Unknotting operations, crosscap numbers, and volume bounds

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Main Result 1.

Let $C(K)$ be the crosscap number of K.

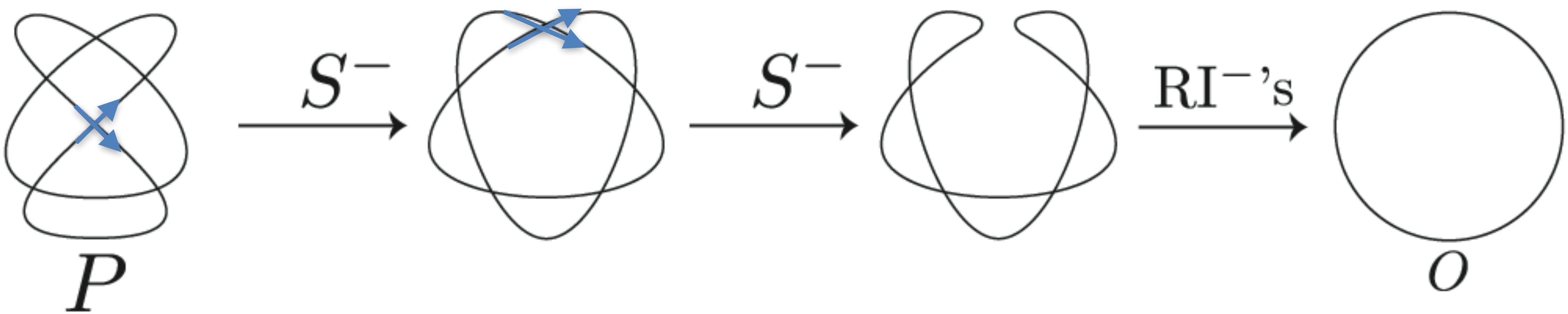
For any prime alternating knot K,

$$C(K) = u^{-}(K).$$

Recalling definition: $u^-(P)$, $u^-(K)$

- $u^-(P)$ is the minimum number of necessary

<u>splices</u> of type S^- among any sequences of S^- and RI^- to obtain O. $u^-(K) := \min_P u^-(P)$.



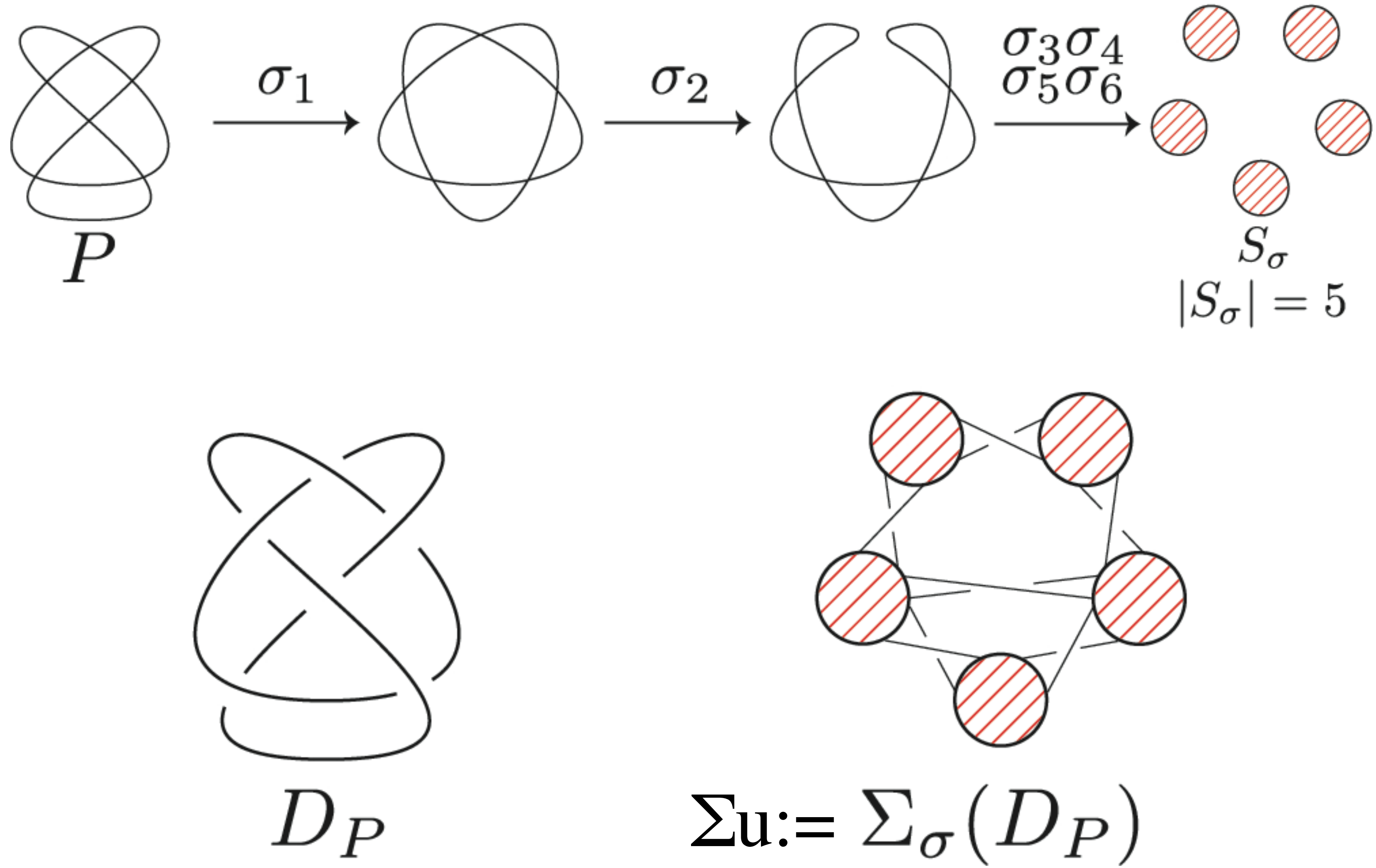
Plan of proof for $u^-(D) \leqq C(K)$

We will compare

Σ_u : a non-orientable state surface realizing $u^-(D)$

with

Σ_{AK} : a surface realizing $C(K)$ or $g(K)$
(Adams-Kindred).

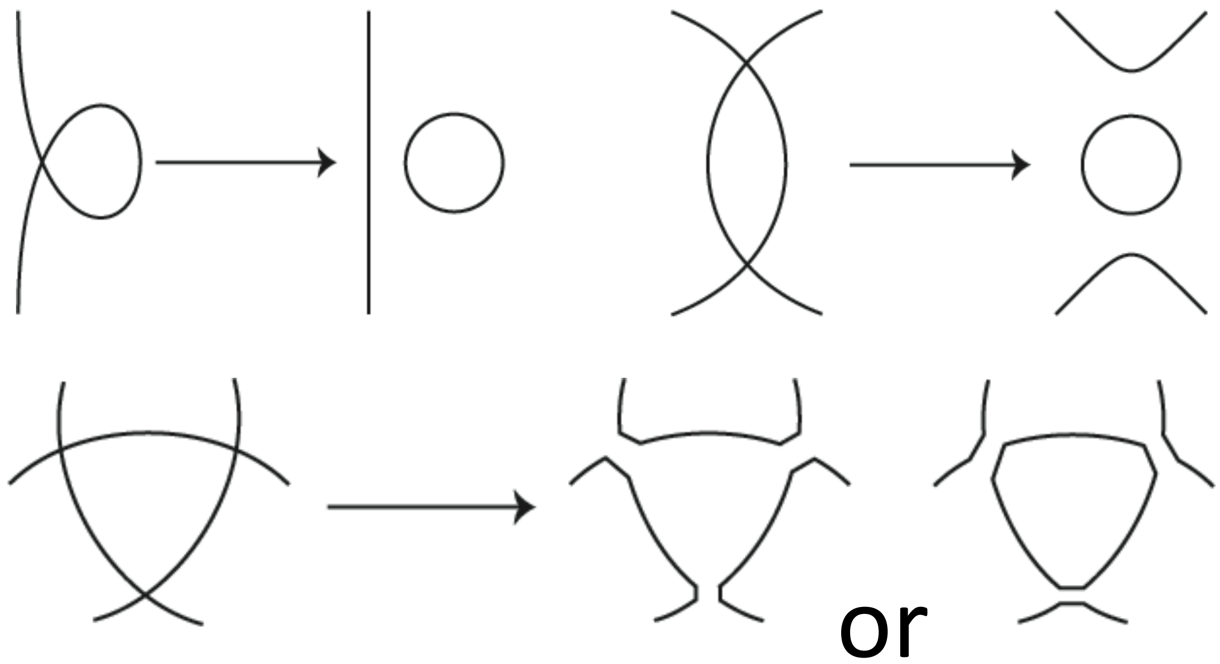


P
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
S_σ
$|S_\sigma| = 5$
D_P
$\Sigma u := \Sigma_\sigma(D_P)$

Construction of Σ_{AK}

Find m-gon of the smallest m and splice as follows

(P has a 3-gon if 3≦ m Eliahou-Harary-Kauffman, 2008)



Notation 1. Σ_{AK} *gives a sequence of splices* $(\sigma_i)_{i=1}^{n(D)}$:

$$D = D_0 \overset{\sigma_1}{\to} D_1 \overset{\sigma_2}{\to} D_2 \overset{\sigma_3}{\to} \cdots \overset{\sigma_{n(D)}}{\longrightarrow} D_{n(D)}$$

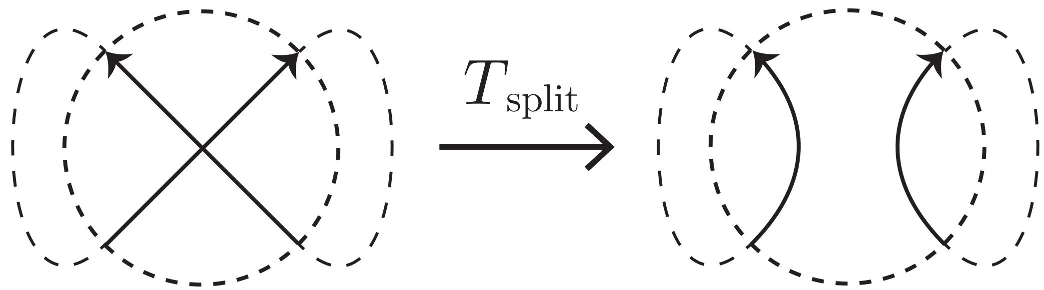
Each orientation of D_i *is of* σ_i. ($=$ *ori.* S^-, S^-_{join}, T_{split}, T_{join}).

