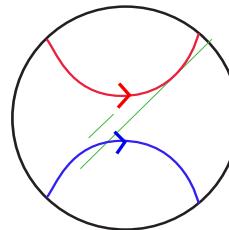
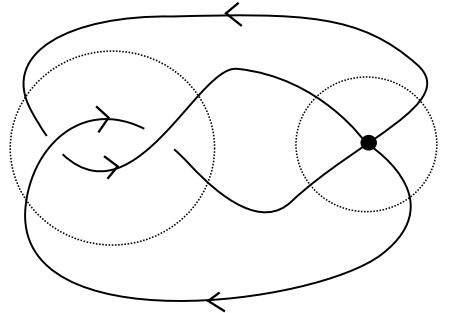
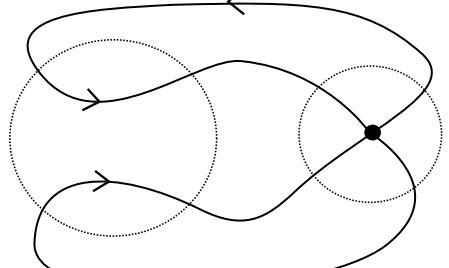
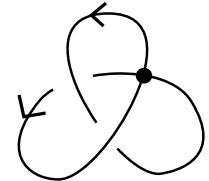
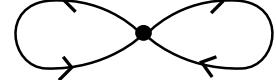
2-bouquet から得られる基点付き2-component link

2-bouquet

2-string tangle



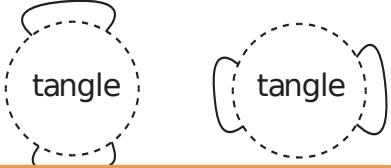
Knot 型



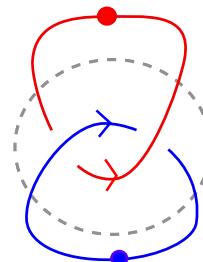
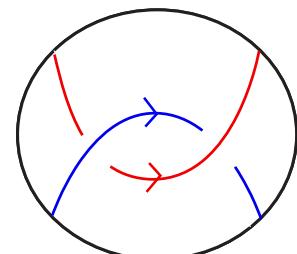
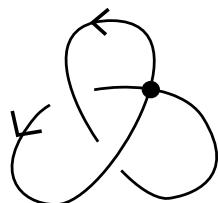
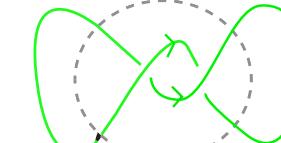
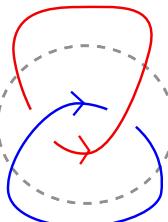
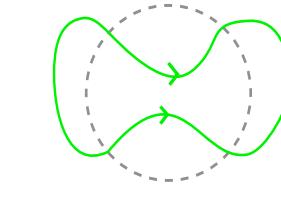
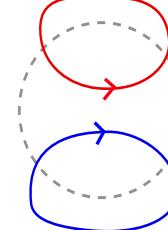
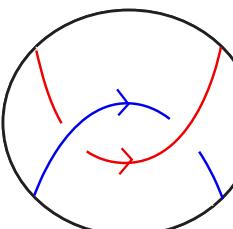
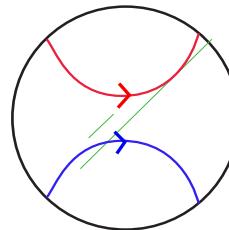
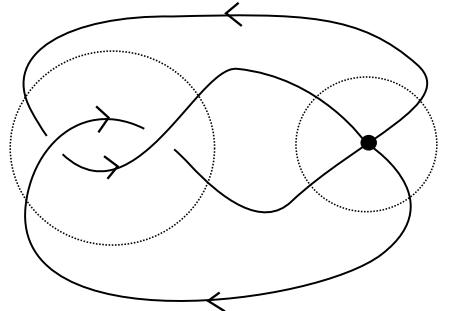
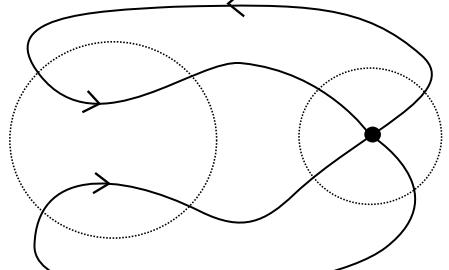
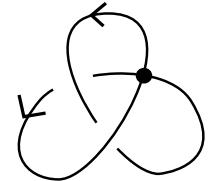
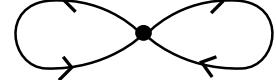
2-bouquet から得られる基点付き2-component link

2-bouquet

2-string tangle



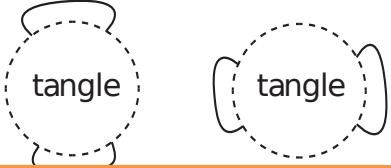
Knot 型



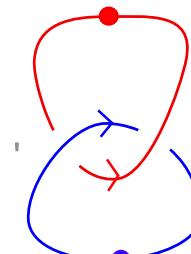
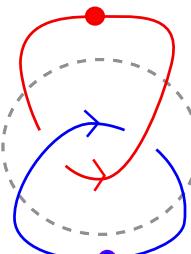
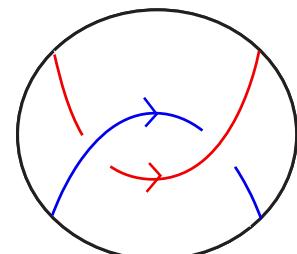
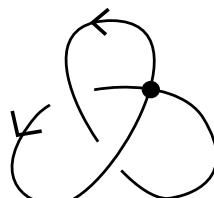
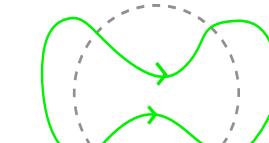
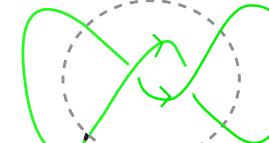
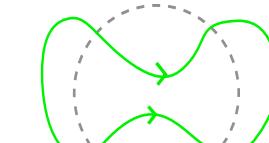
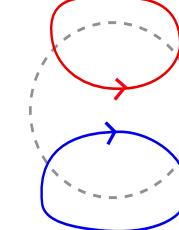
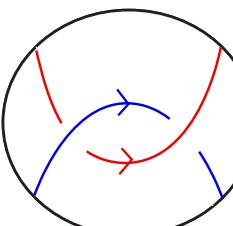
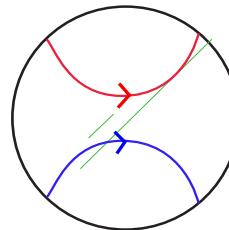
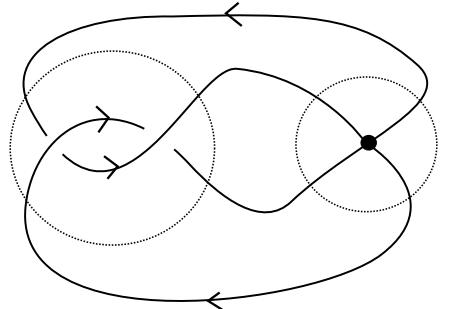
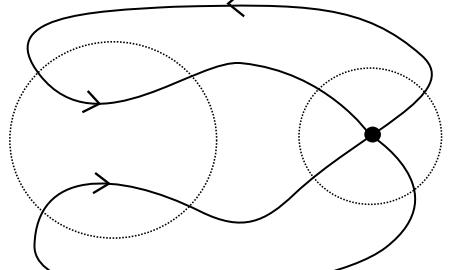
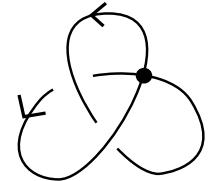
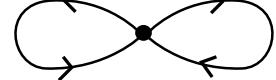
2-bouquet から得られる基点付き2-component link

2-bouquet

2-string tangle



Knot 型



Knot 型の2-bouquetからは、基点付きの2-component linkが一意に得られる

主定理

Theorem 1

Each of $\langle \text{Diagram } 1, \cdot \rangle$, $\langle \text{Diagram } 2, \cdot \rangle$, $\langle \text{Diagram } 3, \cdot \rangle$, and $\langle \text{Diagram } 4, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 3 of two-component links.

