Unknotting operations, crosscap numbers, and volume bounds

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MSCS Quantum Topology Seminar
organized by L.H.Kauffman
August 20, 2020

PDF Information

non-arXiv, I will mention the reason today.

- A lower bound of crosscap numbers of alternating knots, JKTR, March 2020
- Crosscap number of knots and volume bounds, submitted to IJM, revised under review.

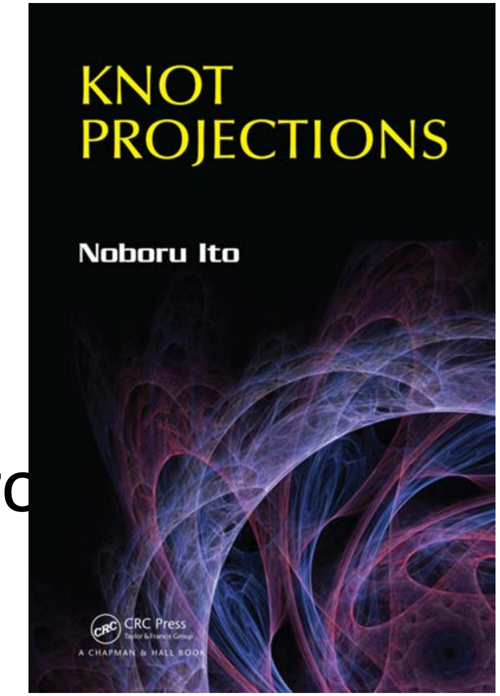
To access them,

https://researchmap.jp/noboru_ito?lang=en

(from JST researchmap, my portal)

Knot Projections (by NI, 2016/12)

This book includes Perko’s pair, introductory contents quoting from Kauffman Lecture note and results of papers Takimura-I. from 2013.



Mr. Perko gave “5 stars” for this book,

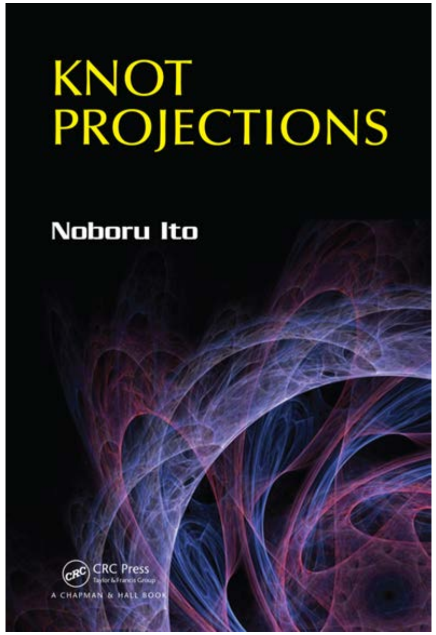
also positive opinions for the paper of

“Thirty-two equivalence relations on knot pro

(2015)”, but commented

—“mostly not Knot theory” (Zbl 06720087).

Knot Projections (by NI, 2016/12)

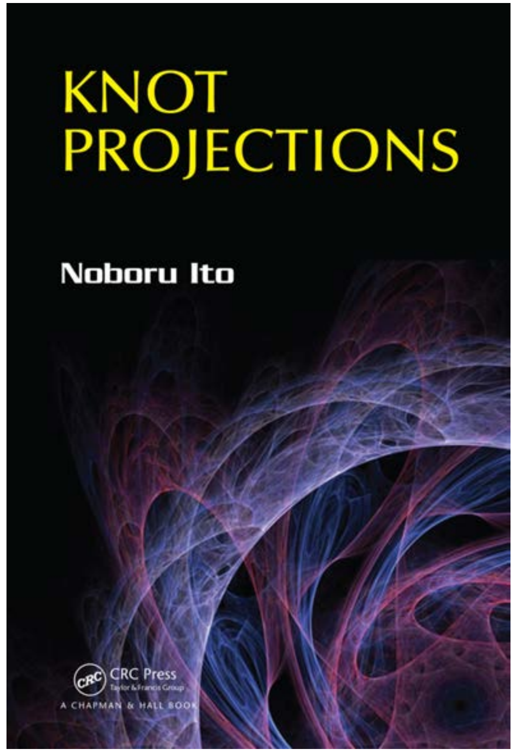


I agreed his opinions and replied to his message.

NI to Perko (July 25, 2017):

"I think I can write new application to knot from our knot projection theory. Please be looking forward to it."

Knot Projections (NI, 2016)



Prof. Kauffman gave a comment.

—"But the point of doing mathematics is follow the trail as far as you can. He is doing that."

→ This message encouraged me.

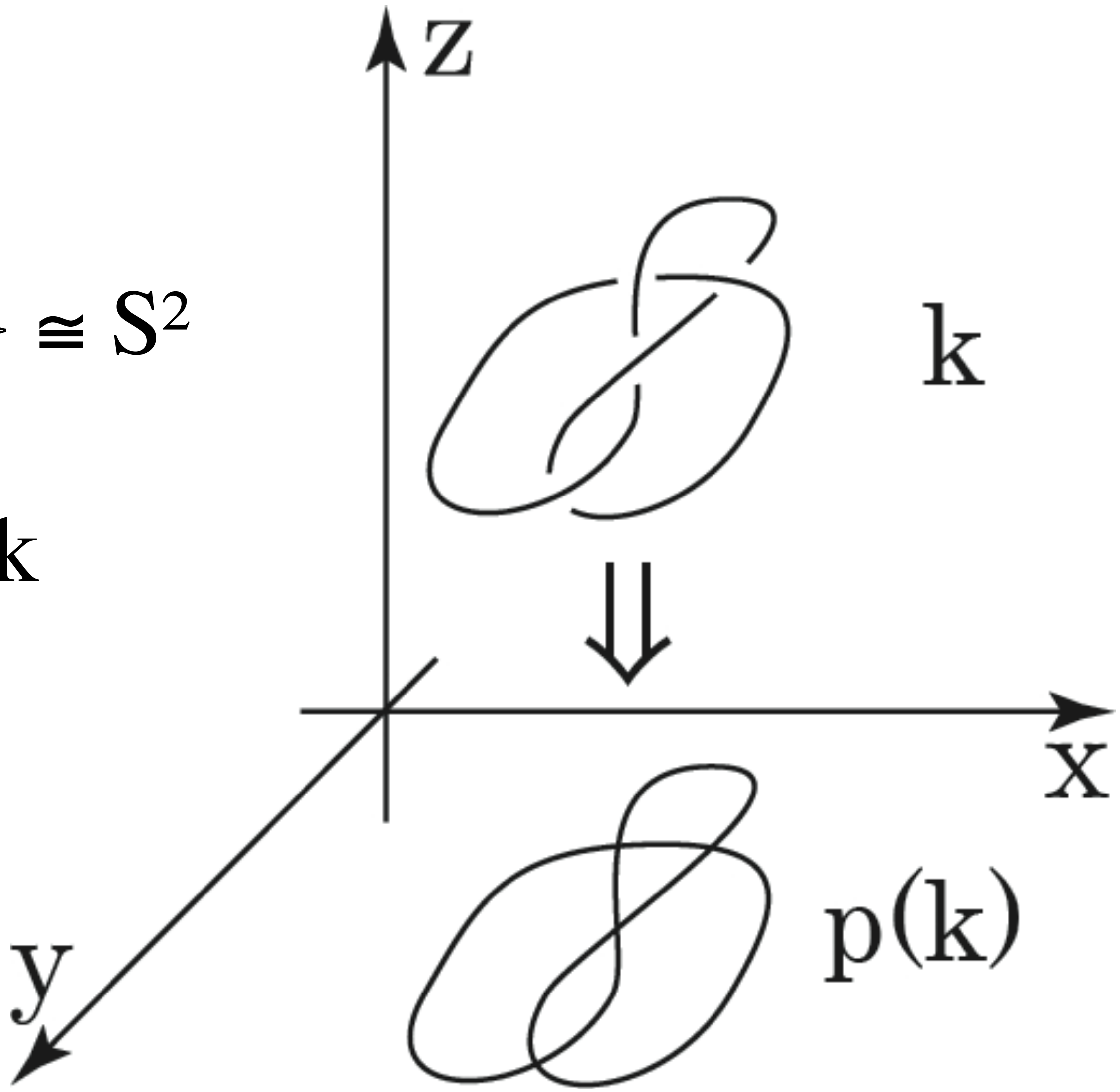
Definition (knot projection)

k: knot in R^3

$p : R^3 \to R^2 \subset R^2 \cup \{\infty\} \cong S^2$

p(k) : knot projection of k

Let P = p(K),
called knot projection.