It induces

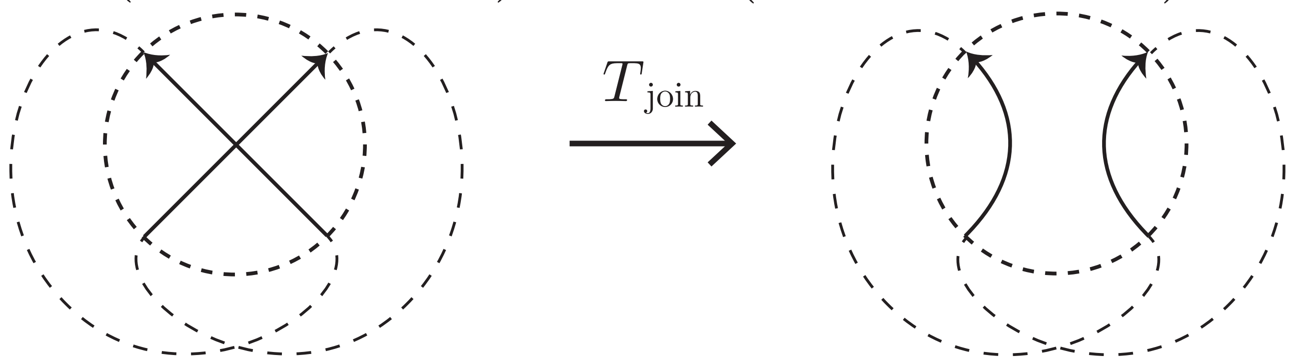
$$CD_D = CD_0 \overset{\sigma_1}{\to} CD_1 \overset{\sigma_2}{\to} CD_2 \overset{\sigma_3}{\to} \cdots \overset{\sigma_{n(D)}}{\longrightarrow} CD_{n(D)}.$$

(CD_D is a Gauss diagram of D; it will be defined.)

Notation 2. • *Oriented* T_{split}, T_{join}. *Seifert splices.*

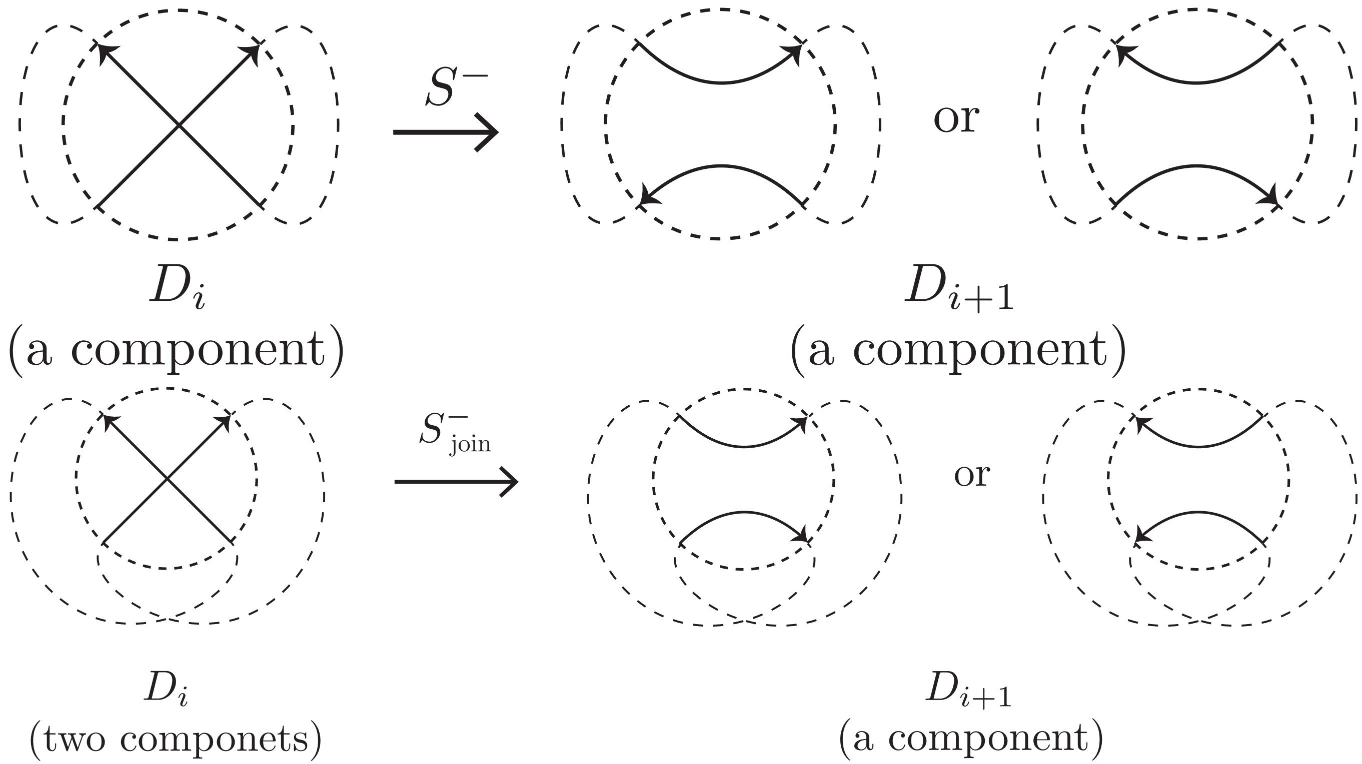


D_i (a component) $\qquad$ D_{i+1} (two componets)



D_i (two componets) $\qquad$ D_{i+1} (a component)

- *Oriented* RI^-. *1st Reidemeister move.*
- *Oriented* S^-, S^-_{join}. *Target orientation must be chosen.*