As a corollary, each of them is also an invariant of order 3 of spatial graphs in \mathbb{R}^3 .

Theorem 2

Each of $\langle \text{Diagram } 1, \cdot \rangle$, $\langle \text{Diagram } 2, \cdot \rangle$, and $\langle \text{Diagram } 3, \cdot \rangle - \frac{1}{3}\langle \text{Diagram } 4, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 3 of two-component links (, and of order 3 of spatial graphs).

Corollary (Östlund-Polyak-Viro formula)

Östlund-Polyak-Viro function $\langle T, \cdot \rangle$, which is

$$\langle \text{Diagram } 1, \cdot \rangle + \langle \text{Diagram } 2, \cdot \rangle + \langle \text{Diagram } 3, \cdot \rangle - \frac{1}{3}\langle \text{Diagram } 4, \cdot \rangle,$$

becomes a link invariant of order 3 of two-component links.

主定理

Theorem 3

\exists an infinitely many pairs (i, j) of 2-component links L_i, L_j ($i \neq j$) s. t.

$$\langle \text{Diagram } 1, L_i \rangle \neq \langle \text{Diagram } 1, L_j \rangle, \langle \text{Diagram } 2, L_i \rangle \neq \langle \text{Diagram } 2, L_j \rangle,$$

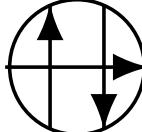
$$\langle \text{Diagram } 3, L_i \rangle \neq \langle \text{Diagram } 3, L_j \rangle, \text{ and } \langle \text{Diagram } 4, L_i \rangle \neq \langle \text{Diagram } 4, L_j \rangle$$

by our invariants, whereas $\forall i, j, \langle T, L_i \rangle = \langle T, L_j \rangle$ on Östlund-Polyak-Viro formula $\langle T, \cdot \rangle$ as in Corollary.

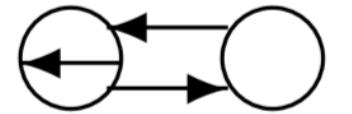
既知の式 (a_2 :Conwayの2次)

	deg 2
knot	a_2
θ (金信)	$-5 + 24 \sum_{i=1}^3 a_2(K_i(T)).$ (半谷千春が正規化 (2008))
H-C	[Kanenobu-Sugita (2012)]
Bouquet	無

既知の式 ($V^{(3)}$:Jonesの3次)

	$\deg 3$
knot	 + 2 
θ (金信)	$-132 \sum_{i=1}^3 a_2(K_i(T)) + \frac{22}{3} \sum_{i=1}^3 V^{(3)}(K_i(T); 1)$ $-144a_3(S)$ (吉田英生 2014). (cf. [Ishii-Kishimoto])
H-C	cf. [Kanenobu-Sugita (2012)]
Bouquet	<p>[講演者ら] $\langle \begin{array}{c} \oplus \\[-1ex] \leftarrow \end{array} \rightarrow \circ, \cdot \rangle, \langle \begin{array}{c} \oplus \\[-1ex] \rightarrow \end{array} \leftarrow \circ, \cdot \rangle,$</p> <p>$\langle \begin{array}{c} \ominus \\[-1ex] \leftarrow \end{array} \rightarrow \circ, \cdot \rangle, \langle \begin{array}{c} \ominus \\[-1ex] \rightarrow \end{array} \leftarrow \circ, \cdot \rangle,$</p> <p>$\langle \begin{array}{c} \circ \\[-1ex] \leftarrow \end{array} \rightarrow \circ, \cdot \rangle - \frac{1}{3} \langle \begin{array}{c} \circ \\[-1ex] \rightarrow \end{array} \leftarrow \circ, \cdot \rangle.$</p>

証明 : Theorem 2

 Ω_{II+-}

Theorem 1

Each of $\langle \text{Diagram } 1, \cdot \rangle$, $\langle \text{Diagram } 2, \cdot \rangle$, $\langle \text{Diagram } 3, \cdot \rangle$, and $\langle \text{Diagram } 4, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 3 of two-component links.

As a corollary, each of them is also an invariant of order two of spatial graphs in \mathbb{R}^3 up to flat vertex isotopy.

Theorem 2

Each of $\langle \text{Diagram } 1, \cdot \rangle$, $\langle \text{Diagram } 2, \cdot \rangle$, and $\langle \text{Diagram } 3, \cdot \rangle - \frac{1}{3}\langle \text{Diagram } 4, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 3 of two-component links (, and of order 3 of spatial graphs).

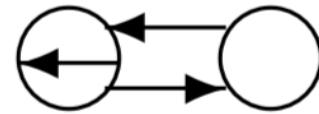
Corollary (Östlund-Polyak-Viro formula)

Östlund-Polyak-Viro function $\langle T, \cdot \rangle$, which is

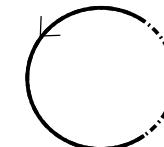
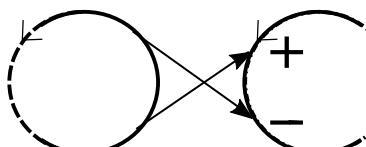
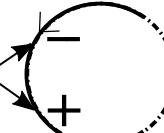
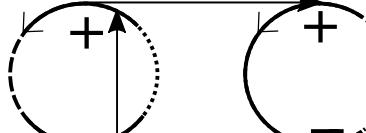
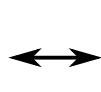
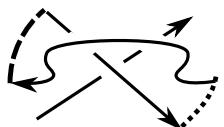
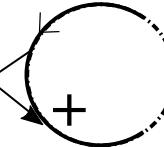
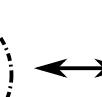
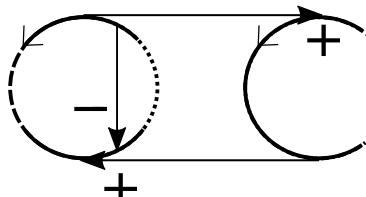
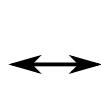
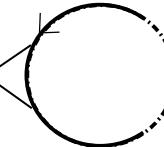
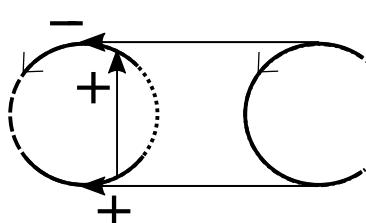
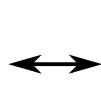
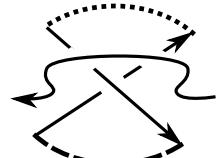
$$\langle \text{Diagram } 1, \cdot \rangle + \langle \text{Diagram } 2, \cdot \rangle + \langle \text{Diagram } 3, \cdot \rangle - \frac{1}{3}\langle \text{Diagram } 4, \cdot \rangle,$$

becomes a link invariant of order 3 of two-component links.