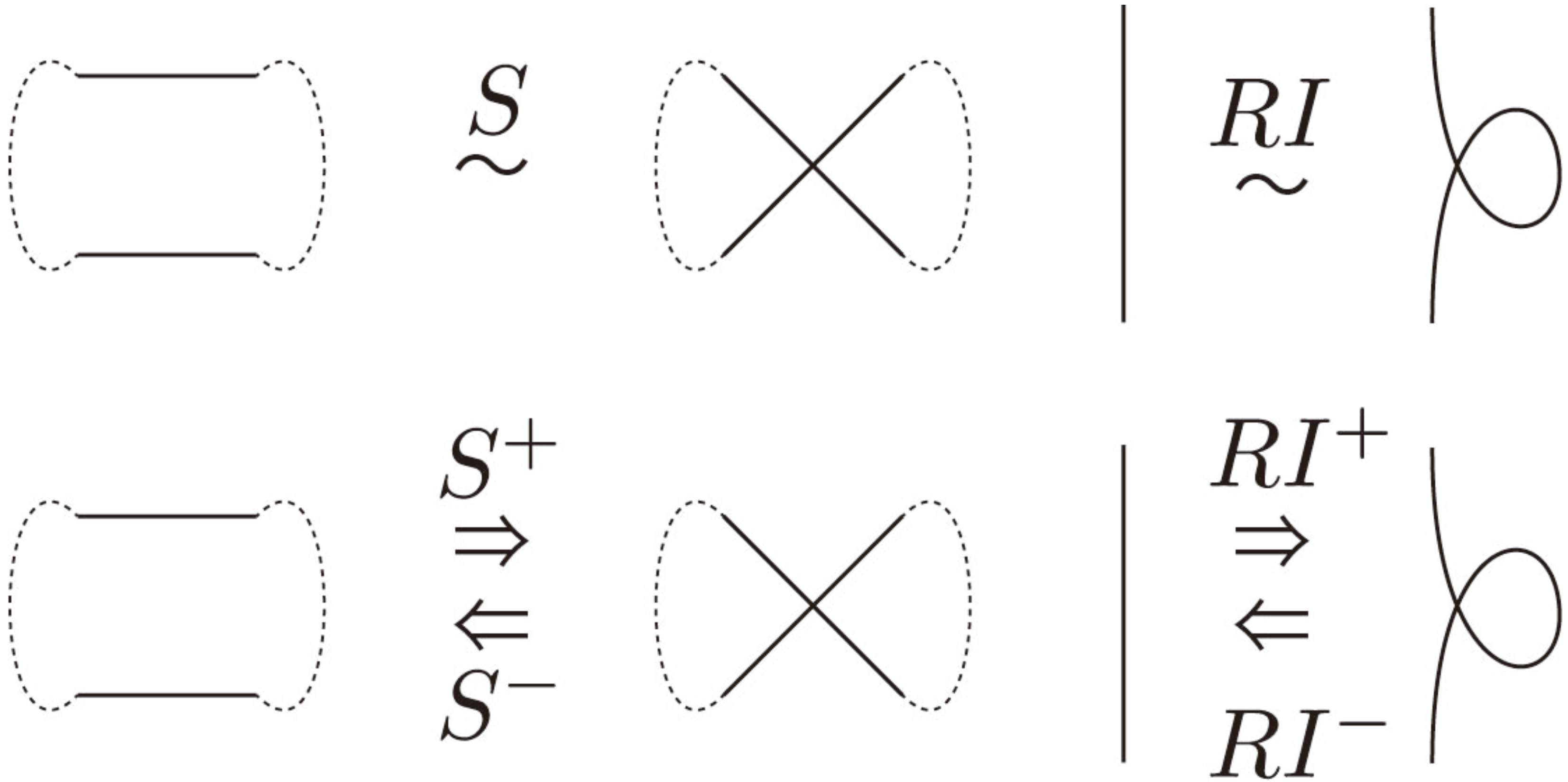
First Question of NI to Takimura (2016)

What is an appropriate "unknotting-type" number for knot projection?

Dr. Yusuke Takimura: a young person who is a teacher of junior high school, took PhD on 2020.

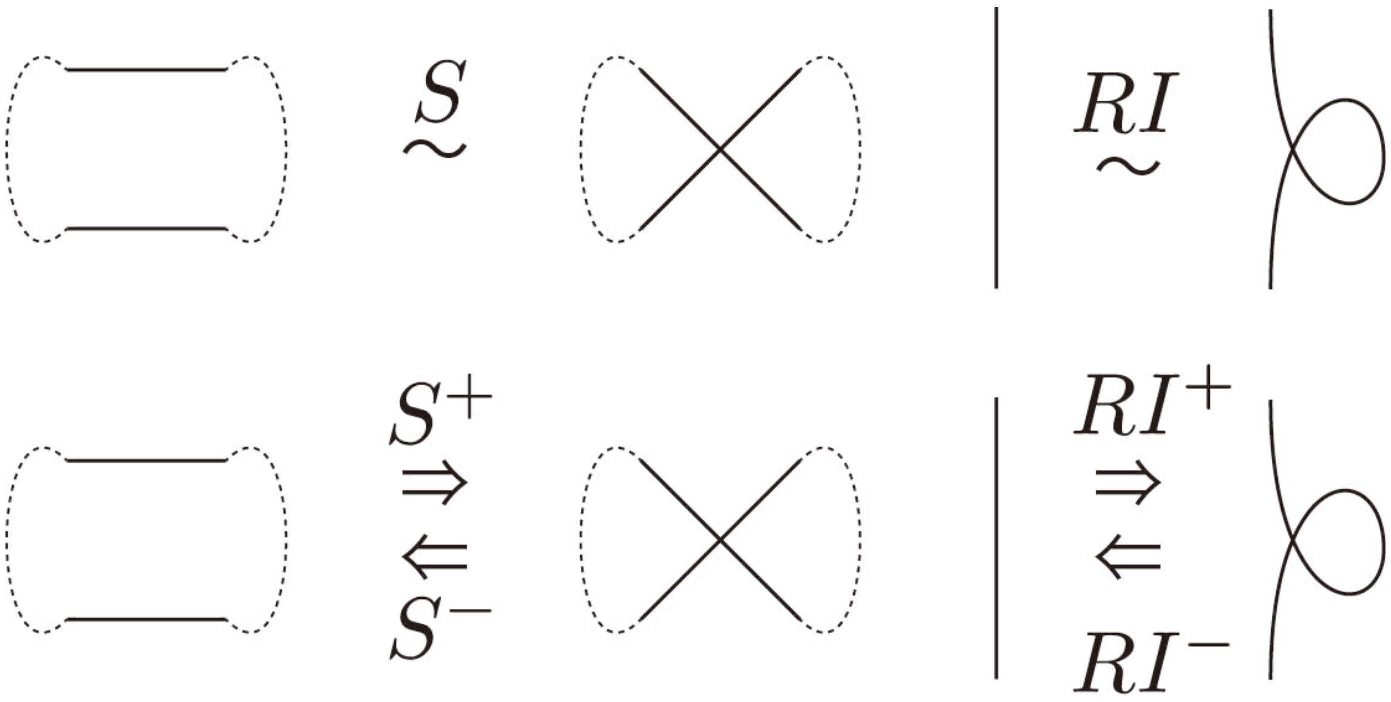
Takimura's Answer

Let us consider splices and inverses of type S & RI:

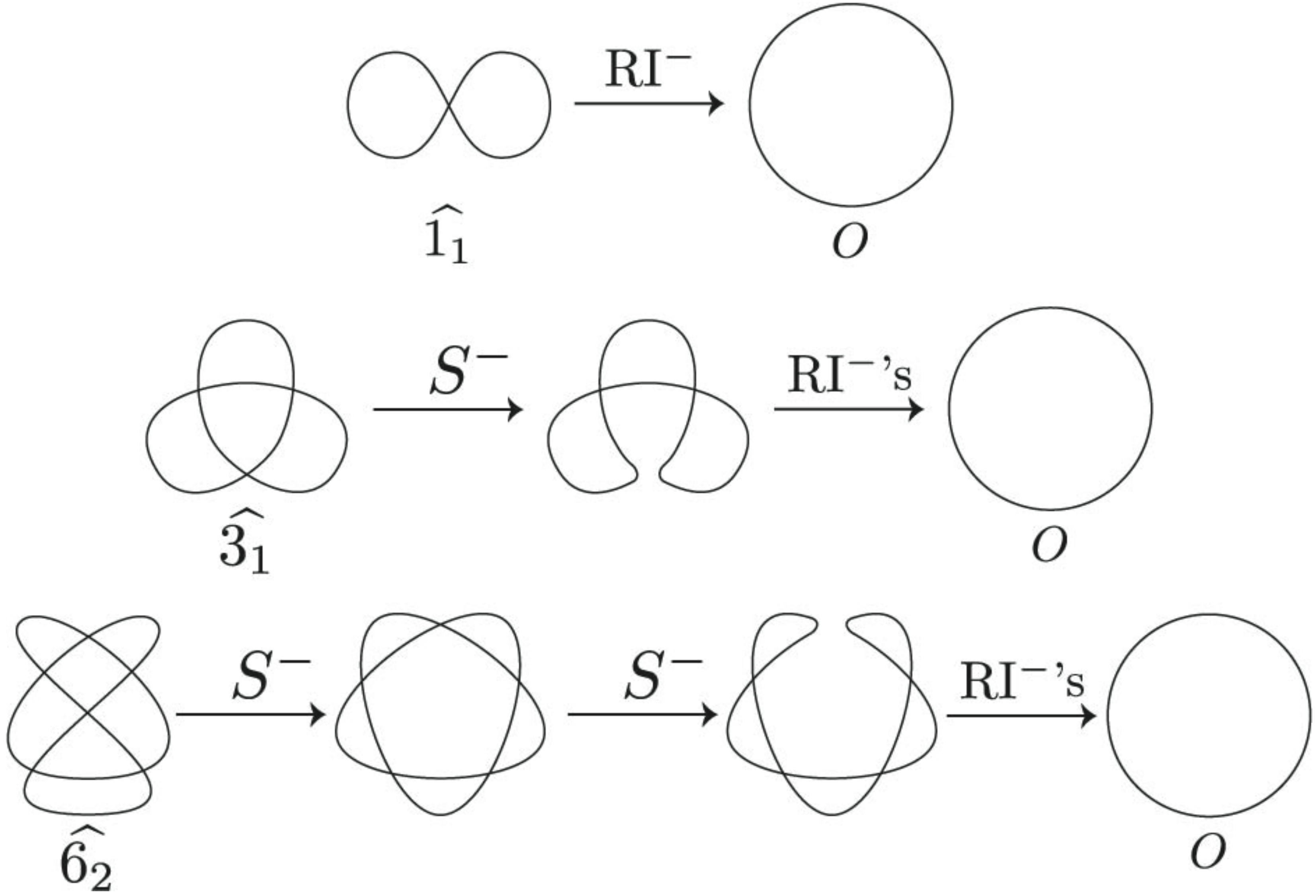


Takimura's u(P)

u(P) is the minimum number of necessary operations of type S among any sequences of S and RI to obtain 0-crossing knot projection O.