Property S^-

Claim

Key Lemma

Main Result 1

S^-

Property S^-

Claim

Key Lemma

Main Result 1

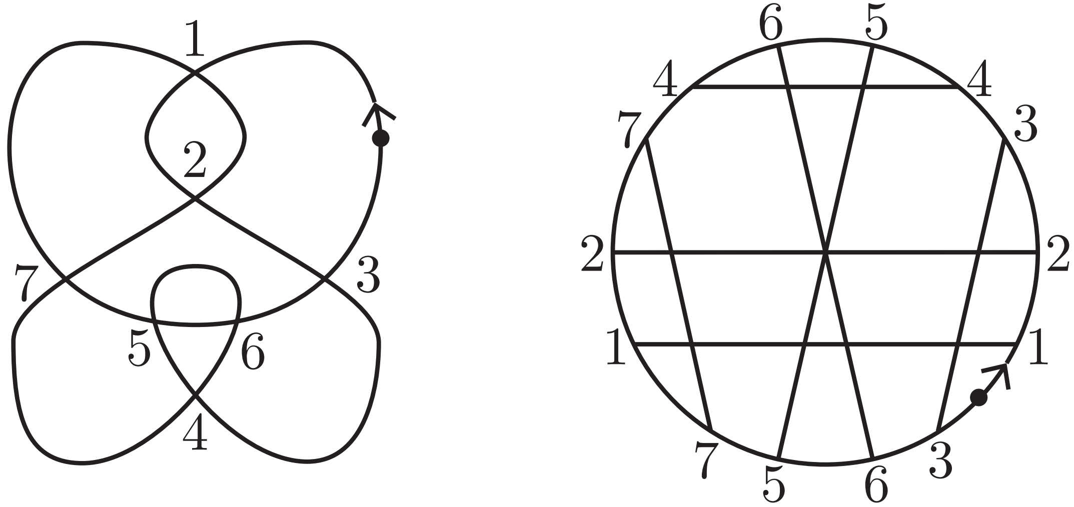
S^-

S^- T T

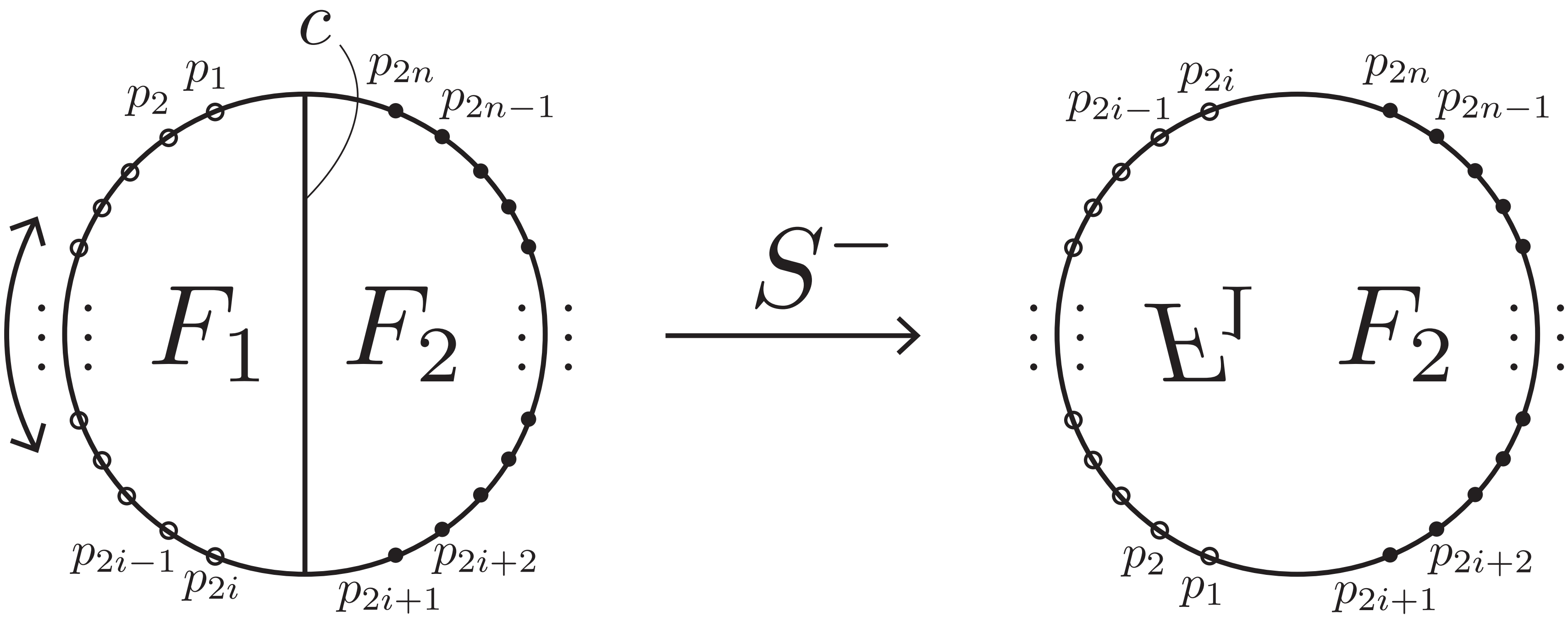
S^- T T

to $S^-S^-S^-$

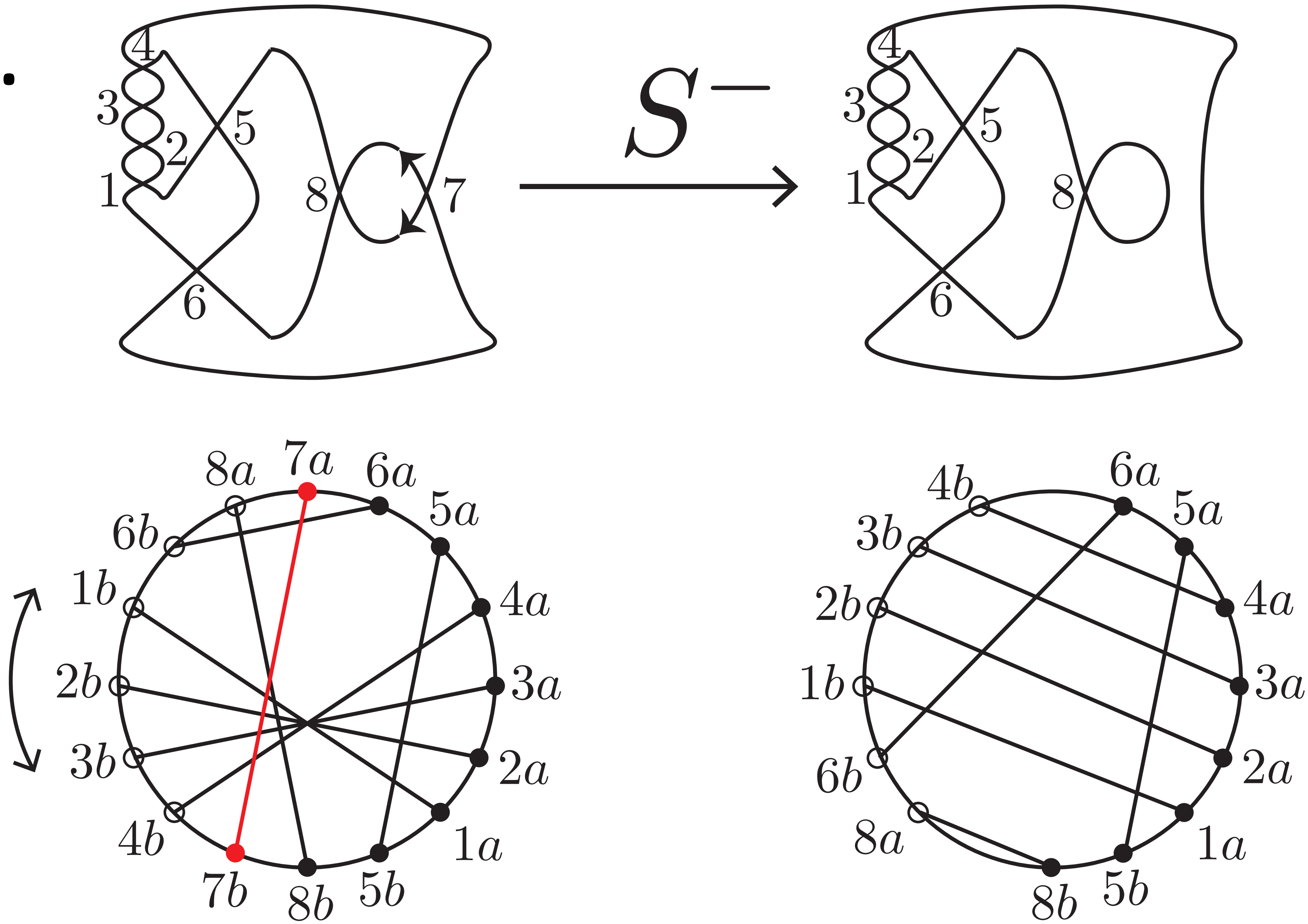
Definition 1. Let D be a knot diagram whose projection is P. Then there is a generic immersion $g : S^1 \to S^2$ such that $g(S^1) = P$. ***It is denoted by*** $\boldsymbol{CD_D}$**.**



Property S⁻. *The behavior of S^- in CD_D is as follows.*
(*The difference of cyclic Gauss words is presented as:*
$cp_1p_2\ldots p_{2i}cp_{2i+1}\ldots p_{2n} \longrightarrow p_{2i}p_{2i-1}\ldots p_1p_{2i+1}p_{2i+2}\ldots p_{2n}$.)



e.g.



4
3
2
5
1
8
7
6
S^-
4
3
2
5
1
8
6
8a
7a
6a
5a
6b
1b
4a
2b
3a
3b
2a
4b
1a
7b
8b
5b
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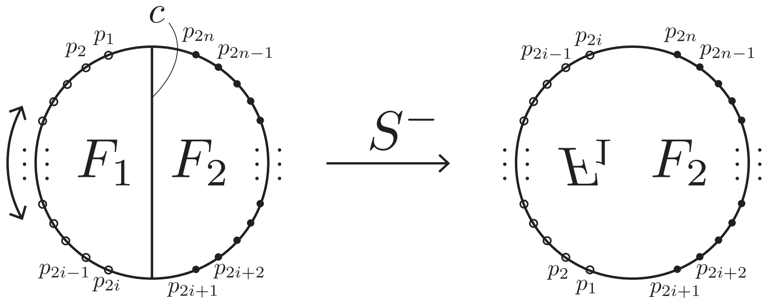
8a
1a
8b
5b