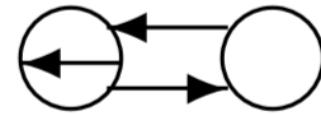
証明 : Theorem 2


 Ω_{II+-}

2-components link の Reidemeister moves

 Ω_{II+-}

 $\Omega_{III++-b}$

 $\Omega_{III++-m}$

 $\Omega_{III++-t}$


証明 : Theorem 2

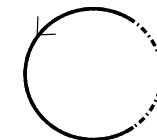
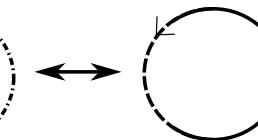
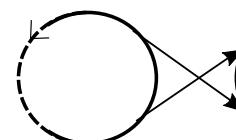
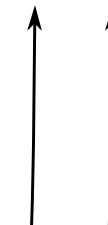


Ω_{II+-}

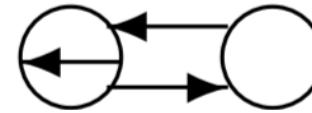
定義から,

$$\begin{aligned} & \langle \text{Diagram } 1, D_l^{\Omega_{II+-}} \rangle - \langle \text{Diagram } 2, D_r^{\Omega_{II+-}} \rangle \\ &= \sum_{z^{(l)} \in \text{Sub}(D_l^{\Omega_{II+-}})} \text{sign}(z^{(l)}) (\text{Diagram } 1, z^{(l)}) - \sum_{z^{(r)} \in \text{Sub}(D_r^{\Omega_{II+-}})} \text{sign}(z^{(r)}) (\text{Diagram } 2, z^{(r)}). \end{aligned}$$

Ω_{II+-}



証明 : Theorem 2



Ω_{II+-}

このとき, Reidemeister II で増える “2” 本に注目し, $z_2^{(l)}$ が, まさにその 2 本を含む sub diagram とする.

$z_1^{(l)}, z_0^{(l)}$ も同じように記号を定義する.

すると,

$$LHS = \sum_{i=0}^2 \sum_{z_i^{(l)} \in \text{Sub}(D_l^{\Omega_{II+-}})} \text{sign}(z_i^{(l)}) (\text{Diagram}, z_i^{(l)}) - \sum_{z^{(r)} \in \text{Sub}(D_r^{\Omega_{II+-}})} \text{sign}(z^{(r)}) (\text{Diagram}, z^{(r)}).$$

定義から $z_0^{(l)}$ は当然 $z^{(r)}$ と同型.

言い換えれば,

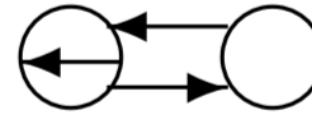
$$\sum_{z_0^{(l)} \in \text{Sub}(D_l^{\Omega_{II+-}})} \text{sign}(z_0^{(l)}) (\text{Diagram}, z_0^{(l)}) = \sum_{z^{(r)} \in \text{Sub}(D_r^{\Omega_{II+-}})} \text{sign}(z^{(r)}) (\text{Diagram}, z^{(r)}).$$

(もちろん, これは一般でいえる.)

証明 : Theorem 2

Reidemeister II で増減する “2” 本の関係は符号が $+, -$ だったから、

証明 : Theorem 2



Ω_{II+-}

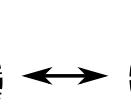
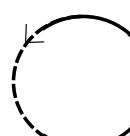
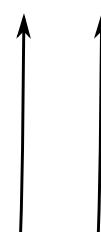
$$\sum_{z_2^{(l)} \in \text{Sub}\left(D_l^{\Omega_{II}+-}\right)} \text{sign}(z_2^{(l)})\left(\begin{smallmatrix} \leftarrow & \rightarrow \\ \oplus & \ominus \end{smallmatrix}, z_2^{(l)}\right) = 0.$$

まとめると、与式の RII の差は

$$\sum_{i=0}^2 \sum_{z_i^{(l)} \in \text{Sub}\left(D_l^{\Omega_{II}+-}\right)} \text{sign}(z_i^{(l)})\left(\begin{smallmatrix} \leftarrow & \rightarrow \\ \oplus & \ominus \end{smallmatrix}, z_i^{(l)}\right) - \sum_{z^{(r)} \in \text{Sub}\left(D_r^{\Omega_{II}+-}\right)} \text{sign}(z^{(r)})\left(\begin{smallmatrix} \leftarrow & \rightarrow \\ \oplus & \ominus \end{smallmatrix}, z^{(r)}\right) = 0.$$

よって RII の不变性を得る (RIII も同様).

Ω_{II+-}



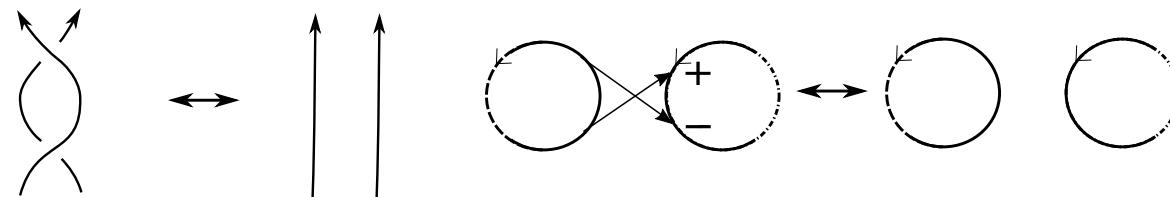
証明

同様にして、

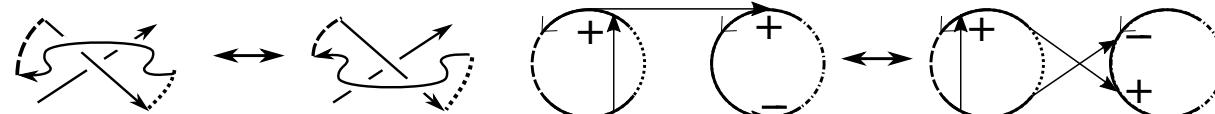
$$\langle \text{○} \xrightarrow{\leftarrow} \text{○}, \cdot \rangle, \langle \text{○} \xrightarrow{\rightarrow} \text{○}, \cdot \rangle, \langle \text{○} \xleftarrow{\leftarrow} \text{○} \xrightarrow{\rightarrow} \text{○}, \cdot \rangle, \langle \text{○} \xleftarrow{\leftarrow} \text{○} \xleftarrow{\rightarrow} \text{○}, \cdot \rangle,$$

$$\langle \text{○} \xrightarrow{\leftarrow} \text{○} \xrightarrow{\rightarrow} \text{○}, \cdot \rangle - \frac{1}{3} \langle \text{○} \xrightarrow{\rightarrow} \text{○} \xrightarrow{\leftarrow} \text{○}, \cdot \rangle$$