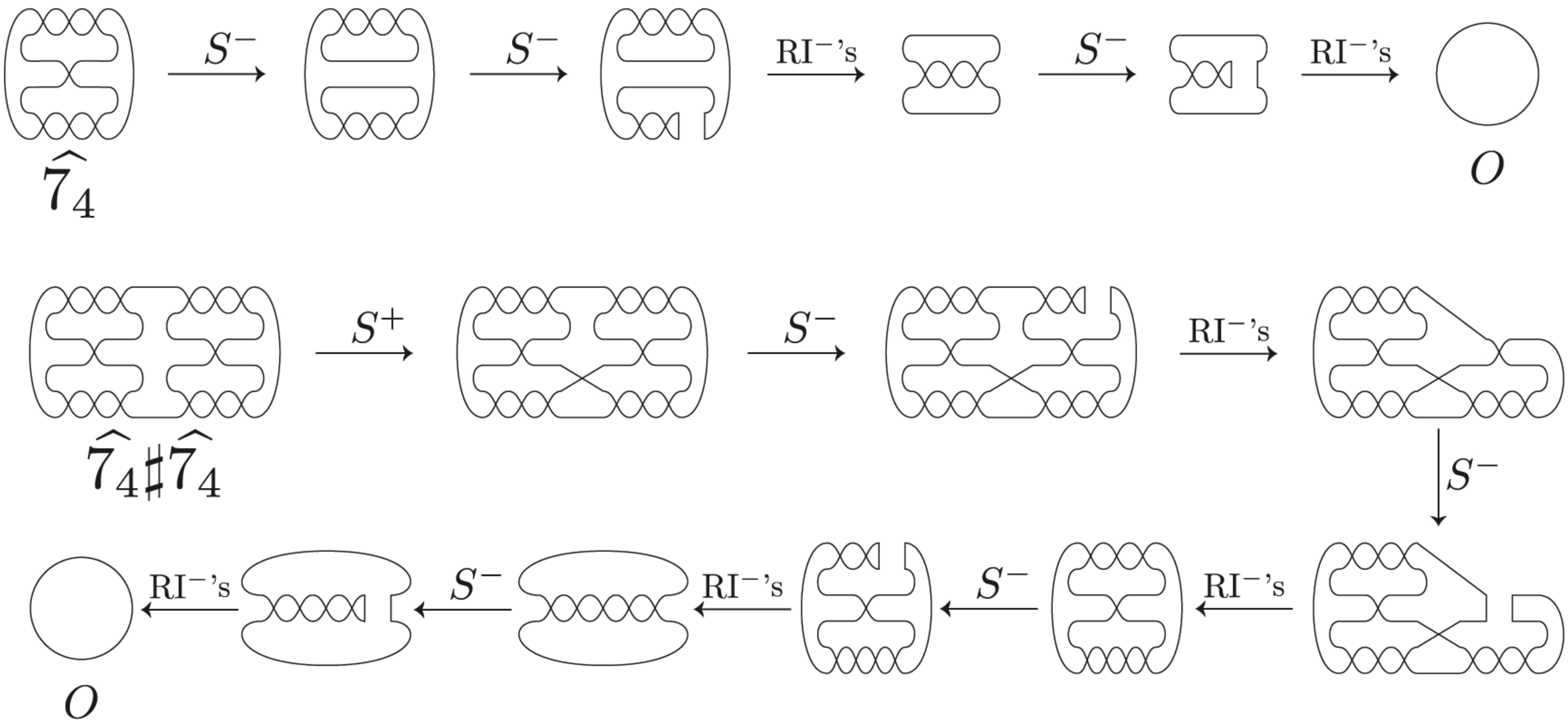


e.g.

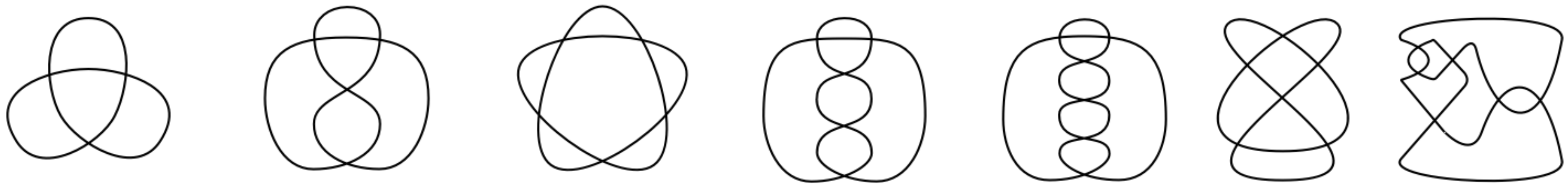


$$u\ (\widehat{1_1})=0 \qquad u\ (\widehat{3_1})=1 \qquad u\ (\widehat{6_2})=2$$

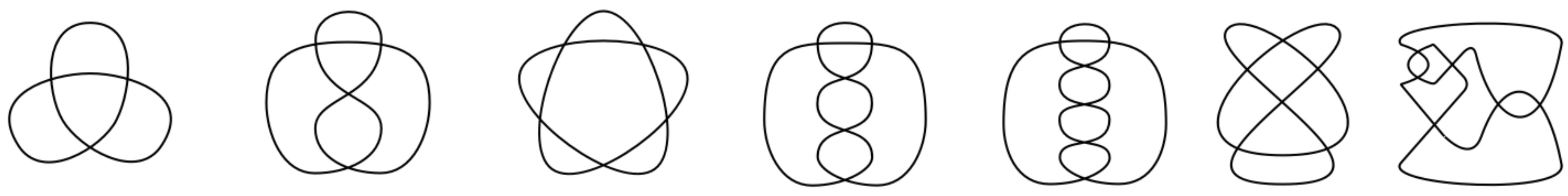
Example: u($\widehat{7}_4 \sharp \widehat{7}_4$) ≦ 5 whereas u($\widehat{7}_4$) = 3.



u (P) (Takimura made table up to 8 crossings)

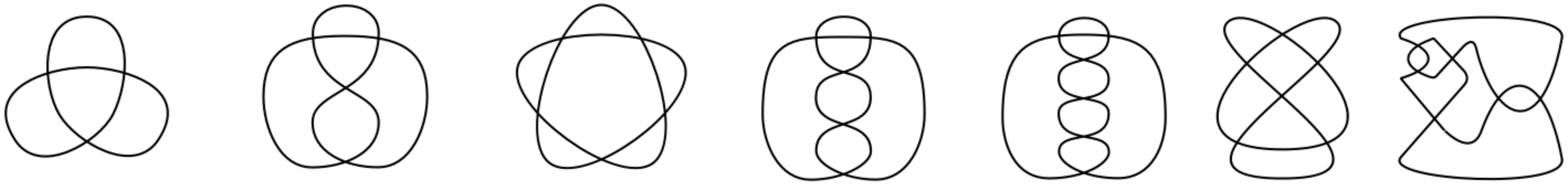


1 2 1 2 2 2 3



1 2 1 2 2 2 3

Seeing it, NI noticed that this table matched
Crosscap numbers of prime alt. knots.



1 2 1 2 2 2 3

Seeing it, NI noticed that this table matched Crosscap numbers of prime alt. knots.

→NI took S^- and RI^- only from Takimura's u(P).

Modified Definition: $u^{-}(P)$ by NI.

・$u^{-}(P)$ is the minimum number of necessary splices of type S^{-} among any sequences of S^{-} and RI^{-} to obtain O.

・$u(P)$ is the minimum number of necessary operations of type S among any sequences of S and RI to obtain O.

*Fact (Khovanov, 1997) Any RI's sequence is arranged to a seq. of only RI^{-}'s / RI+'s.