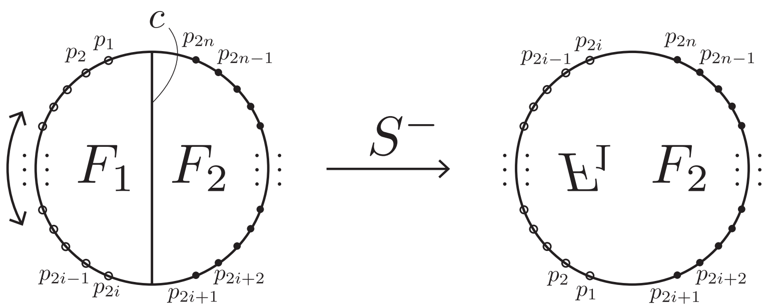
Property S⁻

S⁻

Claim

Key Lemma

Main Result 1



Property S⁻

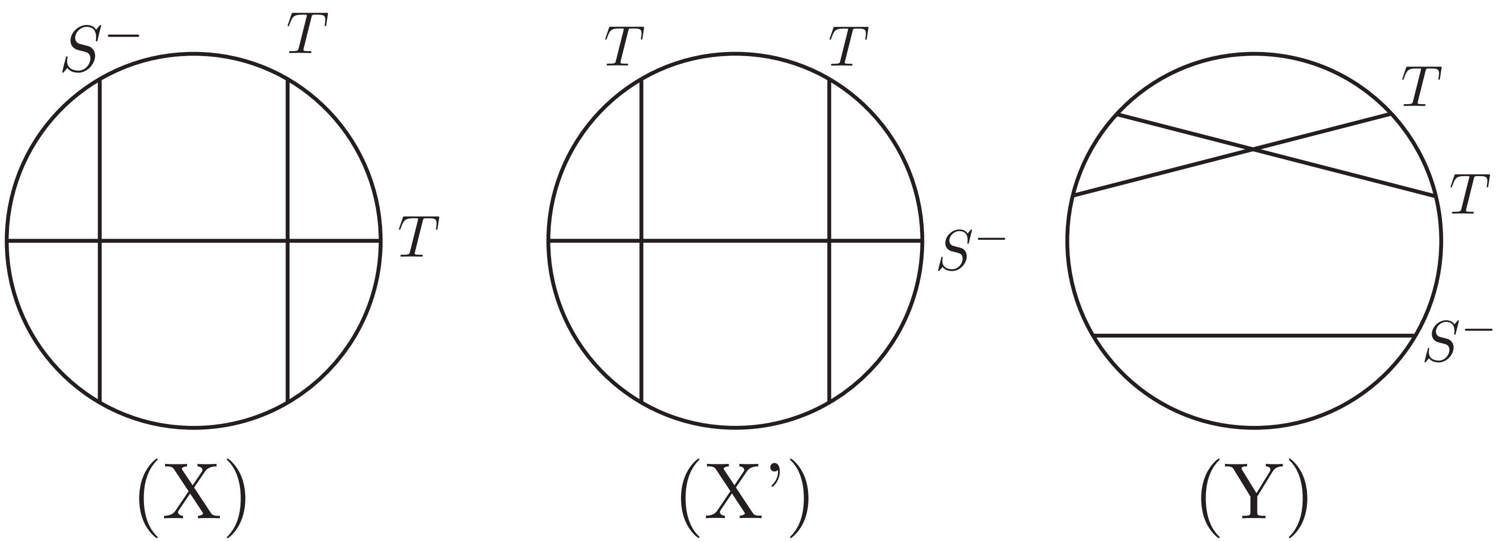
S⁻

Claim

Key Lemma

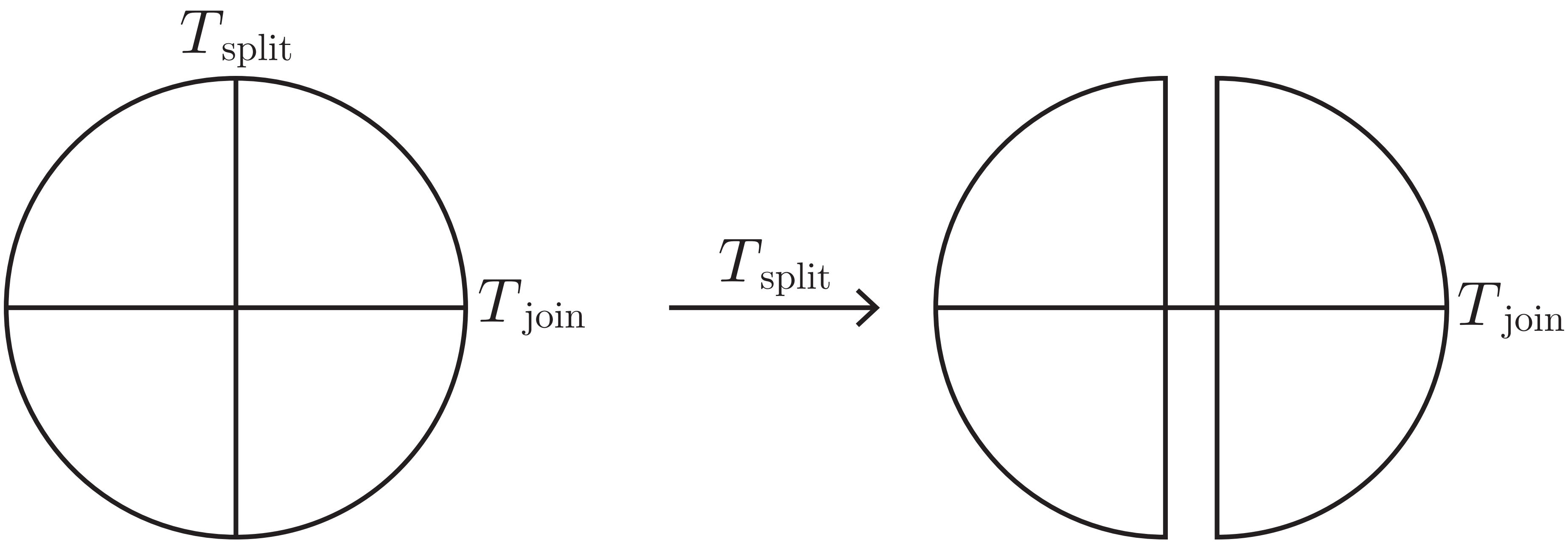
Main Result 1

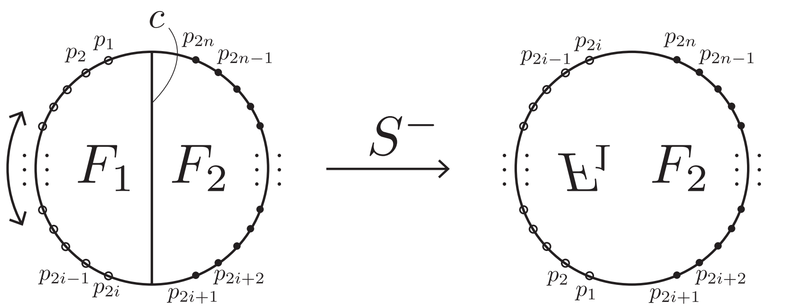
Claim. *Suppose that* $(\sigma_i)_{i=1}^{n(D)}$ *satisfies* $\sigma_1 = S^-$, $\sigma_2 = T_{\text{split}}$, *and* $\sigma_3 = T_{\text{join}}$. *Then, the three chords in* CD_D *corresponding to* σ_1, σ_2, *and* σ_3 *are as in*

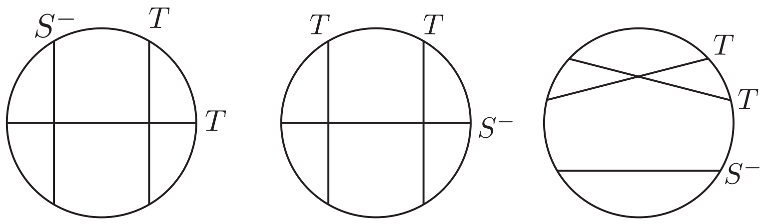


Observation 1

Component-preserving successive "T T" should have a chord intersection.







(X) (X') (Y)

Property S^-

S^-

Claim

S^- T T

Key Lemma

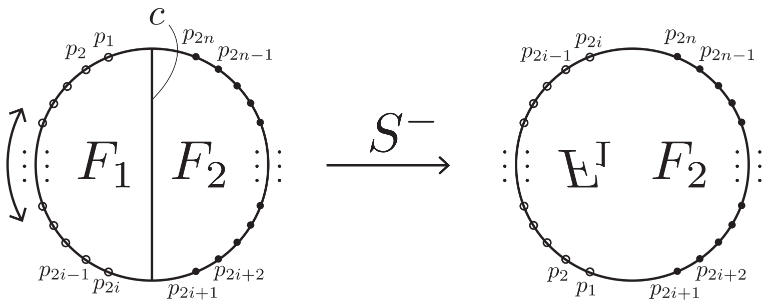
Main Result 1

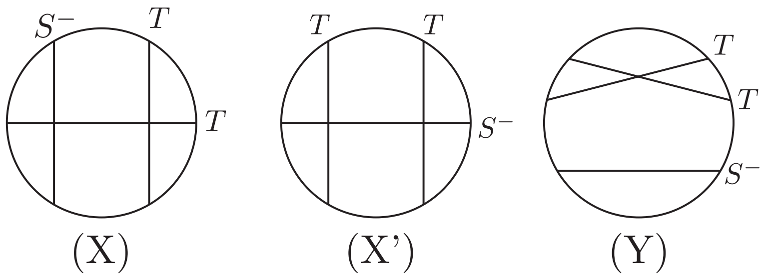
Key Lemma . *Let D be a prime (alternating or non-alternating) knot diagram with exactly $n(D)$ (> 1) crossings with $\sigma_i \neq \mathrm{RI}^-$ $(\forall i)$. Suppose that $\sigma_1 = S^-$ and that $(\sigma_i)_{i=2}^{n(D)}$ includes at least one T_{join}, S^-_{join}, or S^-.*

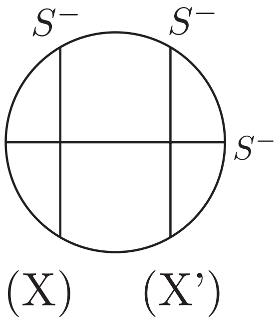
Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ such that $\sigma'_1 = S^-$ and $\sigma'_2 = S^-$, and $\sigma'_i \neq \mathrm{RI}^-$ $(\forall i)$.

Key Lemma . *Let D be a prime (alternating or non-alternating) knot diagram with exactly $n(D)$ (>1) crossings with $\sigma_i \neq \mathrm{RI}^-$ $(\forall i)$. Suppose that $\sigma_1 = S^-$ and that $(\sigma_i)_{i=2}^{n(D)}$ includes at least one T_{join}, S^-_{join}, or S^-.*
Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ such that $\sigma'_1 = S^-$ and $\sigma'_2 = S^-$, and $\sigma'_i \neq \mathrm{RI}^-$ $(\forall i)$.

Roughly speaking, suppose that AK-sequence starts from one S⁻. if "join" or more S⁻ appears in the seq., S⁻ … → S⁻ S⁻….by reordering.