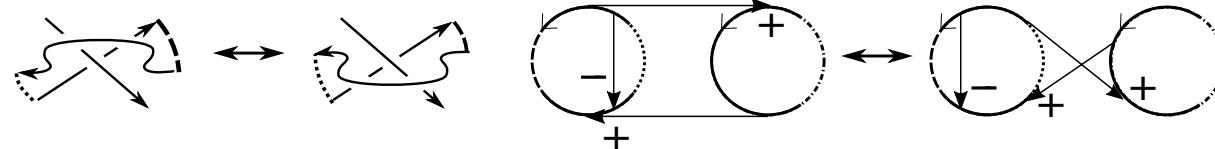
$$\Omega_{II+-}$$



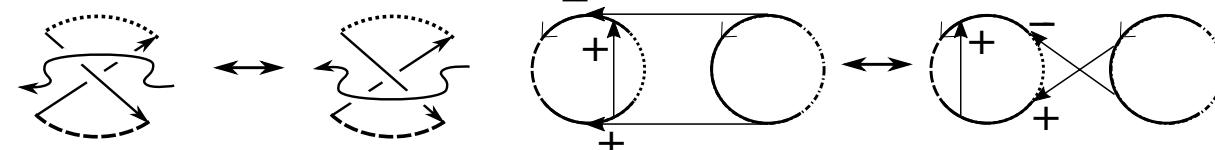
$$\Omega_{III+-+b}$$



$$\Omega_{III+-+m}$$



$$\Omega_{III+-+t}$$



Theorem 3 の例

Theorem 3

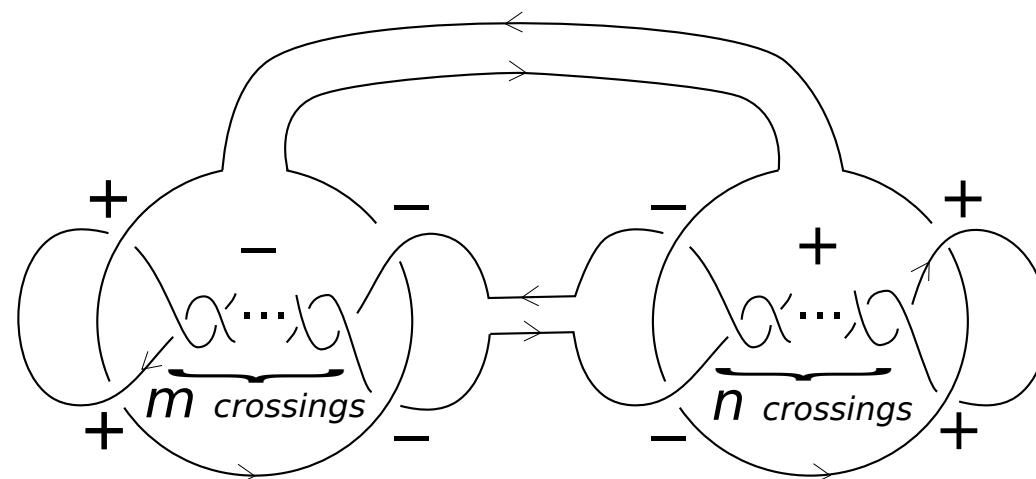
\exists an infinitely many pairs (i, j) of 2-component links L_i, L_j ($i \neq j$) s. t.

$$\langle \text{Diagram } 1, L_i \rangle \neq \langle \text{Diagram } 1, L_j \rangle, \langle \text{Diagram } 2, L_i \rangle \neq \langle \text{Diagram } 2, L_j \rangle,$$

$$\langle \text{Diagram } 3, L_i \rangle \neq \langle \text{Diagram } 3, L_j \rangle, \text{ and } \langle \text{Diagram } 4, L_i \rangle \neq \langle \text{Diagram } 4, L_j \rangle$$

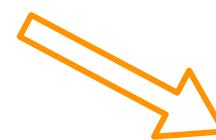
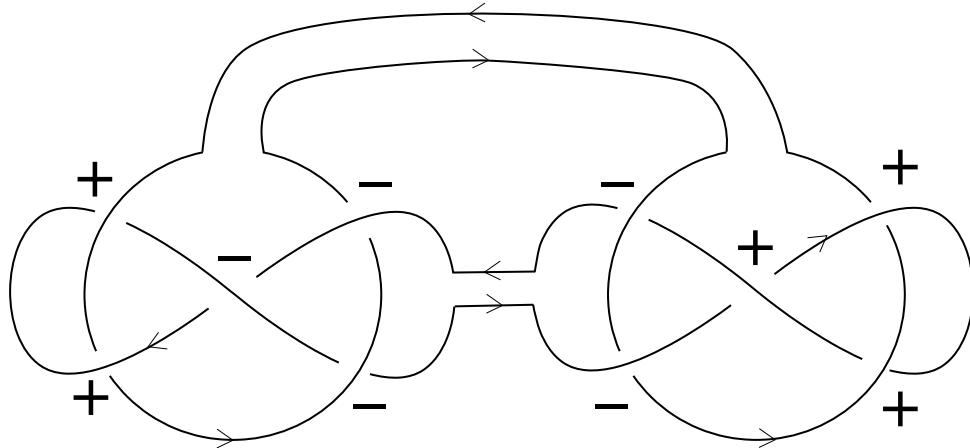
by our invariants, whereas $\forall i, j, \langle T, L_i \rangle = \langle T, L_j \rangle$ on Östlund-Polyak-Viro formula $\langle T, \cdot \rangle$ as in Corollary.

m+n+8交点の2成分絡み目 (m, n: 奇数、 m=n)