Modified Def. and Main result

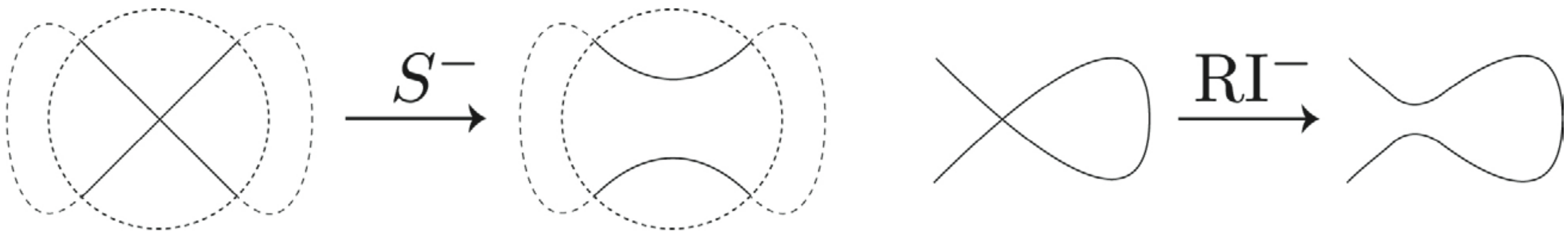
Main Result 1 (our preprint, 2nd revised, under review).

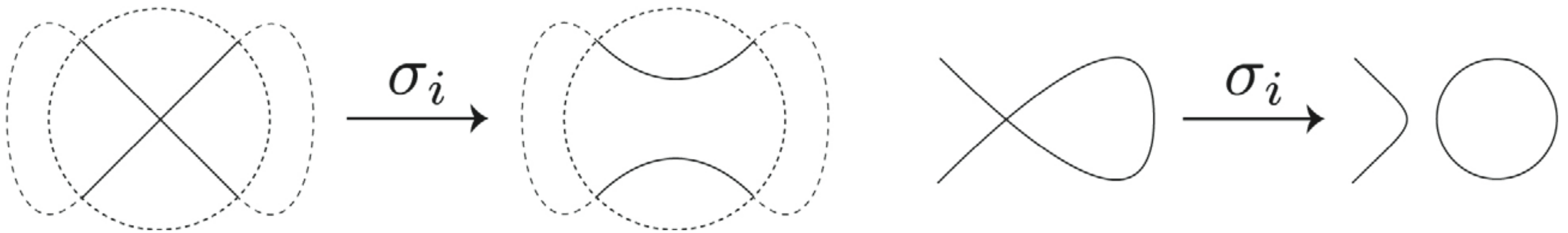
Let $u^{-}(K) = \min \{ u^{-}(D) \mid D : \text{knot diagram of } K \}$.

For any prime alternating knot K,

$$u^{-}(K) = C(K).$$

We define a splice (= removing band) σ_i corresponding to S^- or RI^- ($1 \leq i \leq n(P)$).



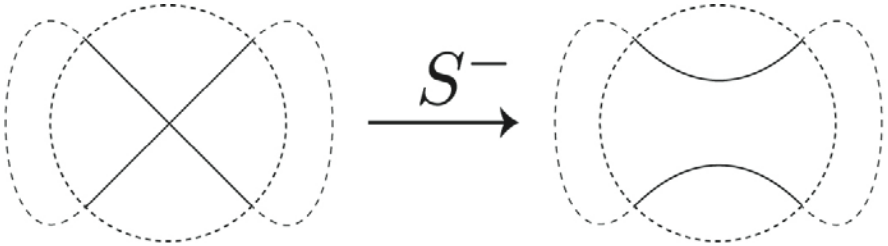


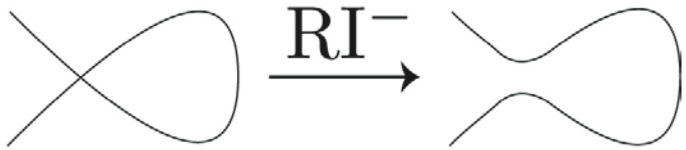
S_σ : Kauffman state by applying $\sigma = (\sigma_1, \sigma_2 \cdots \ \sigma_{n(\mathrm{P})})$ to P.

$|S_\sigma|$: the number circles in S_σ

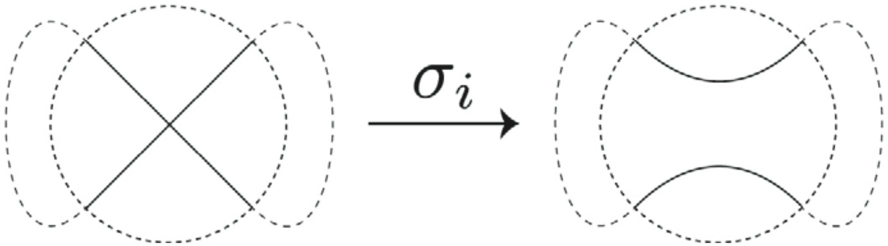
We define a splice (= removing band) σ_i corresponding to S^- or RI^- ($1 \leq i \leq n(\mathrm{P})$).