Property S^- — **S^-**

Claim — **S^- T T**

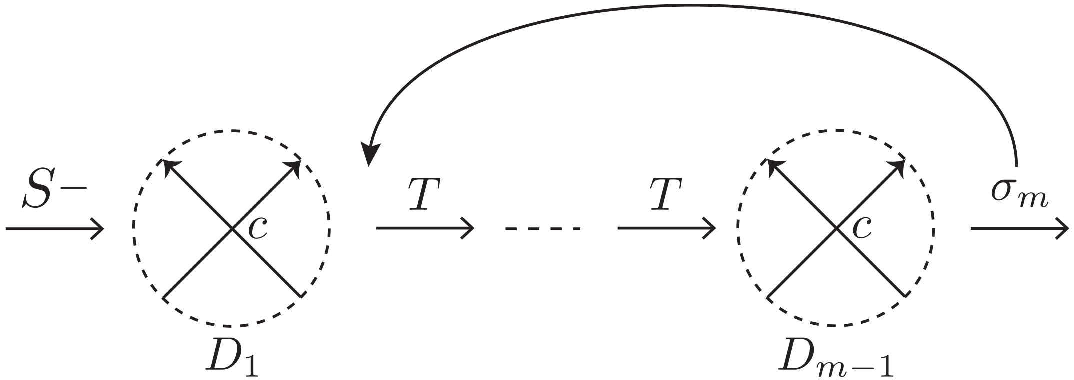
Key Lemma — **S^- T T to $S^-S^-S^-$**

Main Result 1

Proof of Key Lemma

Case (1): $(\sigma_i)_{i=2}^{n(D)}$ includes at least one S^- or S^-_{join}.

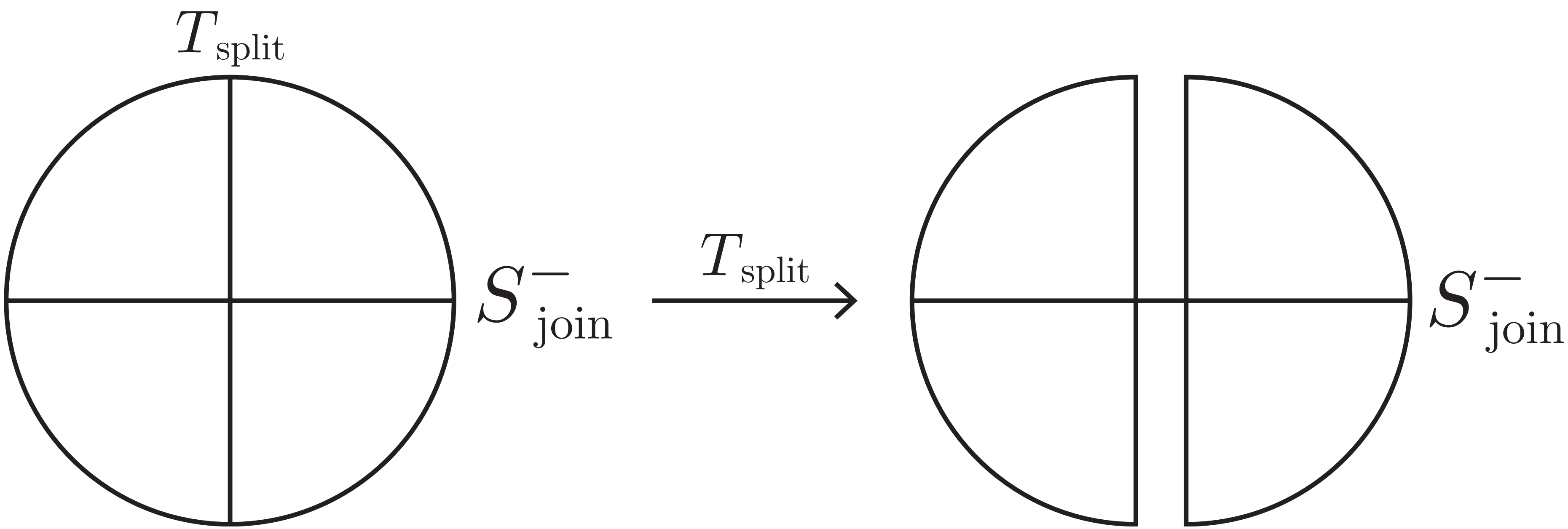
$S^-T\cdots TS^-\cdots$, or $S^-T\cdots TS^-_{\mathrm{join}}\cdots$. Moving σ_m ($= S^-$ or S^-_{join}) to σ'_2,



we obtain $S^-S^-\cdots$.

Observation 1'

Component-preserving pair "T S" should have a chord intersection.



Proof of Key Lemma

Case (2): $(\sigma_i)_{i=2}^{n(D)}$ includes no splice S^- and no splice S^-_{join}, but includes a splice T_{join}.

We have reordering:

$$S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \to S^- T_{\text{split}} T_{\text{join}} T \cdots T.$$

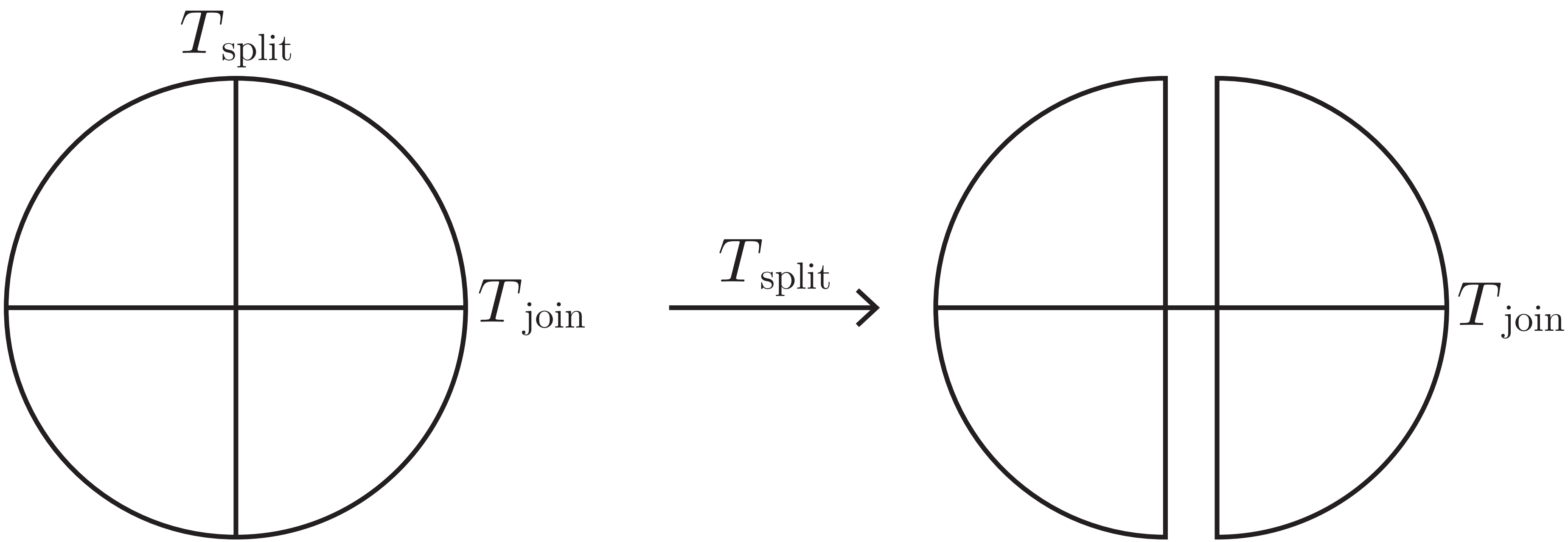
- Case: either (X) or (X') is included:

By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^- \ldots$

- Case: there is no (X) and no (X'), but (Y) appears:

Observation 1

<u>Component-preserving pair “T T” should have a chord intersection.</u>



Proof of Key Lemma

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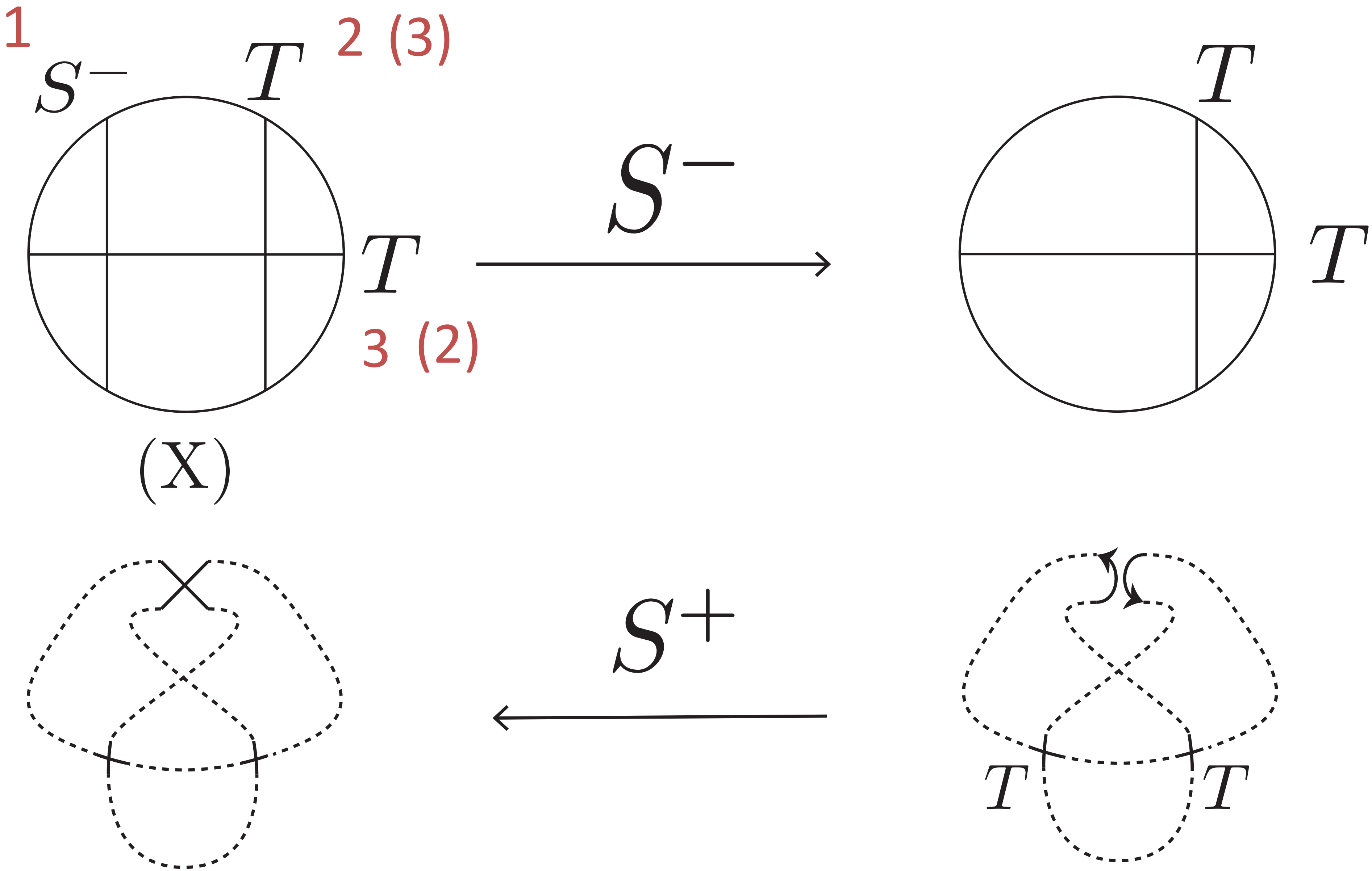
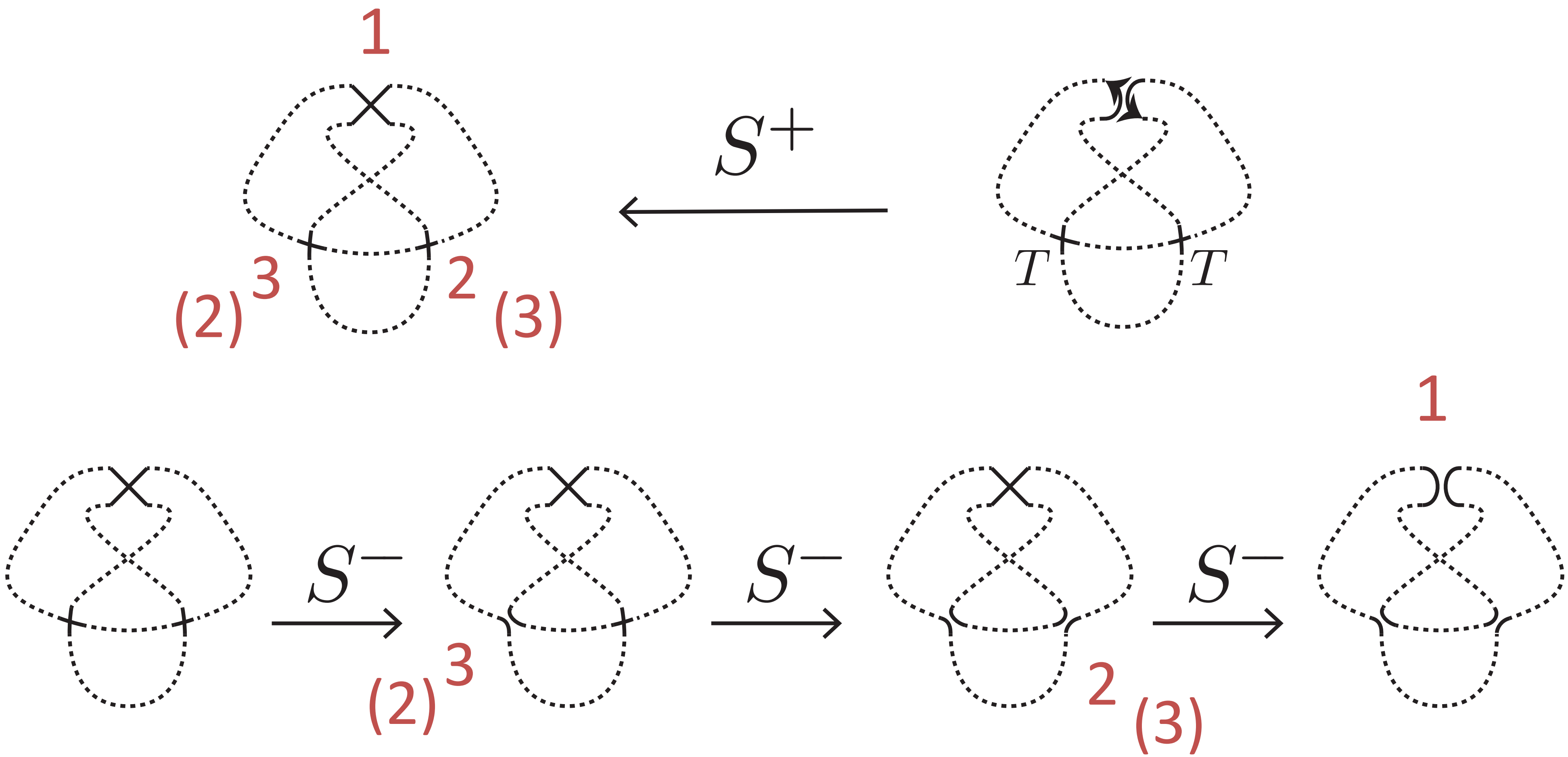
- Case: there is no (X) and no (X'), but (Y) appears:


1
S^-
T
2 (3)
T
3 (2)
(X)
S^-
T
T
S^+
T
T



<u>Reordering: 123 —> 321 or 231</u>

Proof of Key Lemma

Case (2): $(\sigma_i)_{i=2}^{n(D)}$ includes no splice S^- and no splice S^-_{join}, but includes a splice T_{join}.

We have reordering:

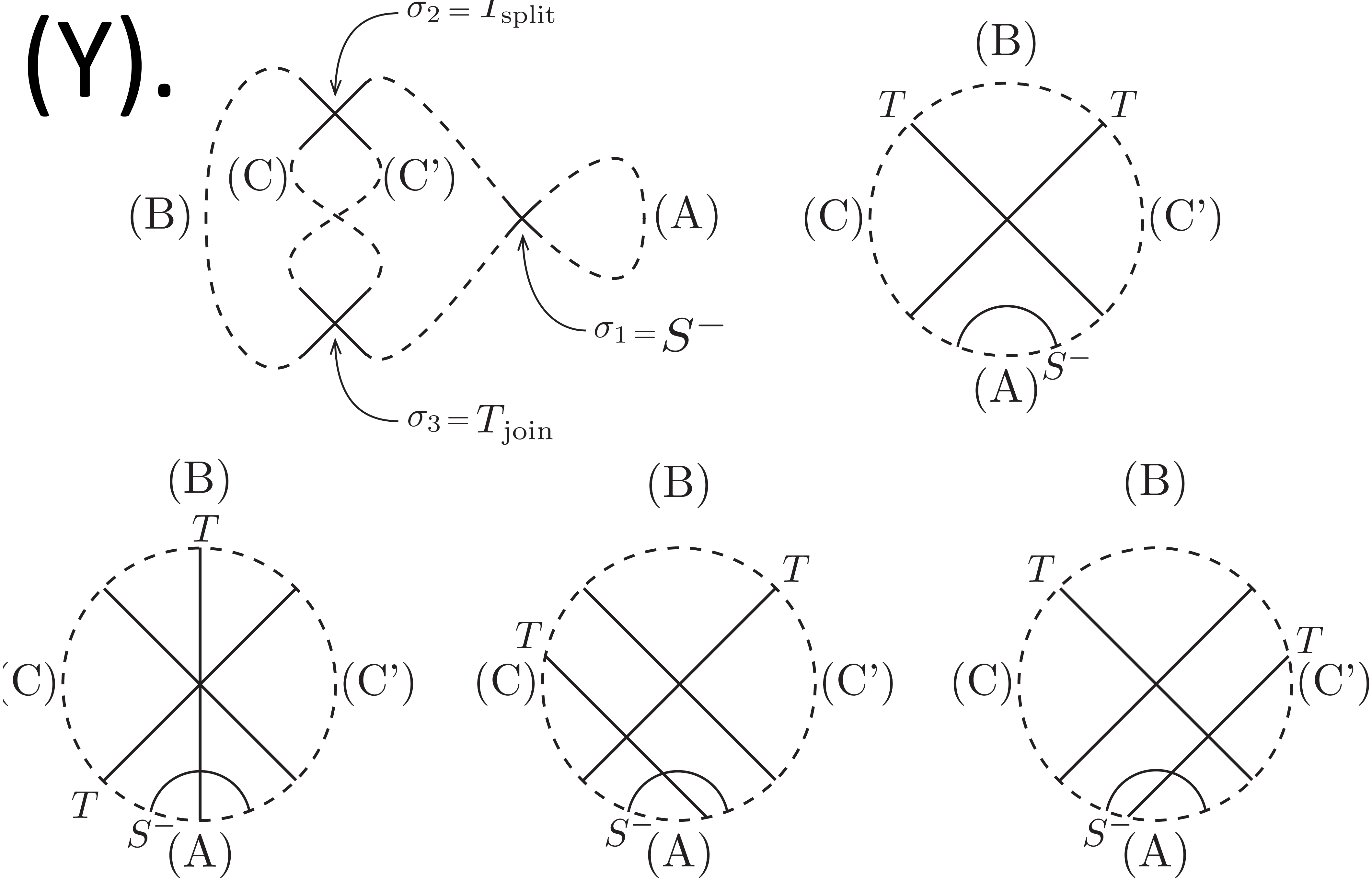
$$S^- T_{\mathrm{split}} \cdots T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T \to S^- T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T.$$

- Case: either (X) or (X') is included:

By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^- \ldots$ It's the highest point of the proof, we'll go to the next slide!