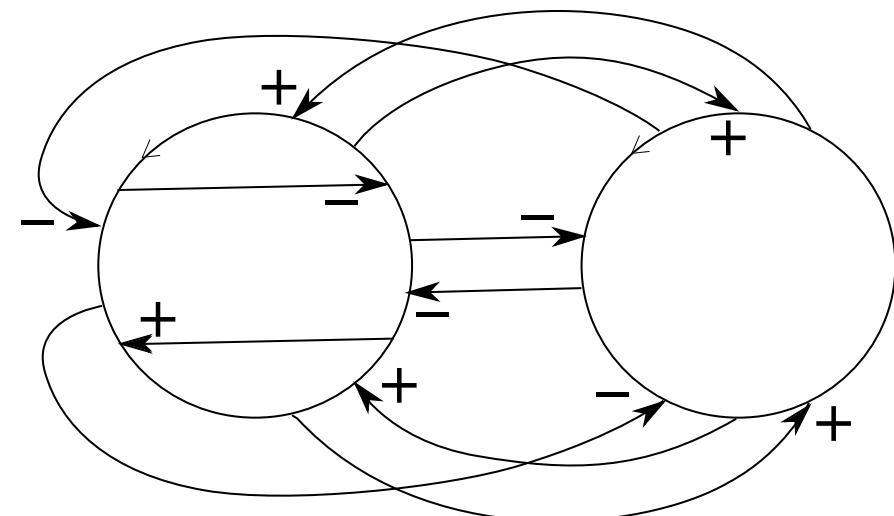


例 : $m=1$, $n=1$

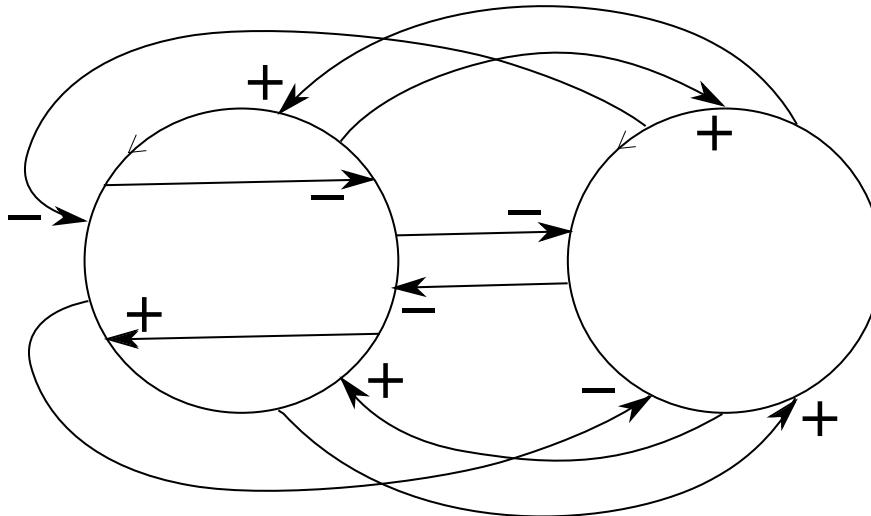
2-component link



arrow diagram



例 : $m=1, n=1$



Theorem 1

Each of $\langle \text{---} \rightarrow \circlearrowleft, \cdot \rangle$, $\langle \text{---} \rightarrow \circlearrowright, \cdot \rangle$, $\langle \text{---} \leftarrow \circlearrowleft, \cdot \rangle$, and $\langle \text{---} \leftarrow \circlearrowright, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 2 of two-component links.

As a corollary, each of them is also an invariant of order two of spatial graphs in \mathbb{R}^3 up to flat vertex isotopy.

Theorem 2

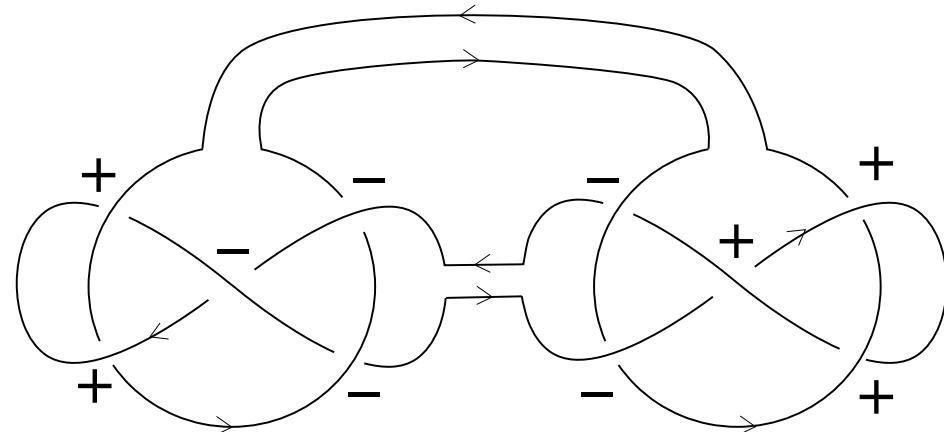
Each of $\langle \text{---} \rightarrow \circlearrowleft, \cdot \rangle$, $\langle \text{---} \rightarrow \circlearrowright, \cdot \rangle$, and $\langle \text{---} \leftarrow \circlearrowleft, \cdot \rangle - \frac{1}{3} \langle \text{---} \leftarrow \circlearrowright, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 2 of two-component links (, and of order 2 of spatial graphs).

Corollary (Östlund-Polyak-Viro formula)

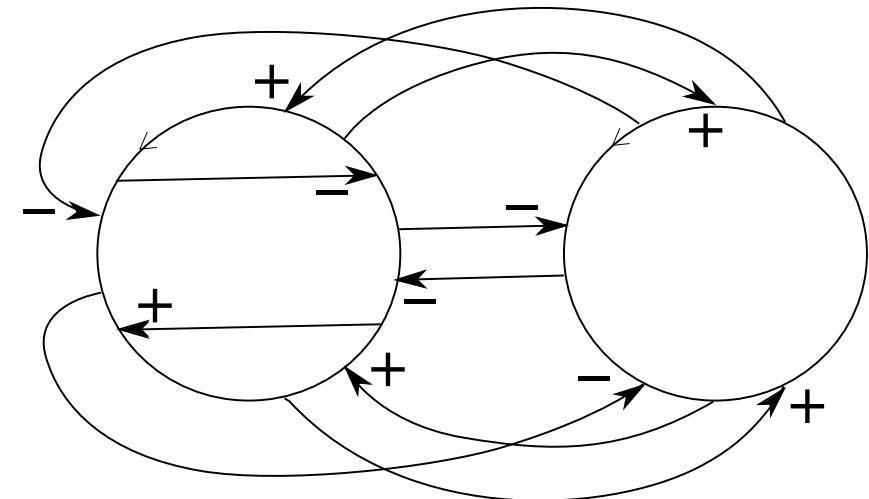
Östlund-Polyak-Viro function $\langle T, \cdot \rangle$, which is $\langle \text{---} \rightarrow \circlearrowleft, \cdot \rangle + \langle \text{---} \rightarrow \circlearrowright, \cdot \rangle + \langle \text{---} \leftarrow \circlearrowleft, \cdot \rangle - \frac{1}{3} \langle \text{---} \leftarrow \circlearrowright, \cdot \rangle$, becomes a link invariant of order 3 of two-component links.

例 : $m=1, n=1$

2-component link



arrow diagram

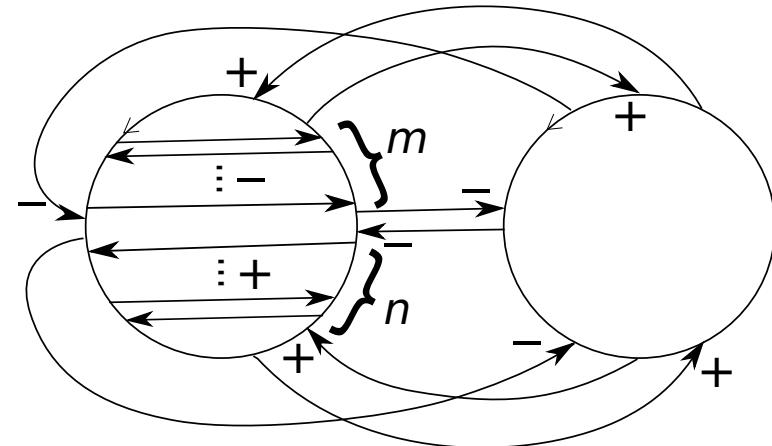
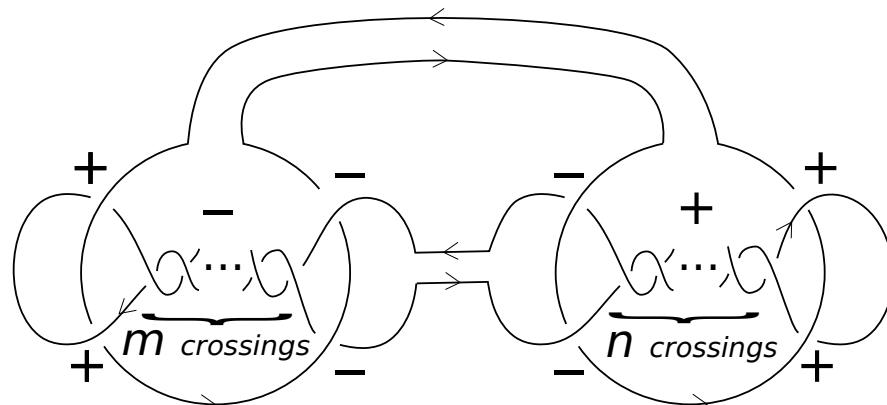