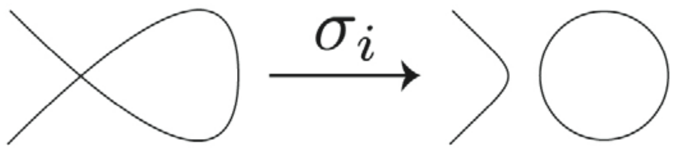
Non-counting





Euler char. unchanged

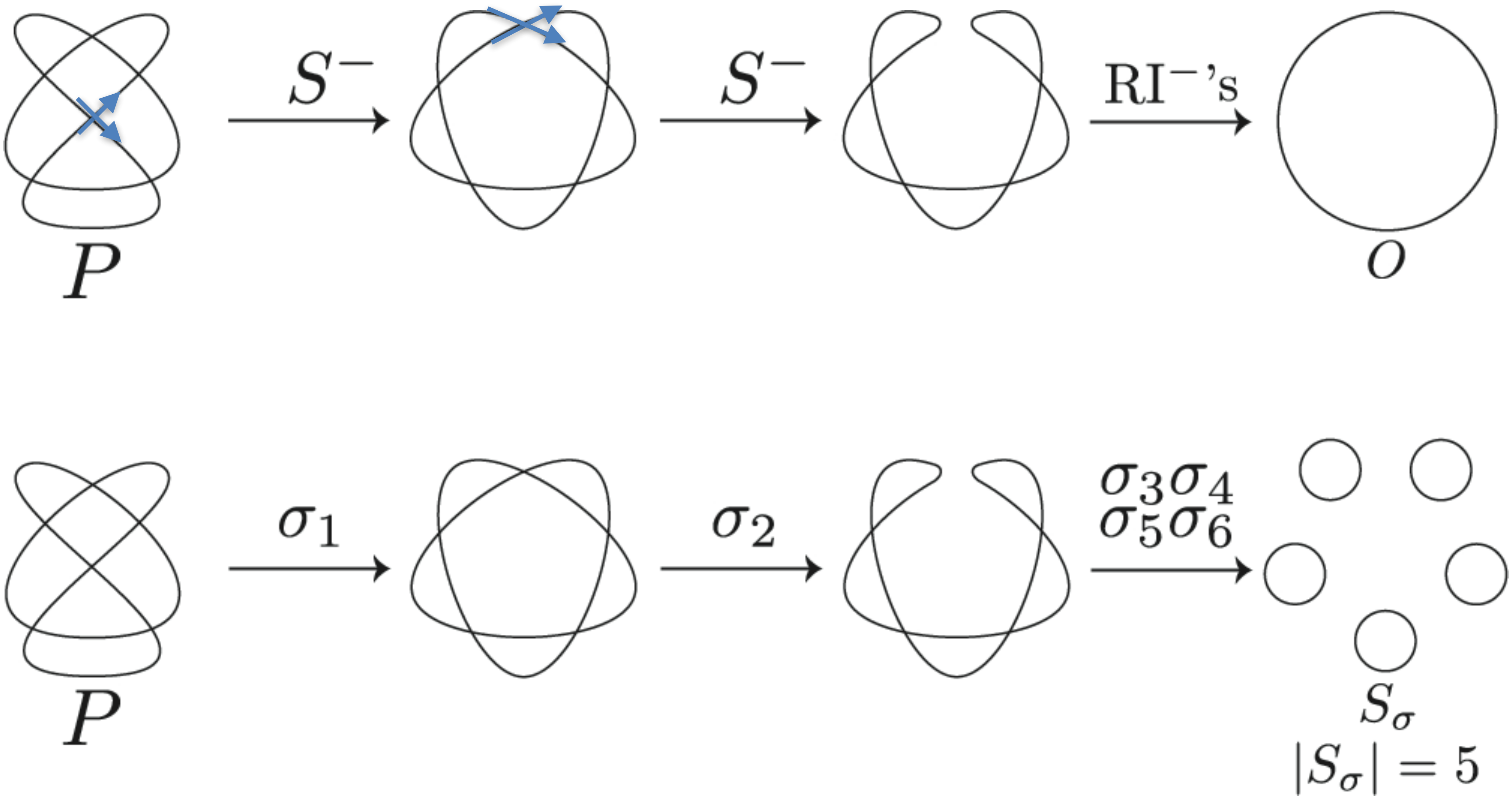


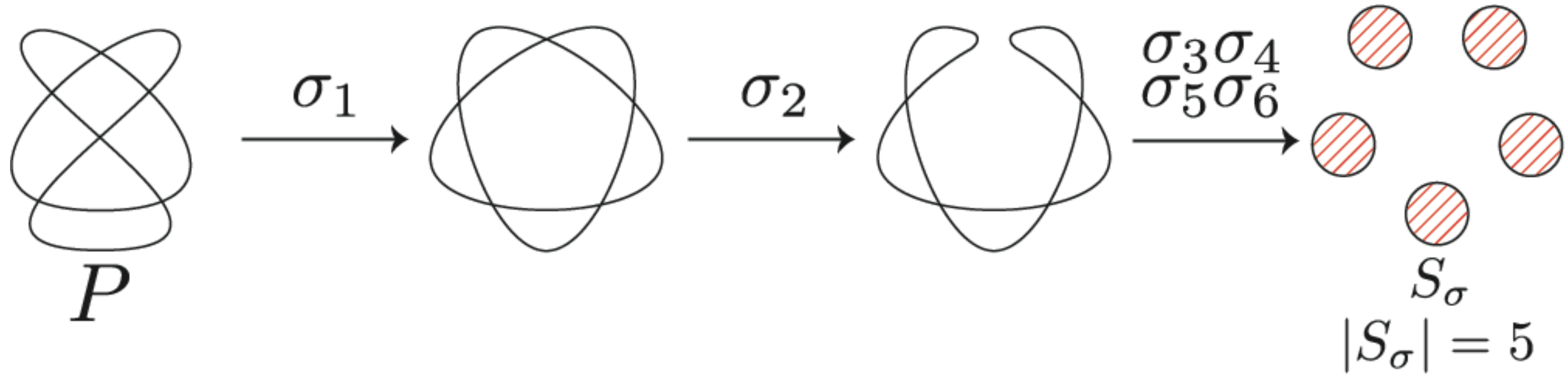


S_σ : Kauffman state by applying $\sigma = (\sigma_1, \sigma_2 \cdots \ \sigma_{n(\mathrm{P})})$ to P.

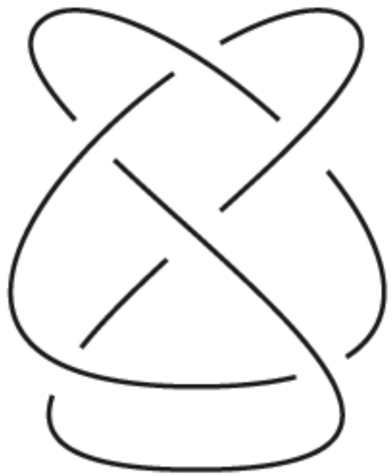
$|S_\sigma|$: the number circles in S_σ .

Example

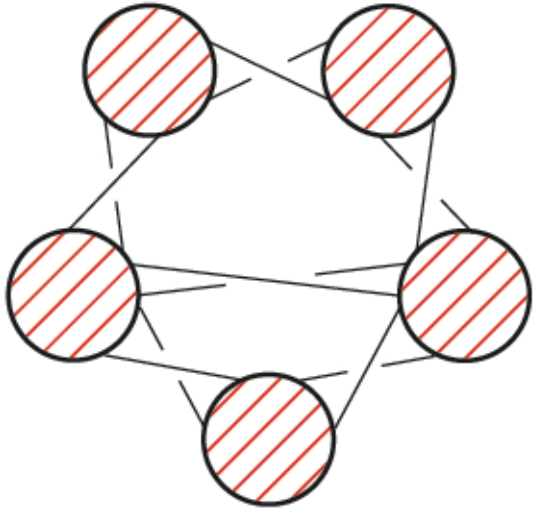

S^-
S^-
RI^-'s
P
O
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
P
S_σ
$|S_\sigma| = 5$



*In general, we do not need alt. diag.



D_P



$\Sigma_\sigma(D_P)$

Upper bound

Result (Takimura-I. 2018, IJM).
For any knot K,

$$C(K) \leqq u^{-}(K).$$

Proof of Upper bound

$$\chi(\Sigma_{max}) \geqq \chi(\Sigma_\sigma)$$

where

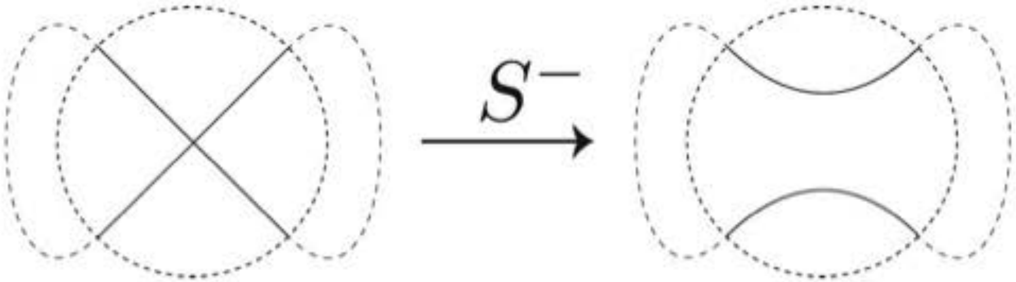
Σ_{max} : non-ori. surf. with maximal Euler char.

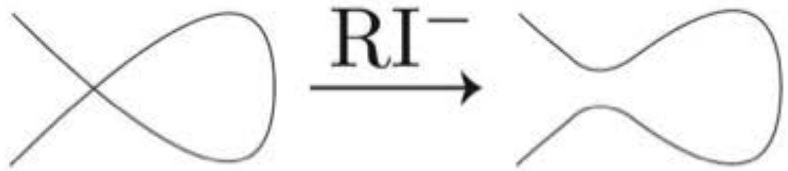
Σ_σ : non-ori. surf. given by σ .

Letting n(P) := # crossings,

$$P = P_0 \xrightarrow{\mathrm{Op}_1} P_1 \xrightarrow{\mathrm{Op}_2} P_2 \xrightarrow{\mathrm{Op}_3} \cdots \xrightarrow{\mathrm{Op}_{n(P)}} P_{n(P)} = O$$

where $\mathrm{Op_i} = S^- \text{ or } RI^- \ (1 \leq \mathrm{Op_i} \leq n(P))$.





$$|S_\sigma| = \#\{Op_i \mid Op_i = \mathrm{RI}^-\} + 1$$

$$n(P) = \#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = \mathrm{RI}^-\}$$

$$1 - C(K) = \chi(\Sigma_{max}) \geqq \chi(\Sigma_\sigma) = |S_\sigma| - n(P)$$

$$= (\#\{Op_i \mid Op_i = \mathrm{RI}^-\} + 1) - (\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = \mathrm{RI}^-\})$$

Proof of Upper bound

$$1 - C(\quad K \quad) \geq |S_\sigma| - n(P)$$

$$= (\#\{Op_i \mid Op_i = RI^-\} + 1) - (\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = RI^-\})$$

$$= 1 - \#\{Op_i \mid Op_i = S^-\}$$

$$= 1 - u^-(P) .$$

Proof of Upper bound

$$1 - C(\quad K \quad) \geq |S_\sigma| - n(P)$$

$$= (\#\{Op_i \mid Op_i = RI^-\} + 1) - (\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = RI^-\})$$

$$= 1 - \#\{Op_i \mid Op_i = S^-\}$$

$$= 1 - u^-(P).$$

$$C(\quad K \quad) \leq u^-(P).$$

Lower bound (non-short proof))

$C(K) \leqq u^{-}(K)$ (Takimura-I. 2018, IJM).

If K is prime alternating, this equality always exists.

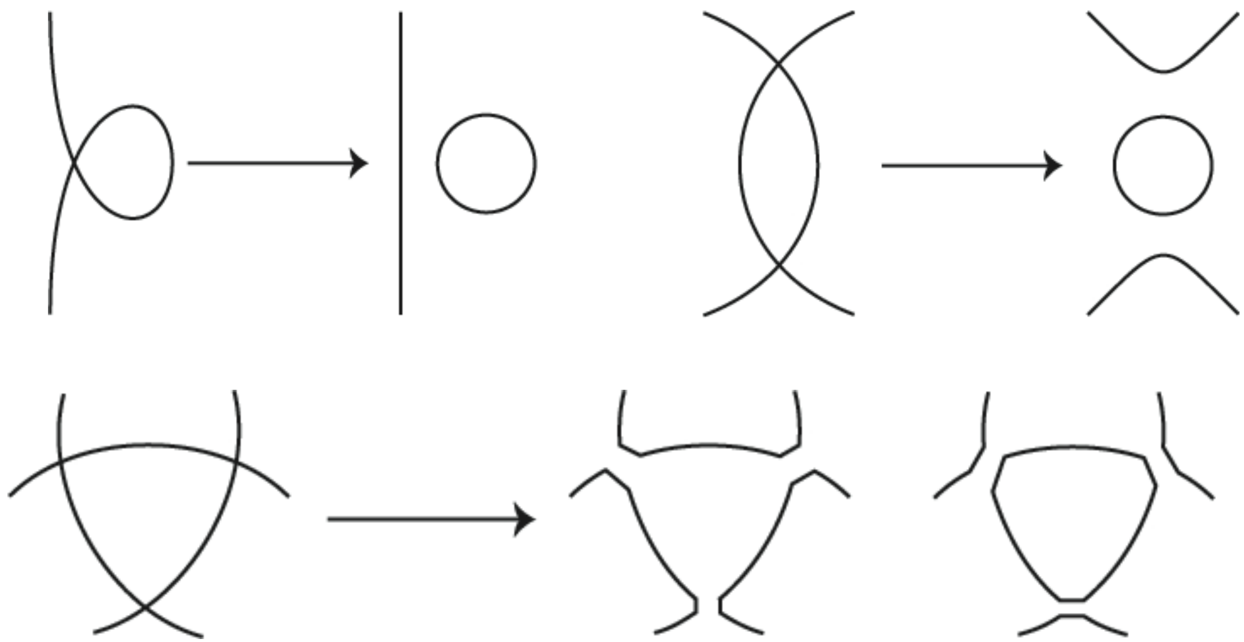
Adams-Kindred Theorem (2013, AGT)

For any n-crossing alternating diagram, there exists

a state giving the minimal genus among 2^n candidates.

