Unknotting operations, crosscap numbers, and volume bounds

Noboru Ito

(National Institute of Technology, Ibaraki College)

MSCS Quantum Topology Seminar
organized by L.H. Kauffman
August 27, 2020
Main Result 1.
Let \( C(K) \) be the crosscap number of \( K \).

For any prime alternating knot \( K \),

\[
C(K) = u^-(K).
\]
Recalling definition: $u^-(P), u^-(K)$

- $u^-(P)$ is the minimum number of necessary splices of type $S^-$ among any sequences of $S^-$ and $RI^-$ to obtain $O$. $u^-(K) := \min_P u^-(P)$. 
Plan of proof for $u^{-}(D) \leq C(K)$

We will compare

$\sum u : \text{a non-orientable state surface realizing } u^{-}(D)$

with

$\sum_{AK} : \text{a surface realizing } C(K) \text{ or } g(K)$ (Adams-Kindred).
$\Sigma u := \Sigma_\sigma(D_P)$

$P \xrightarrow{\sigma_1} \xrightarrow{\sigma_2} \xrightarrow{\sigma_3\sigma_4\sigma_5\sigma_6} S_\sigma$

$|S_\sigma| = 5$
Construction of $\Sigma_{AK}$

Find m-gon of the smallest $m$ and splice as follows

(P has a 3-gon if $3 \leq m$ Eliahou-Harary-Kauffman, 2008)

or
Notation 1. $\Sigma_{AK}$ gives a sequence of splices $(\sigma_i)_{i=1}^{n(D)}$:

$$D = D_0 \xrightarrow{\sigma_1} D_1 \xrightarrow{\sigma_2} D_2 \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n(D)}} D_{n(D)}$$

Each orientation of $D_i$ is of $\sigma_i$. ($=\text{ori. } S^-, S_{\text{join}}^-, T_{\text{split}}, T_{\text{join}}$).

It induces

$$CD_D = CD_0 \xrightarrow{\sigma_1} CD_1 \xrightarrow{\sigma_2} CD_2 \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n(D)}} CD_{n(D)}.$$

$(CD_D$ is a Gauss diagram of $D$; it will be defined.)
Notation 2. • Oriented $T_{\text{split}}$, $T_{\text{join}}$. Seifert splices.
• Oriented $\text{RI}^-$. 1st Reidemeister move.

• Oriented $S^-$, $S_{\text{join}}^-$. Target orientation must be chosen.
Property $S^{-}$

Claim

Key Lemma

Main Result 1
Property $S^-$

Claim

Key Lemma

to $S^-S^-S^-$

Main Result 1
Definition 1. Let $D$ be a knot diagram whose projection is $P$. Then there is a generic immersion $g : S^1 \to S^2$ such that $g(S^1) = P$. It is denoted by $CD_D$. 
Property $S^-$. The behavior of $S^-$ in $CD_D$ is as follows. (The difference of cyclic Gauss words is presented as:

$$c p_1 p_2 \cdots p_{2i} c p_{2i+1} \cdots p_{2n} \rightarrow p_{2i} p_{2i-1} \cdots p_1 p_{2i+1} p_{2i+2} \cdots p_{2n}.$$ )
e.g.
Property $S^-$

Claim

Key Lemma

Main Result 1
Property $S^-$

Claim

Key Lemma

Main Result 1
Claim. Suppose that $(\sigma_i)_{i=1}^{n(D)}$ satisfies $\sigma_1 = S^-$, $\sigma_2 = T_{\text{split}}$, and $\sigma_3 = T_{\text{join}}$. Then, the three chords in $CD_D$ corresponding to $\sigma_1$, $\sigma_2$, and $\sigma_3$ are as in
Observation 1

Component-preserving successive “T T” should have a chord intersection.
**Property $S^-$**

**S^-**

**S^- T T**

**Claim**

**Key Lemma**

**Main Result 1**
Key Lemma. Let $D$ be a prime (alternating or non-alternating) knot diagram with exactly $n(D) (> 1)$ crossings with $\sigma_i \neq RI^- (\forall i)$. Suppose that $\sigma_1 = S^-$ and that $(\sigma_i)_{i=2}^{n(D)}$ includes at least one $T_{\text{join}}$, $S^-_{\text{join}}$, or $S^-$. Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ such that $\sigma'_1 = S^-$ and $\sigma'_2 = S^-$, and $\sigma'_i \neq RI^- (\forall i)$. 
Key Lemma. Let $D$ be a prime (alternating or non-alternating) knot diagram with exactly $n(D)$ ($>1$) crossings with $\sigma_i \neq R\Gamma^-$ ($\forall i$). Suppose that $\sigma_1 = S^-$ and that $(\sigma_i)_{i=2}^{n(D)}$ includes at least one $T_{\text{join}}$, $S_{\text{join}}^-$, or $S^-$. Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ such that $\sigma'_1 = S^-$ and $\sigma'_2 = S^-$, and $\sigma'_i \neq R\Gamma^-$ ($\forall i$).