Invariants of Theorem 1				
Values of $L(1, 1)$	-1	-1	1	1

Invariants of Theorem 2			
Values of $L(1, 1)$	$1 - 1 = 0$	$1 - 1 = 0$	0

Östlund-Polyak-Viro formula	+ + - $\frac{1}{3}$
Values of $L(1, 1)$	$2(1 - 1) = 0$

例： $m+n+8$ 交点の2成分絡み目 (m, n : 奇数、 $m=n$)



Theorem 1

Each of $\langle \text{---} \circlearrowleft \text{---}, \cdot \rangle$, $\langle \text{---} \circlearrowright \text{---}, \cdot \rangle$, $\langle \text{---} \circlearrowleft \text{---}, \cdot \rangle$, and $\langle \text{---} \circlearrowright \text{---}, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 2 of two-component links.

As a corollary, each of them is also an invariant of order two of spatial graphs in \mathbb{R}^3 up to flat vertex isotopy.

Theorem 2

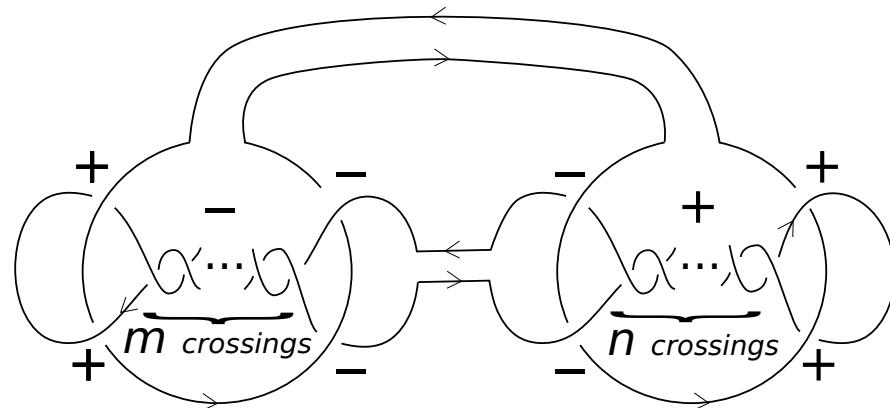
Each of $\langle \text{---} \circlearrowleft \text{---}, \cdot \rangle$, $\langle \text{---} \circlearrowright \text{---}, \cdot \rangle$, and $\langle \text{---} \circlearrowleft \text{---}, \cdot \rangle - \frac{1}{3} \langle \text{---} \circlearrowright \text{---}, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 2 of two-component links (, and of order 2 of spatial graphs).

Corollary (Östlund-Polyak-Viro formula)

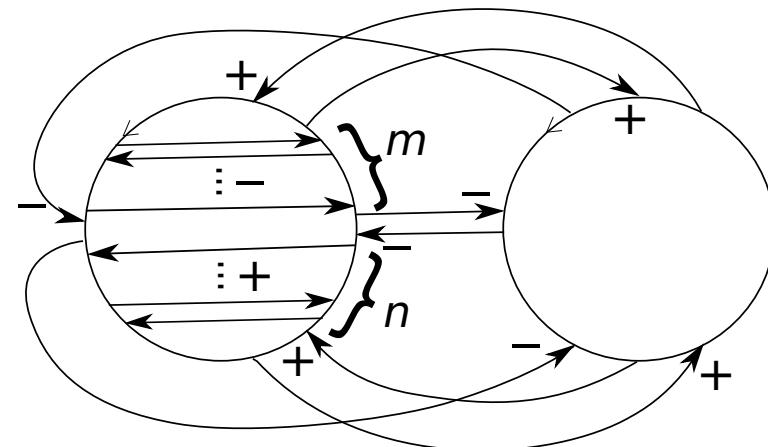
Östlund-Polyak-Viro function $\langle T, \cdot \rangle$, which is $\langle \text{---} \circlearrowleft \text{---}, \cdot \rangle + \langle \text{---} \circlearrowright \text{---}, \cdot \rangle + \langle \text{---} \circlearrowleft \text{---}, \cdot \rangle - \frac{1}{3} \langle \text{---} \circlearrowright \text{---}, \cdot \rangle$, becomes a link invariant of order 3 of two-component links.

例： $m+n+8$ 交点の2成分絡み目 (m, n : 奇数、 $m=n$)

2-component link



arrow diagram



Invariants of Theorem 1	$\oplus \leftarrow \rightarrow$	$\oplus \leftarrow \rightarrow$	$\ominus \leftarrow \rightarrow$	$\ominus \leftarrow \rightarrow$
Values of $L(m, n)$	$-n$	$-n$	m	m

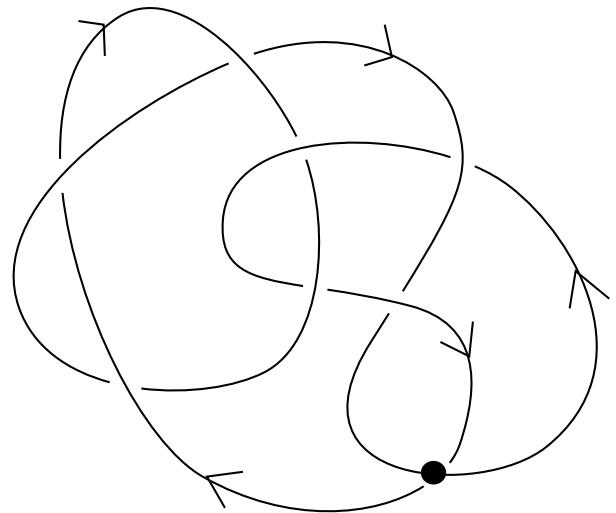
Invariants of Theorem 2	$\ominus \leftarrow \rightarrow$	$\leftarrow \leftarrow \rightarrow$	$\ominus \leftarrow \rightarrow - \frac{1}{3} \ominus \leftarrow \rightarrow$
Values of $L(m, n)$	$m - n$	$m - n$	0

Östlund-Polyak-Viro formula	$\ominus \leftarrow \rightarrow + \ominus \leftarrow \rightarrow + \ominus \leftarrow \rightarrow - \frac{1}{3} \ominus \leftarrow \rightarrow$
Values of $L(m, n)$	$2(m - n)$