Adams-Kindred algorithm to obtain (ori/non-ori) state surfaces with the maximal Euler char.

Find m-gon of the smallest m and splice as follows (Note that any knot projection P has a 3-gon if 3≦ m by Eliahou-Harary-Kauffman, 2008):



Lower bound (non-short proof)

Compare

Σu : non-ori. state surface realizing u^{-}(D)

with

Σ_{AK} : a surf. with maximal χ by Adams-Kindred.

Sketch

Proof. Σ_{AK} gives a state $\sigma = (\sigma_1, \sigma_2 \cdots \ \sigma_{n(P)})$, and we give reordered $(\sigma'_1, \sigma'_2 \cdots \ \sigma'_{n(P)})$ to recover Σu giving S^{-}...S^{-} RI^{-}... RI^{-} using an argument of Gauss chord of S^{-}.

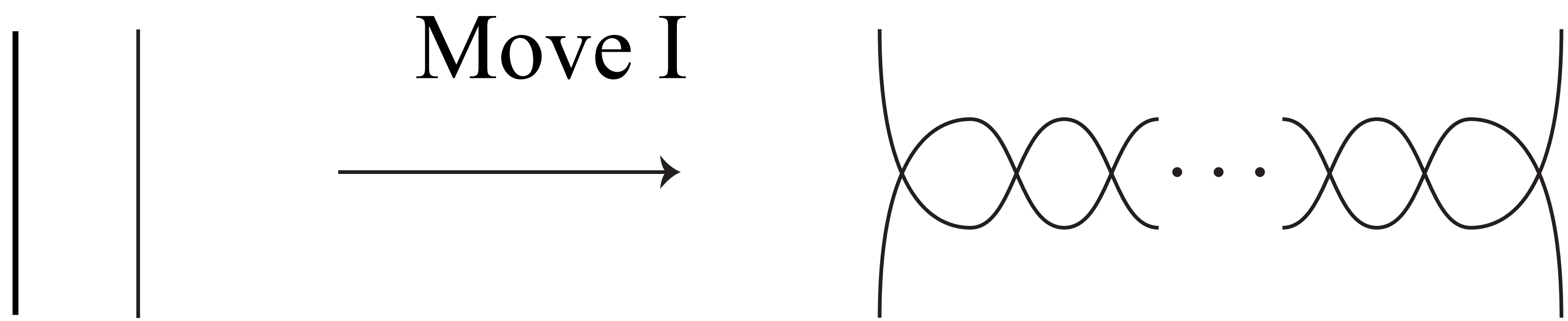
Is $u^{-}(K)$ computable?

$u^{-}(P)$, Computation

Conditions (A) and (B) are equivalent.

(A) $u^{-}(P) = n$.

(B) P is obtained from O by applying Move I, “n” times followed by RI+’s.



Component Preserving

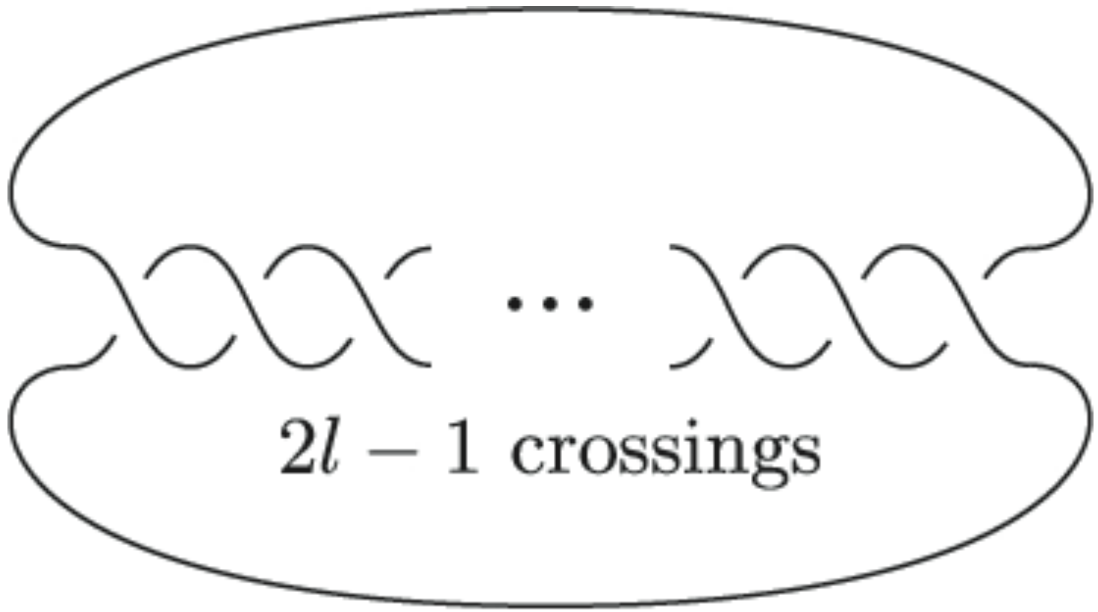
$u^{-}(K)$, still computable for some K.

If K is alternating,
there exists $u^{-}(P)$ such that $C(K) = u^{-}(P)$.

First Example. (Takimura-I., IJM, 2018)

For a prime alternating knot, (A) ⇔ (B) ⇔ (C).

(A) $C(K) = 1$.



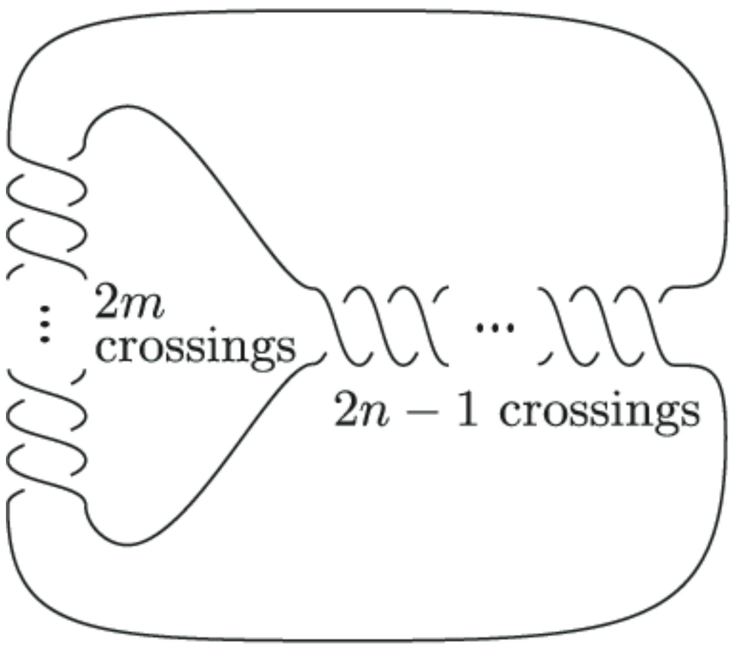
(B) $u^{-}(K) = 1$.

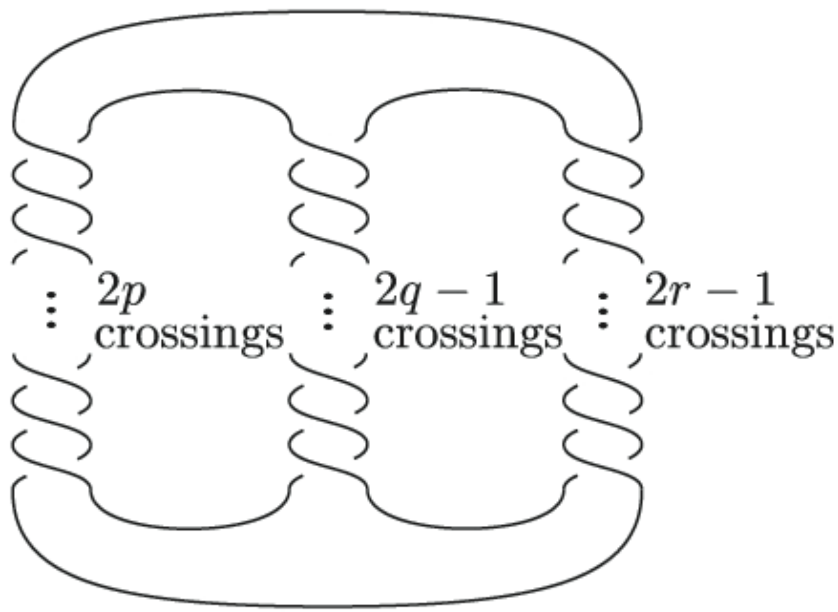
(C) K is as in the right.