Roughly speaking, suppose that AK-sequence starts from one $S^-$. If "join" or more $S^-$ appears in the seq., $S^- \ldots \rightarrow S^- S^- \ldots$ by reordering.
Property $S^-$

Claim

Key Lemma

Main Result 1
Proof of Key Lemma

Case (1): \((\sigma_i)_{i=2}^{n(D)}\) includes at least one \(S^-\) or \(S_{\text{join}}^-\).

\(S^-T \ldots TS^- \ldots\), or \(S^-T \ldots TS_{\text{join}}^- \ldots\). Moving \(\sigma_m\) (= \(S^-\) or \(S_{\text{join}}^-\)) to \(\sigma'_2\),

\[
\begin{align*}
S^- & \quad \xrightarrow{c} \quad D_1 \\
T & \quad \xrightarrow{\ldots} \quad T \\
\sigma_m & \quad \xrightarrow{c} \quad D_{m-1}
\end{align*}
\]

we obtain \(S^-S^- \ldots\).
Component-preserving pair “T S” should have a chord intersection.

\[ T_{\text{split}} \quad S_{\text{join}}^{-} \xrightarrow{T_{\text{split}}} \quad S_{\text{join}}^{-} \]
Proof of Key Lemma

Case (2): \((\sigma_i)_{i=2}^{n(D)}\) includes no splice \(S^-\) and no splice \(S_{\text{join}}^-\), but includes a splice \(T_{\text{join}}\).

We have reordering:

\[ S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \rightarrow S^- T_{\text{split}} T_{\text{join}} T \cdots T. \]

- Case: either \((X)\) or \((X')\) is included:

By property of \(S^-\), reordering \(123 \rightarrow 231\) or \(321\) obtains a sequence \(S^- S^- S^- \cdots\)

- Case: there is no \((X)\) and no \((X')\), but \((Y)\) appears:
Observation 1

Component-preserving pair “T T” should have a chord intersection.
Proof of Key Lemma

Case (2): \((\sigma_i)_{i=2}^{n(D)}\) includes no splice \(S^-\) and no splice \(S_{\text{join}}^-\), but includes a splice \(T_{\text{join}}\).

We have reordering:

\[
S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \rightarrow S^- T_{\text{split}} T_{\text{join}} T \cdots T.
\]

- Case: either (X) or (X') is included:
  By property of \(S^-\), reordering \(123 \rightarrow 231\) or \(321\) obtains a sequence \(S^- S^- S^- \cdots\).  

- Case: there is no (X) and no (X'), but (Y) appears:
Reordering: 123 $\rightarrow$ 321 or 231
Case (2): \((\sigma_i)_{i=2}^{n(D)}\) includes no splice \(S^\sim\) and no splice \(S_{\text{join}}^\sim\), but includes a splice \(T_{\text{join}}\).

We have reordering:

\[
S^\sim \cdot T_{\text{split}} \cdot \cdot \cdot T_{\text{split}} T_{\text{join}} T \cdot \cdot \cdot T \rightarrow S^\sim \cdot T_{\text{split}} T_{\text{join}} T \cdot \cdot \cdot T.
\]

- Case: either \((X)\) or \((X')\) is included:
  By property of \(S^\sim\), reordering \(123 \rightarrow 231\) or \(321\) obtains a sequence \(S^\sim S^\sim S^\sim \ldots\) It’s the highest point of the proof, we’ll go to the next slide!

- Case: there is no \((X)\) and no \((X')\), but \((Y)\) appears:
(Y). \[ \sigma_2 = T_{\text{split}} \]

\[ \sigma_1 = S^- \]

\[ \sigma_3 = T_{\text{join}} \]
Proof of Key Lemma

Case (2): \( (\sigma_i)_{i=2}^{n(D)} \) includes no splice \( S^- \) and no splice \( S^{-}_{\text{join}} \), but includes a splice \( T_{\text{join}} \).