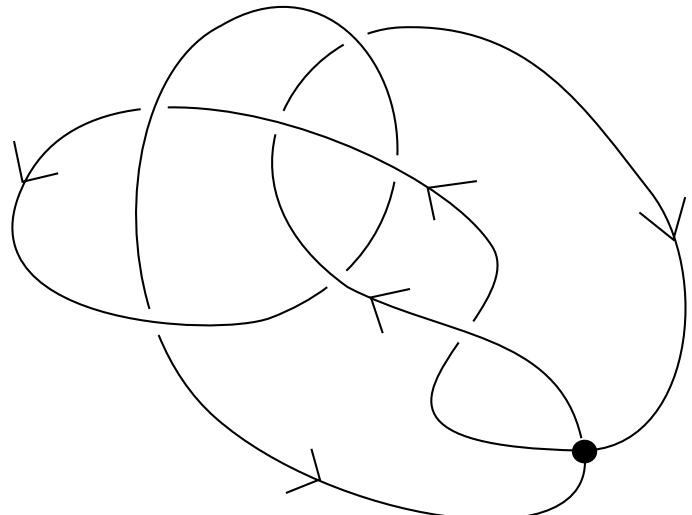
例：7交点 2-bouquet

2-bouquet

G_1



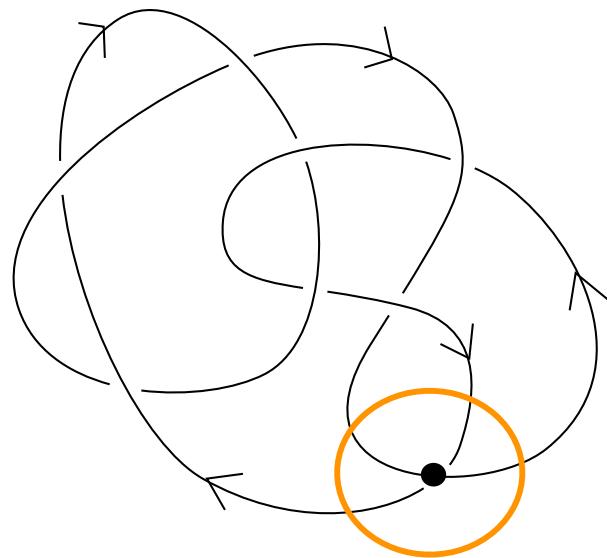
G_2



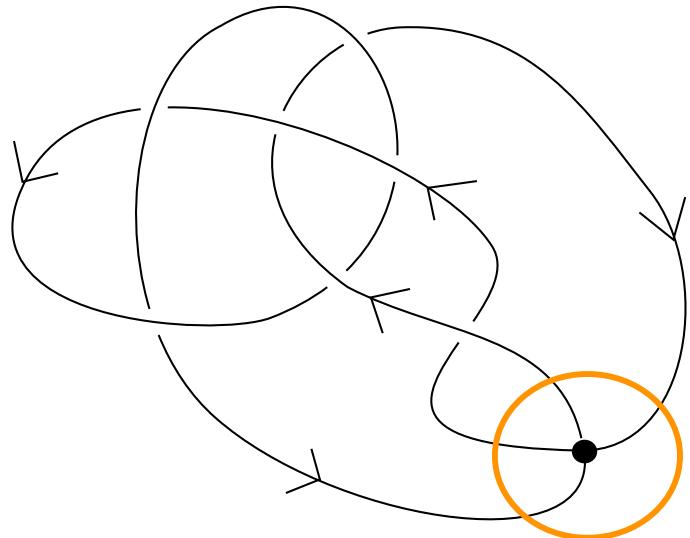
例：7交点 2-bouquet

2-bouquet

G_1

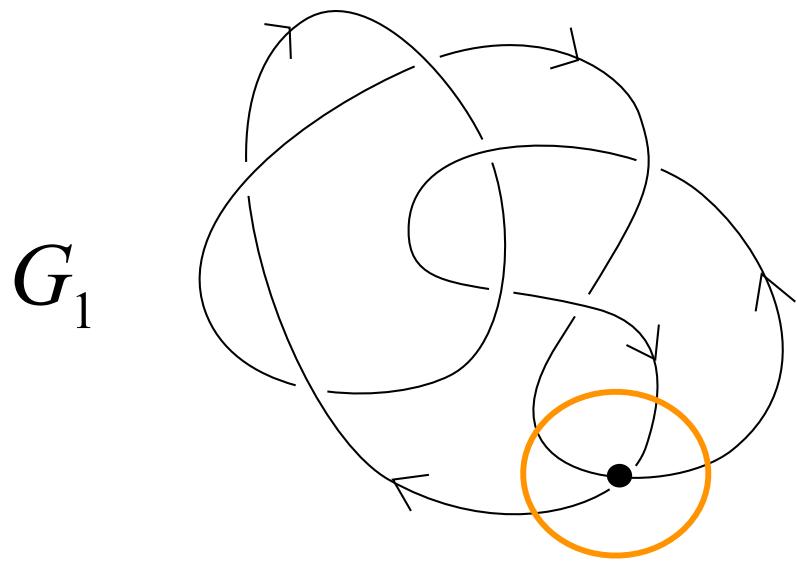


G_2

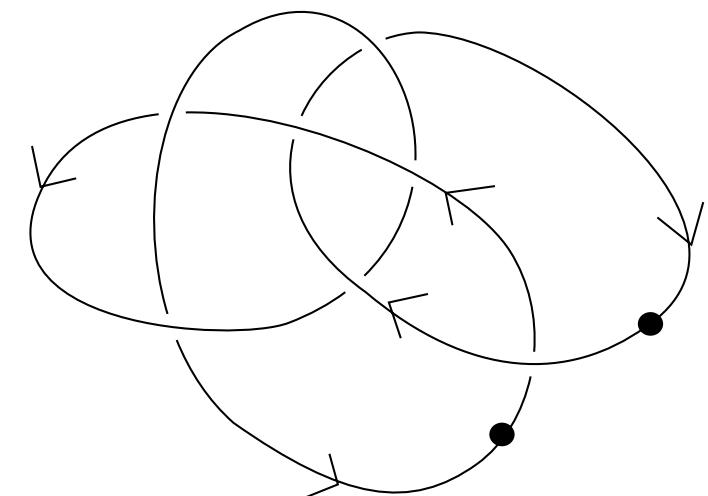
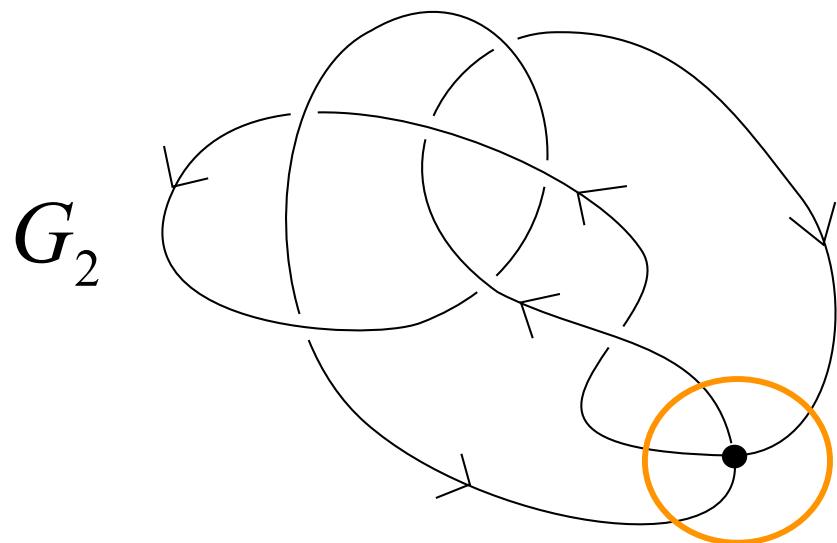
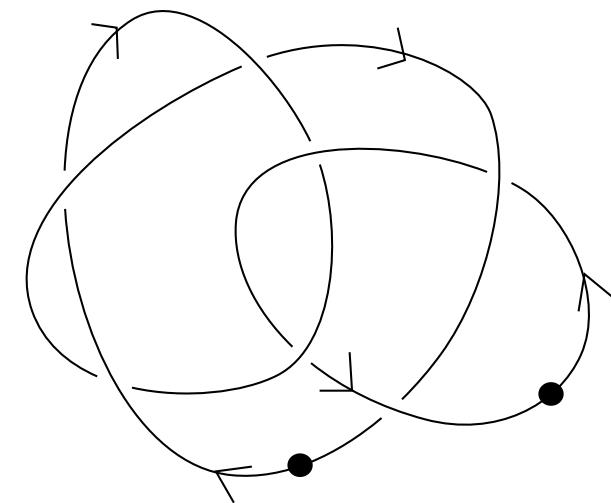