- Case: there is no (X) and no (X'), but (Y) appears:

(Y).



Proof of Key Lemma

Case (2): $(\sigma_i)_{i=2}^{n(D)}$ includes no splice S^- and no splice S^-_{join}, but includes a splice T_{join}.

We have reordering:

$$S^- T_{\mathrm{split}} \cdots T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T \to S^- T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T.$$

- Case: either (X) or (X') is included:

By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^- \ldots$

- Case: there is no (X) and no (X'), but (Y) appears:

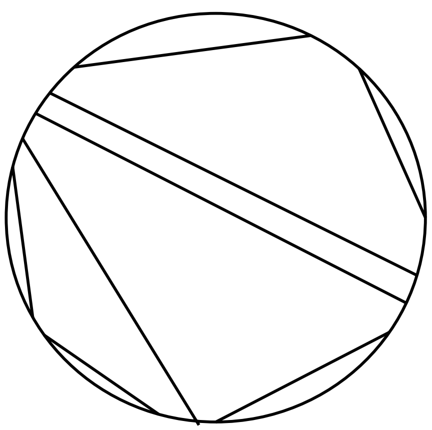
By primeness, (X) should be included $\to$ contradiction. $\square$

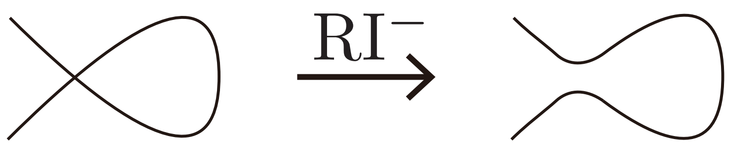
Applying Key Lemma the sequence of splices repeatedly, we have:

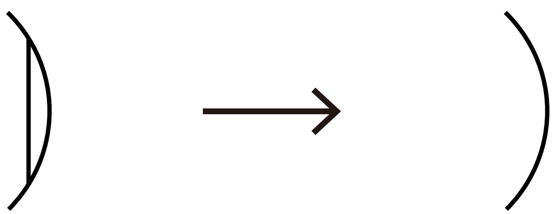
$$S^{-}\cdots S^{-}T_{\mathrm{split}}\cdots T_{\mathrm{split}}$$

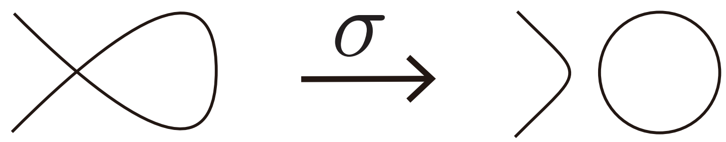
from Σ_{AK}.

Here, in this seq., every T_{split} splits a monogon since there is no chord intersection after $S^{-}S^{-}\ldots S^{-}$ applies.



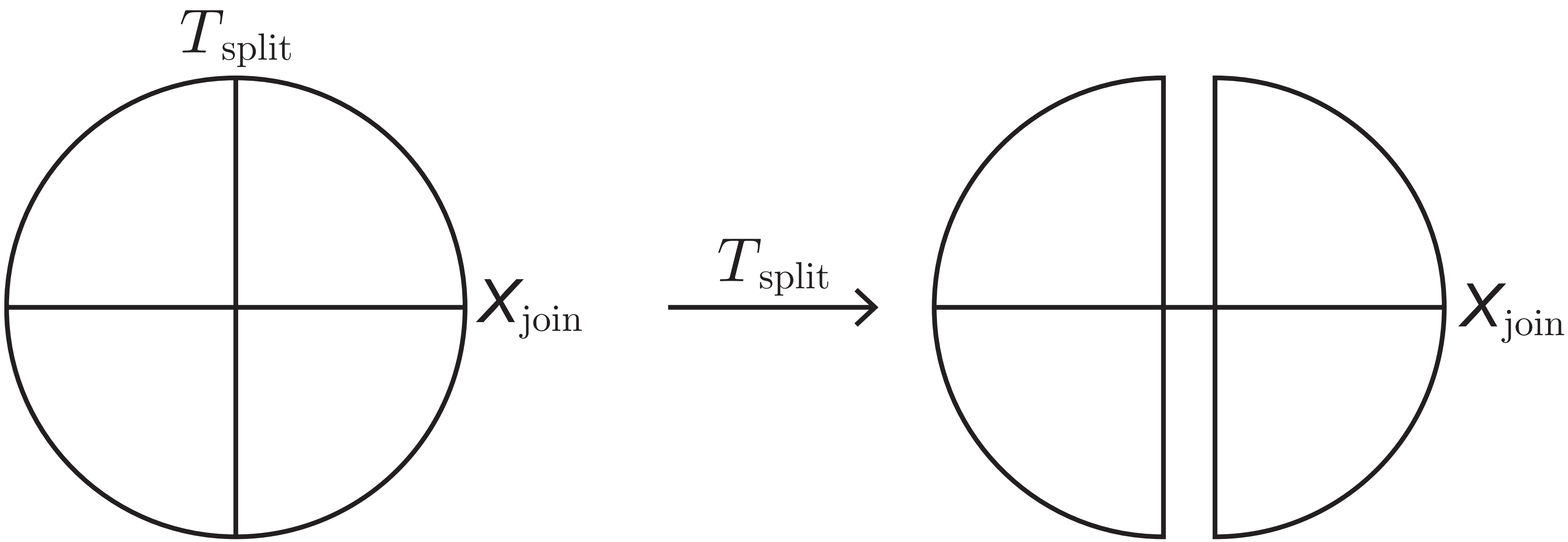


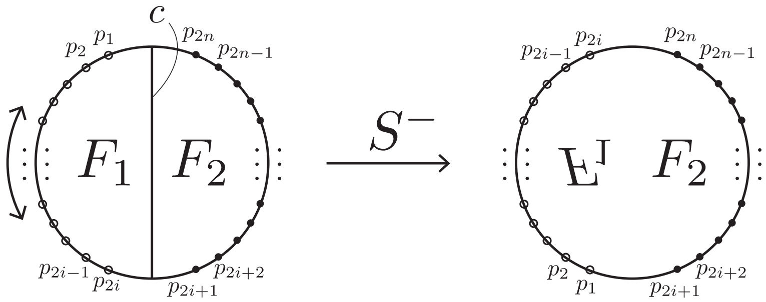


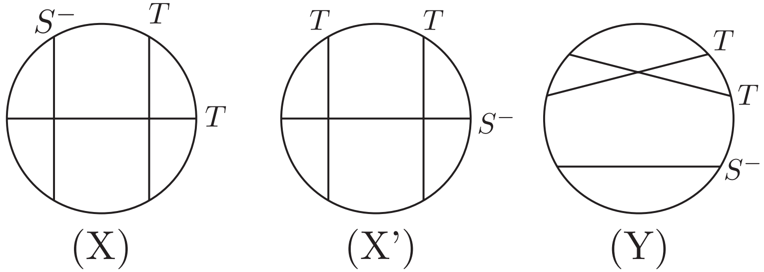


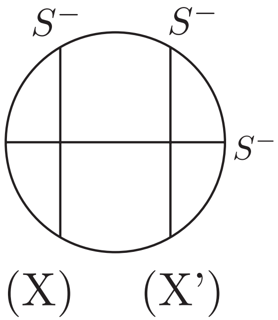
Observation 1''

<u>Any component-preserving pair "T X" should have a chord intersection.</u>









Property S^- — **S^-**

Claim — **S^- T T…**

Key Lemma — **S^- T T… to $S^-S^-S^-$**

Main Result 1

$u^-(D) = C(K)$

Finalizing Proof of Main Result 1 (lower bound)

Case Σ_{AK} is a **non-orientable** surface with the maximal Euler characteristic χ. (**Note**: the seq. has S^-; any $\sigma_i \neq \mathrm{RI}^-$.)
Thus, by **Key Lemma**, this seq. realizes $u^-(D)$ by reordering.

$$S^- S^- \dots S^- T_{\text{split}} T_{\text{split}} \dots T_{\text{split}}.$$

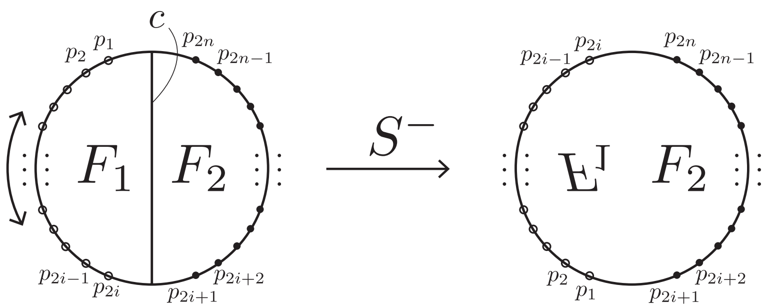
The reordering process implies Observation 2.