Second Example (Takimura-I., IJM, 2018)

For any prime alt. knot K, (A) ⇔ (B) ⇔ (C).

(A) C(K) = 2 , (B) u- (K) = 2, (C) K is as below.





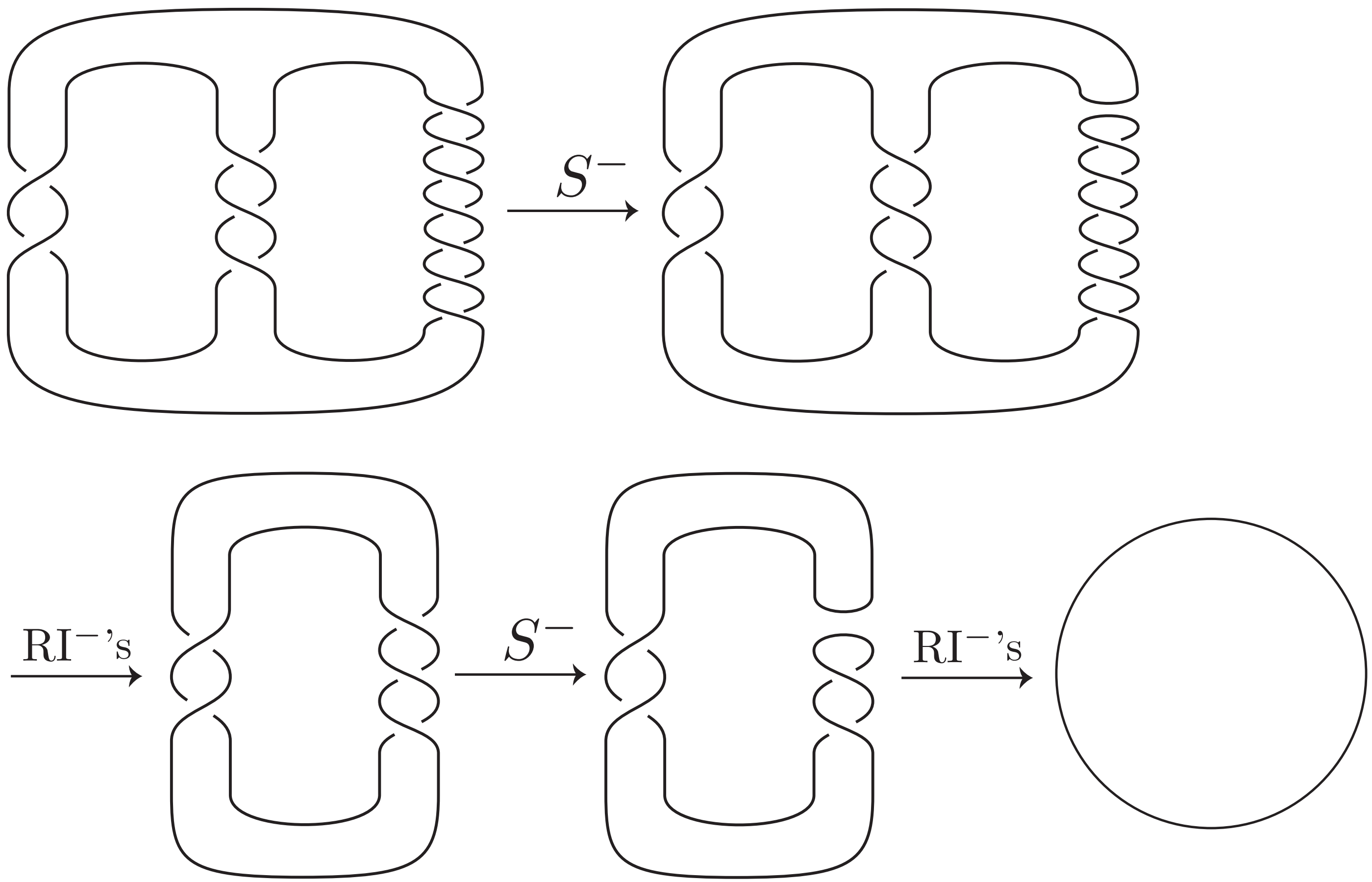
Is there nothing non-alternating knots?

By the argument of this proof, we have:

Main Result 2 (our preprint under review now)
For any knot K, if there exists a state realizing the maximal Euler characteristic,

$$u^{-}(K) = C(K).$$

Example (taught by T. Kindred)



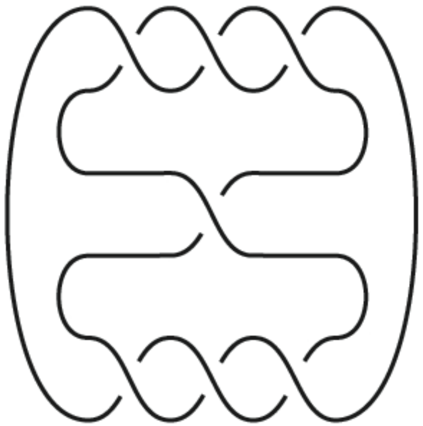
$u^{-}(K)$, still computable for some K.

If K is alternating, everything works.
<sufficient condition>

Comment.
There is a possibility to generalize the above condition (as Main Result 2).

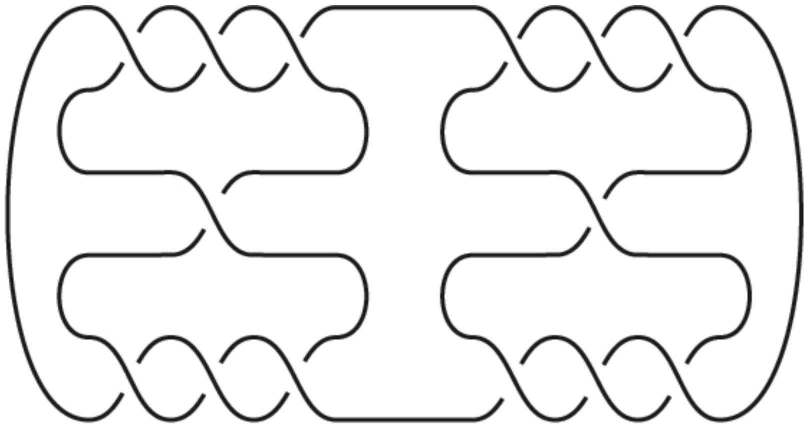
Is “primeness” essential ?