例：7交点 2-bouquet

2-bouquet



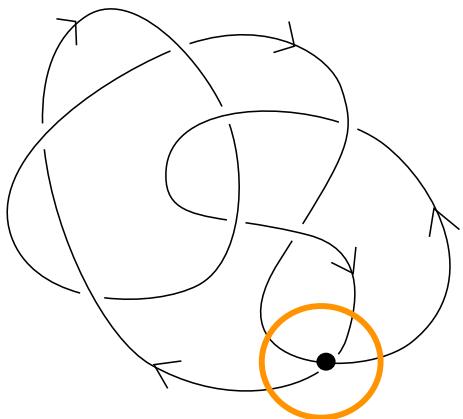
2-component link



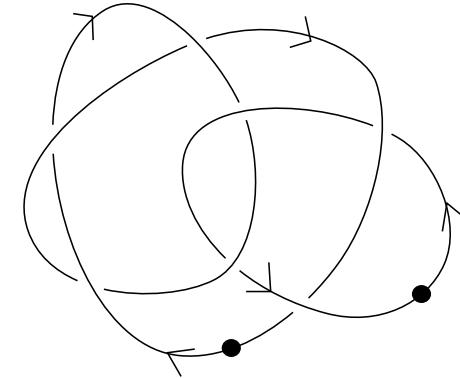
例：7交点 2-bouquet

2-bouquet

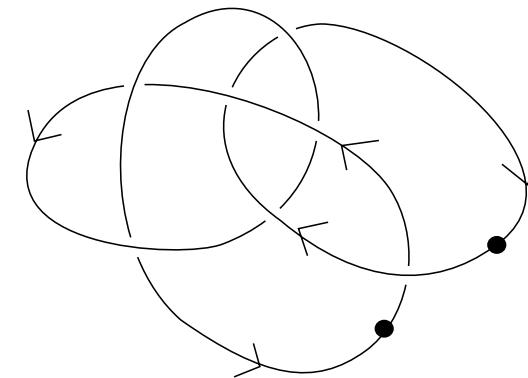
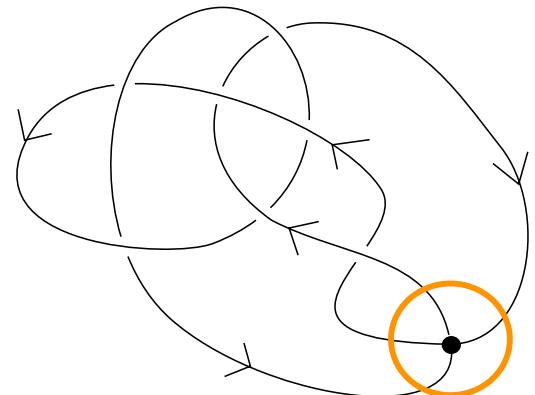
G_1



2-component link



G_2



	Invariants of Theorem 1				Invariants of Theorem 2			Östlund-Polyak-Viro formula
2-bouquet								
G_1	-3	-3	0	0	-3	-3	-2	$-3 - 3 - 2 = -8$
G_2	0	0	-3	-3	-3	-3	-2	$-3 - 3 - 2 = -8$

参考文献

- [1] S. Barthel and D. Buck, Toloidal embeddings of abstractly planar graphs are knotted or linked, *J. Math. Chem.* **53** (2015), 1772–1790.
- [2] T. Deguchi, On numerical applications of Vassiliev invariants to computational problems in physics, *Proceedings of the Conference on Quantum Topology*, 87–98, *World Sci. Publ., River Edge, NJ*, 1994.
- [3] T. Deguchi and K. Tsurusaki, A statistical study of random knotting using the Vassiliev invariants, *J. Knot Theory Ramifications* **3** (1994), 321–353.
- [4] M. Goussarov, M. Polyak, and O. Viro, Finite-type invariants of classical and virtual knots, *Topology* **39** (2000), 1045–1068.
- [5] N. Ito, Space of chord diagrams on spherical curves, *Internat. J. Math.* **30** (2019), 1950060, 25pp.
- [6] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (2) **126** (1987), 335–388.
- [7] H. Murakami, On the derivatives of the Jones polynomial, *Kobe J. Math.* **3** (1986), 61–64.
- [8] N. Oyamaguchi, Enumeration of spatial 2-bouquet graphs up to flat vertex isotopy, *Topology Appl.* **196** (2015), part B, 805–814.
- [9] Östlund, A diagrammatic approach to link invariants of finite degree, *Math. Scand.* **94** (2004), 295–319.
- [10] M. Polyak, Minimal generating sets of Reidemeister moves, *Quantum Topol.* **1** (2010), 399–411.
- [11] M. Polyak and O. Viro, Gauss diagram formulas for Vassiliev invariants, *Internat. Math. Res. Notices* **1994**, 445ff., approx. 8 pp.
- [12] H. Heguri, T. Yamamoto, and Y. Tezuka, Folding construction of a pentacyclic quadruply fused polymer topology with tailed *kyklo*-telechelic precursors, *Angewandte Chemie International Edition* **2015**, 54 (30), 8688–8692.
- [13] N. Sugai, H. Heguri, T. Yamamoto, and Y. Tezuka, A programmed polymer Folding: *Click and Clip* construction of doubly fused tricyclic and triply fused tetracyclic polymer topologies, *J. Am. Chem. Soc.*, 2011, 133, 49, 19694–19697.
- [14] E. Uehara and T. Deguchi, *J. Chem. Phys.* **147**, 094901 (2017).

ご清聴ありがとうございました

Theorem 1

Each of $\langle \text{Diagram 1}, \cdot \rangle$, $\langle \text{Diagram 2}, \cdot \rangle$, $\langle \text{Diagram 3}, \cdot \rangle$, and $\langle \text{Diagram 4}, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 2 of two-component links.

As a corollary, each of them is also an invariant of order two of spatial graphs in \mathbb{R}^3 up to flat vertex isotopy.

Theorem 2

Each of $\langle \text{Diagram 1}, \cdot \rangle$, $\langle \text{Diagram 2}, \cdot \rangle$, and $\langle \text{Diagram 3}, \cdot \rangle - \frac{1}{3}\langle \text{Diagram 4}, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order 2 of two-component links (, and of order 2 of spatial graphs).

Corollary (Östlund-Polyak-Viro formula)

Östlund-Polyak-Viro function $\langle T, \cdot \rangle$, which is $\langle \text{Diagram 1}, \cdot \rangle + \langle \text{Diagram 2}, \cdot \rangle + \langle \text{Diagram 3}, \cdot \rangle - \frac{1}{3}\langle \text{Diagram 4}, \cdot \rangle$, becomes a link invariant of order 3 of two-component links.