We have reordering:

\[
S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \to S^- T_{\text{split}} T_{\text{join}} T \cdots T.
\]

• Case: either (X) or (X’) is included:
  By property of \( S^- \), reordering 123 \( \to \) 231 or 321 obtains a sequence \( S^- S^- S^- \ldots \)

• Case: there is no (X) and no (X’), but (Y) appears:
  By primeness, (X) should be included \( \to \) contradiction.  \( \square \)
Applying Key Lemma the sequence of splices repeatedly, we have:

\[ S^- \cdots S^- T_{split} \cdots T_{split} \]

from \( \Sigma_{AK} \).

Here, in this seq., every \( T_{split} \) splits a monogon since there is no chord intersection after \( S^- S^- \ldots S^- \) applies.
Observation 1’’

Any component-preserving pair “T X” should have a chord intersection.
Property $S^-$

Claim

Key Lemma

$S^-$

$S^-$ to $S^-$

$u^-(D) = C(K)$

Main Result 1
Finalizing Proof of Main Result 1 (lower bound)

Case $\Sigma_{AK}$ is a non-orientable surface with the maximal Euler characteristic $\chi$. (Note: the seq. has $S^{-}$; any $\sigma_i \neq RI^{-}$.)

Thus, by Key Lemma, this seq. realizes $u^{-}(D)$ by reordering.

$$S^{-}S^{-} \ldots S^{-}T_{\text{split}}T_{\text{split}} \ldots T_{\text{split}}.$$

The reordering process implies Observation 2.
Observation 2. Each reordering may cause:

\[ T_{\text{split}}, T_{\text{join}} \leftrightarrow S^-, S^- \quad \text{or} \quad T_{\text{split}}, S^-_{\text{join}} \leftrightarrow S^-, S^- \. \]

Thus,

\[
1 - u^-(D) = 1 - \#\{S^- \text{in seq.}\} \\
= 1 - 2\#T_{\text{join}} - 2\#S^-_{\text{join}} - \#S^- \\
= 1 + (\#T_{\text{split}} - \#T_{\text{join}} - \#S^-_{\text{join}}) - n(D) \\
= \chi(\Sigma_{AK}) = 1 - C(K) .
\]
Claim

Property $S^-$

$S^-$

Claim

Key Lemma

$S^-$ $T$ $T$ ...

to $S^-S^-S^-$

Main Result 1

$u^-(D) = C(K)$
Finalizing Proof of Main Result 1 (lower bound)

Case $\Sigma_{AK}$ is a orientable surface with the maximal Euler characteristic. **Note:** $2g(K) < C(K) \Leftrightarrow C(K) = 2g(K) + 1$.

It returns to the non-orientable case since $\chi (= 1 - 2g(K))$ is changed into $1 - (2g(K) + 1) (= 1 - C(K))$ by the replacement:

![Seifert splice](image)
Then for any prime alternating knot diagram $D$,

$$u^-(K) \leq \min_D u^-(D) = C(K).$$

Recalling that $C(K) \leq u^-(K)$, it completes the proof. \qed
By the argument of this proof, we have:

**Main Result 2**
For any knot $K$, if there exists a state realizing the maximal Euler characteristic,

$$u^-(K) = C(K).$$
Proof of Main result 3 (Band Surgery)

Merit: simpler proof, non-depending on primeness.
Definition ($B(D)$)

Let $D$ be an alternating knot diagram. $B(D)$ is the minimum number of necessary band surgeries $B$ among any sequences of $B$ and $RI^{-}$ to obtain $O$. Let $B(K) = \min_D B(D)$.
Main Result 3 (Takimura-I., JKTR, 2020)

$\Gamma(K)$: min of 1st Betti num. of alt. knot $K$.

(1) $C(K) = B(K) \iff C(K) = \Gamma(K)$.

(2) $C(K) = B(K) + 1 \iff C(K) \neq \Gamma(K)$.

(3) $B(K \neq K') = B(K) + B(K')$. 
Proof. There exists a state, i.e. a family of splices, implying a spanning surface with the maximal Euler characteristic for alter. knot. Thus,

\[ \mathcal{B}(K) \leq \Gamma(K). \]

\( \Gamma(K) \) is 1st Betti num. 
\[ = \min \{C(K), 2g(K)\}. \]
Proof $\Gamma(K) \leq B(K)$ for Case $C(K) = \Gamma(K)$

$C(K) \ (= \Gamma(K))$

$= b_1 ( \quad )$

$= \min \{ \# \text{ necessary bands to obtain a disk} \}$

$\leq \min \{ \# \text{ necessary bands to obtain a disk from "a" state non-ori. surface of D} \}$

$= B(K).$
Proof: Case $C(K) \neq \Gamma(K)$

$2g(K) = \Gamma(K)$

$= b_1 (\ldots)$

$= \min \{\text{# necessary bands to obtain a disk}\}$

$\leq \min \{\text{# necessary bands to obtain a disk from } \text{"a" state ori. surface of } D\}$

$= B(K)$. □
Main Result 3 (Ito-Takimura, JKTR2020).