$u^-(K_1 \sharp K_2) = u^-(K_1) + u^-(K_2)$, but



7_4

$C(7_4) = 3$



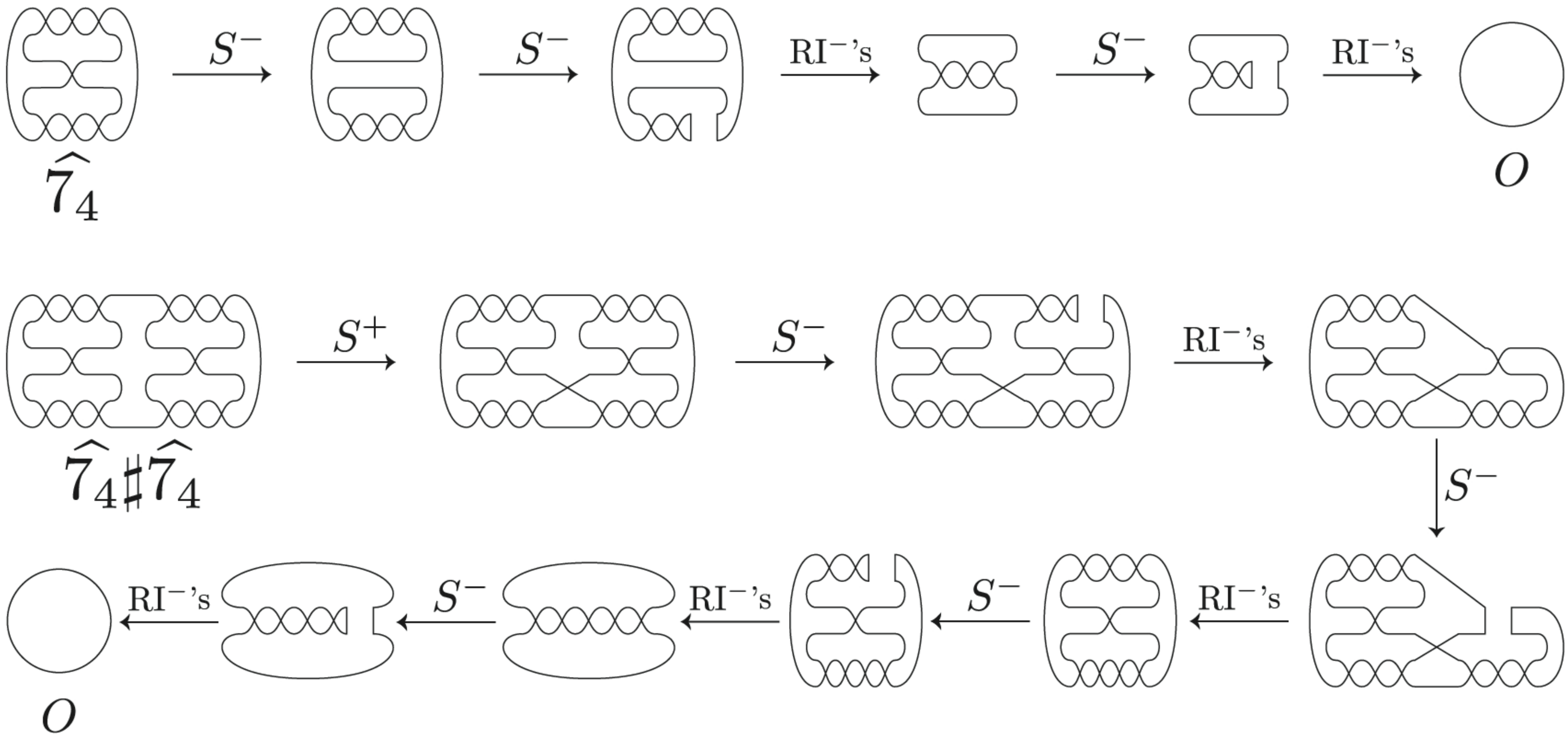
$7_4 \sharp 7_4$

$C(7_4 \# 7_4) = 5$

by Murakami-Yasuhara, PJM, 1997.

Note: there is an interesting example. $u^{-}(7_4 \# 7_4) = 6$, but Takimura's original u satisfies $u(7_4 \# 7_4) \leqq 5$.

<u>Question: Which prime knot K does satisfy $u(K) = u^{-}(K)$?</u>



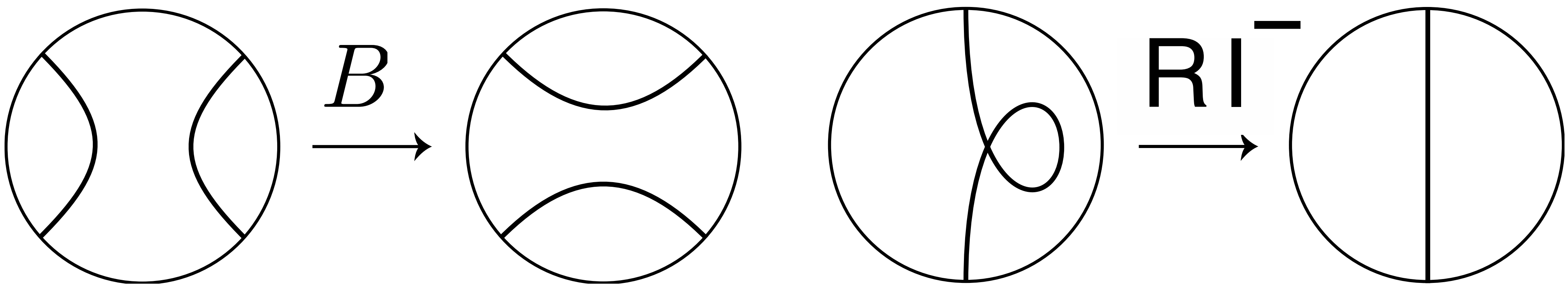
Can we give lower bound even non-prime knots easily?

Can we give lower bound even non-prime knots easily?

—>Yes, we do by “band surgery” though we drop convenience for computation & results for non-alternating knots.

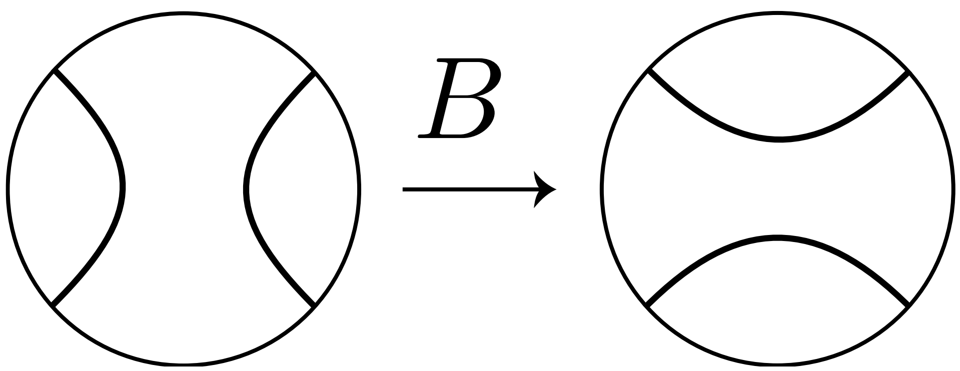
My answer : for alt. knots, via band surgery

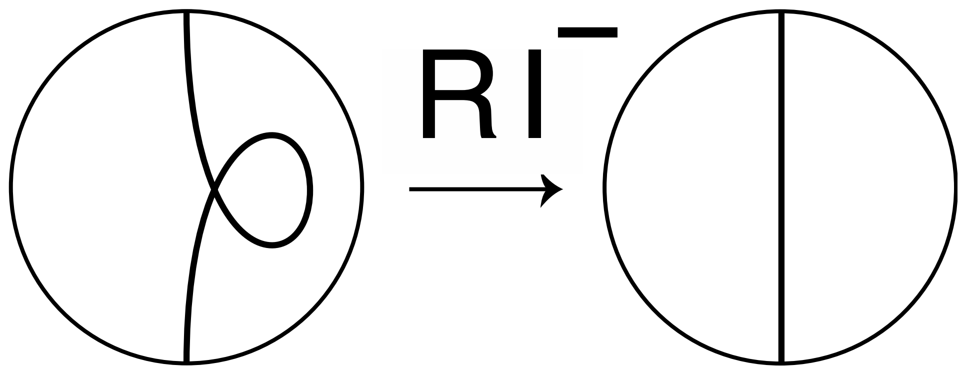
Definition (Band surgery “B” for diagram).



Definition(B(D))

Let D be an alternating knot diagram.
B(D) is the minimum number of necessary band surgeries B among any sequences of B and RI$^-$ to obtain O.





Alternating knot invariant B(K).

K : alternating knot,

Z(K) : the set of alt. knot diag. of K,

$$B(K) := \min_{D \in Z(K)} B(D).$$

Note: Takimura has given B(P) for any knot projection P.
I use it.

Main Result 3 (Takimura-I., JKTR, 2020)

$\Gamma(K)$: min of 1st Betti num. of alt. knot K.

(1) $C(K)=B(K) \Leftrightarrow C(K) = \Gamma(K)$.

(2) $C(K) = B(K) +1 \Leftrightarrow C(K) \neq \Gamma(K)$.

(3) $B(K \# K') = B(K) + B(K')$.

NOTE: $C(K) \neq \Gamma(K) \Leftrightarrow C(K) = 2g(K)+1$.

1978, Clark, Fixed K s.t. $C(K) = 1$.

1996, H. Murakami-Yasuhara, Fixed condi., connected sum.

2004, Teragaito, Fixed $C(K)$, torus knots

Hatcher-Thurston, 1985

2006, Hirasawa-Teragaito, Fixed $C(K)$, 2-bridge knots

2006, Ichihara-Mizushima, Fixed $C(K)$, *many* pretzel knots

2013, Adams-Kindred, Algorithm for alternating knots

(using state surface, introduced by Ozawa)

Hatcher-Ortel, 1989

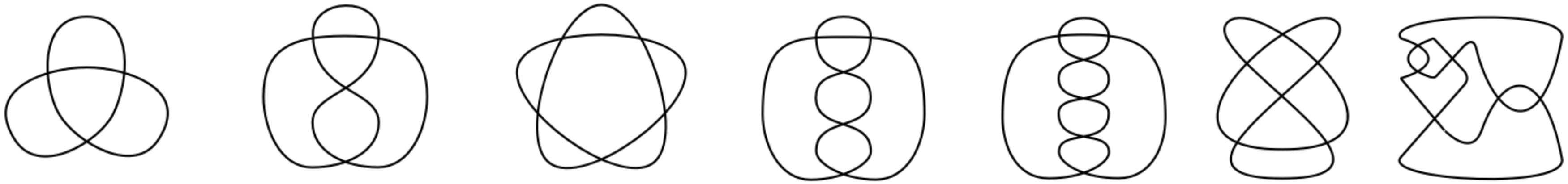
2016, Kalfagianni-Lee, Lower bounds w/ Jones polynomial

(via Adams-Kindred algorithm)

2019, by us, crosscap number n of alt. knots., also by Kindred.

Time line (by us, and Kindred)

- 2019, March 8, we submitted a paper for B(K) to JKTR.
- 2019, April 4, we submitted a paper for $u^{-}(K)$.
- 2019, May 17, Kindred contacted me that he was working for the same equality $C(K) = u^{-}(K)$.
- I like arXiv, but do not like “arXiv competition”, thus, I did not post our papers to arXiv and said OK for him.
- 2019, May 27, Kindred submitted his paper to arXiv.
- 2020, March 20, Our B(K) result was published (JKTR).
- 2020, June 25, Kindred’s $u^{-}(K)$-paper was published (IJM).
- 2020, July 26, we submitted 2nd revised ver. to IJM.



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