For any alternating knot $K$,

$B(K) = C(K)$ if and only if $C(K) \leq 2g(K)$,

$B(K) = C(K)-1$ if and only if $C(K) > 2g(K)$.
Applications

- Relationship with Jones polynomials
- Relationship with hyperbolic volume bounds
- $u^{-}(K)$ is flype invariant
Corollary 1. Let $V_K(q) = a_n q^n + a_{n+1} q^{n+1} + \cdots + a_{m-1} q^{m-1} + a_m q^m$ be the Jones polynomial of a knot $K$. If $K$ is a prime alternating knot, then

$$C(K) = u^-(K) \leq \begin{cases} 
\min\{\left\lfloor \frac{m-n}{2} \right\rfloor, |a_{n+1}| + |a_{m-1}| \} 
& \text{if } C(K) = \Gamma(K), \\
\min\{\left\lfloor \frac{m-n}{2} \right\rfloor, |a_{n+1}| + |a_{m-1}| + 1 \} 
& \text{if } C(K) \neq \Gamma(K). 
\end{cases}$$

Rmk. $C(K) \leq \min\{\left\lfloor n(K)/2 \right\rfloor, t + 1 \}$. $t$: twisted number
Notation.  

• $D$ be a knot diagram of a hyperbolic link $K$.
• $t(D)$: the twist number of $D$ (Lackenby, 2004),
• $v_3$ the volume of a regular hyperbolic ideal tetrahedron,
• $v_8$ the volume of a regular hyperbolic ideal octahedron.
• $\text{vol}(S^3 \setminus K)$: the volume of knot compliment.
Corollary 2. Let $K$ be a prime alternating hyperbolic knot.

$$v_8(u^-(K) - 3)/2 \leq \text{vol}(S^3 \setminus K) \leq 10v_3(3u^-(K) - 4).$$

Corollary 3. Let $K$ be a hyperbolic knot that is the closure of a positive braid with at least three crossings in each twist region.

$$\text{vol}(S^3 \setminus K) \leq 10v_3(3u^-(K) - 4).$$

Corollary 4. Let $K$ be a prime Montesinos hyperbolic knot.

$$\text{vol}(S^3 \setminus K) \leq 6v_8(u^-(K) - 1).$$
Corollary 5.

\[ C(K) \leq u^-(K) \leq \left\lfloor \frac{n(K)}{2} \right\rfloor \]

(the left inequality holds even if \( K \) is non-prime)

when \( K \) is a prime (alternating or non-alternating) knot \( K \).
Corollary 6. Let $K$ be a prime (alternating or non-alternating) knot and for the twist number $t$, suppose $t \geq 2$.
Then, $C(K) \leq u^-(K) \leq \min\{t, \left\lfloor \frac{n(K)}{2} \right\rfloor \}$ if the diagram has a non-orientable state surface whose Euler characteristic is at least as large as that of the diagram’s Seifert state surface, $C(K) \leq u^-(K) \leq \min\{t + 1, \left\lfloor \frac{n(K)}{2} \right\rfloor \}$ otherwise.
Corollary 7. If $D$ and $D'$ are prime reduced (alternating or non-alternating) knot diagrams that are related by flypes, $u^-(D) = u^-(D')$. 
Corollary 8. Let $K$ be a non-alternating knot having the same prime reduced knot projection as that of an alternating knot diagram of an alternating knot $K^{alt}$. Then,

$$C(K) \leq u^-(K) \leq u^-(K^{alt}) = C(K^{alt}).$$
Next target

• Categorification of $C(K)$. Can we relate $\text{sl}(2)$ homology to crosscap? (cf. HFK determines orientable genera.) This relates to the comment by Prof. J.S Carter in this seminar.

• Can we have more refined/new volume bounds?
Next target

• Categorification of $C(K)$. Can we relate sl(2) homology to crosscap? (cf. HFK determines orientable genera.) This relates to the comment by Prof. J.S. Carter in this seminar.

• Can we have more refined/new volume bounds?