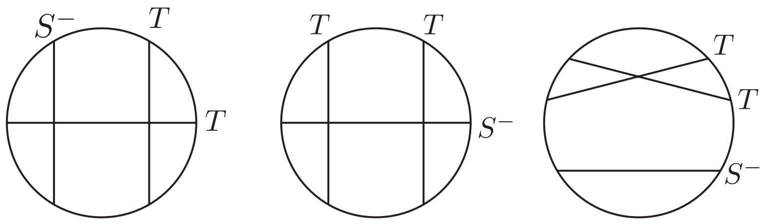
Observation 2. *Each reordering may cause:*

$$T_{\text{split}}, T_{\text{join}} \leftrightarrow S^-, S^- \qquad \textit{or} \qquad T_{\text{split}}, S^-_{\text{join}} \leftrightarrow S^-, S^-.$$

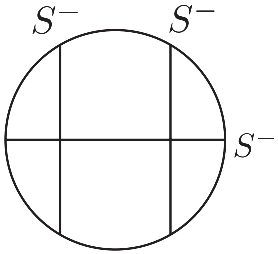
Thus,

$$\begin{aligned} 1 - u^-(D) &= 1 - \sharp\{S^- \text{in seq.}\} \\ &= 1 - 2\sharp T_{\text{join}} - 2\sharp S^-_{\text{join}} - \sharp S^- \\ &= 1 + (\sharp T_{\text{split}} - \sharp T_{\text{join}} - \sharp S^-_{\text{join}}) - n(D) \\ &= \chi(\Sigma_{AK}) = 1 - C(K). \end{aligned}$$





(X) (X') (Y)



(X) (X')

$u^-(D) = C(K)$

Property S^- — **S^-**

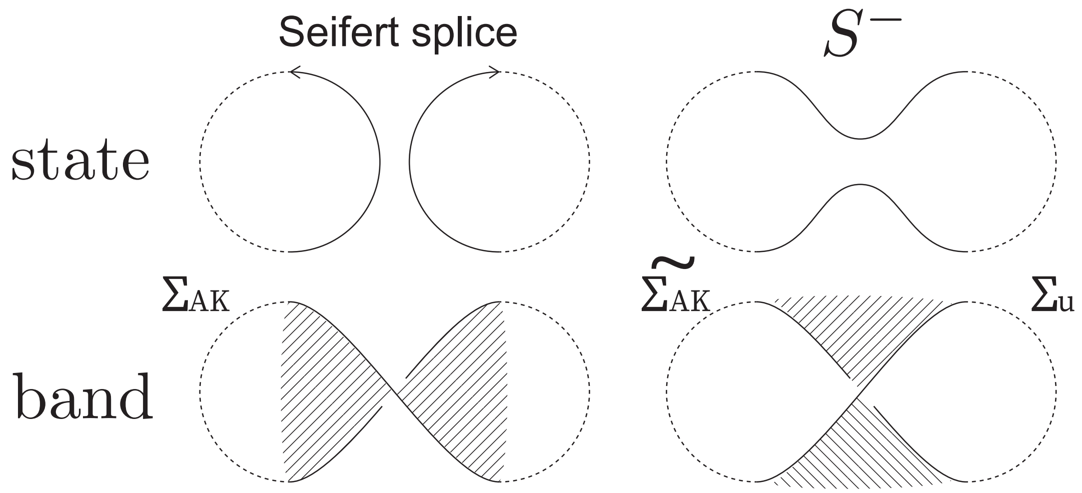
Claim — **S^- T T...**

Key Lemma — **S^- T T... to $S^-S^-S^-$**

Main Result 1

Finalizing Proof of Main Result 1 (lower bound)

Case Σ_{AK} is a orientable surface with the maximal Euler characteristic. **Note**: $2g(K) < C(K) \Leftrightarrow C(K) = 2g(K) + 1$. It **returns to the non-orientable case** since χ $(= 1 - 2g(K))$ is changed into $1 - (2g(K) + 1)$ $(= 1 - C(K))$ by the replacement:



Then for any prime alternating knot diagram D,

$$u^-(K) \leq \min_D u^-(D) = C(K).$$

Recalling that $C(K) \leq u^-(K)$, it completes the proof. $\square$

Ito-Takimura, 2018, arXiv: 2008.11061

By the argument of this proof, we have:

Main Result 2

For any knot K, if there exists a state realizing the maximal Euler characteristic,

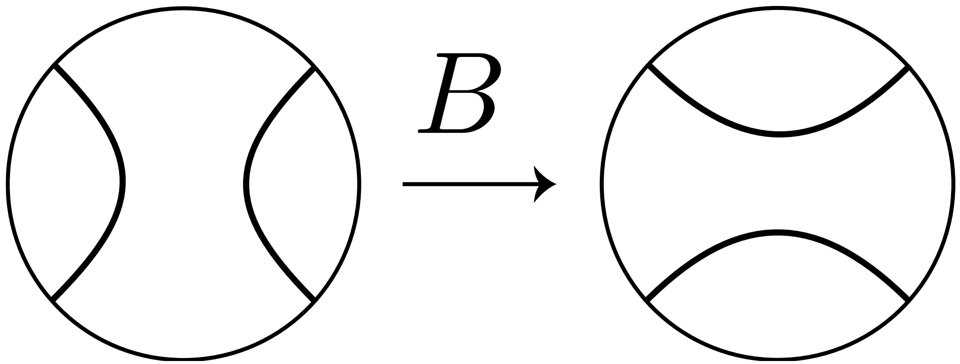
$$u^{-}(K) = C(K).$$

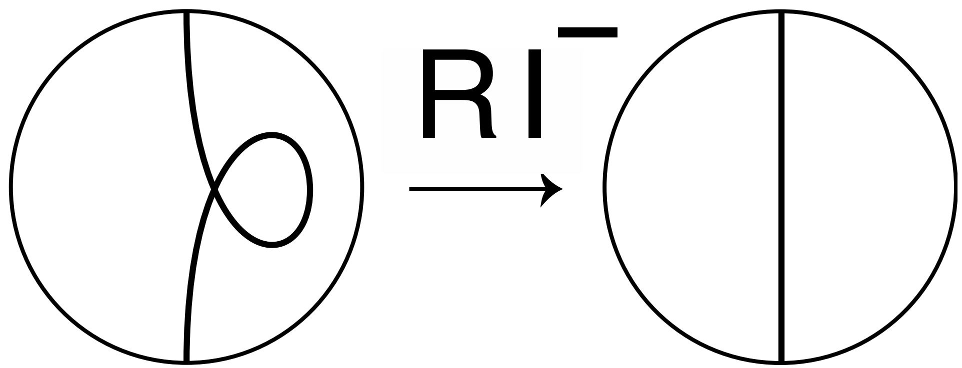
Proof of Main result 3 (Band Surgery)

Merit: simpler proof, non-depending on primeness.

Definition(B(D))

Let D be an alternating knot diagram.
B(D) is the minimum number of necessary band surgeries B among any sequences of B and RI^- to obtain O. Let $B(K) = \min_D B(D)$.





Main Result 3 (Takimura-I., JKTR, 2020)

$\Gamma(K)$: min of 1st Betti num. of alt. knot K.

(1) $C(K)=B(K) \Leftrightarrow C(K) = \Gamma(K)$.

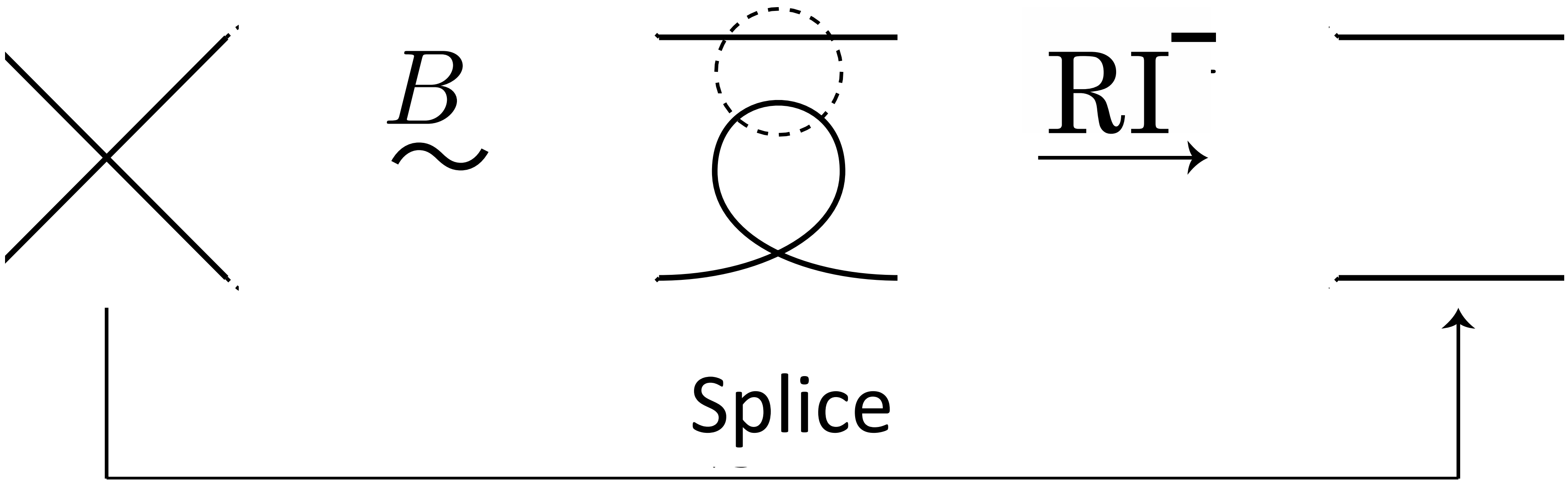
(2) $C(K) = B(K) +1 \Leftrightarrow C(K) \neq \Gamma(K)$.

(3) $B(K \# K') = B(K) + B(K')$.

Proof. There exists a state, i.e. a family of splices, implying a spanning surface with the maximal Euler characteristic for alter. knot. Thus,

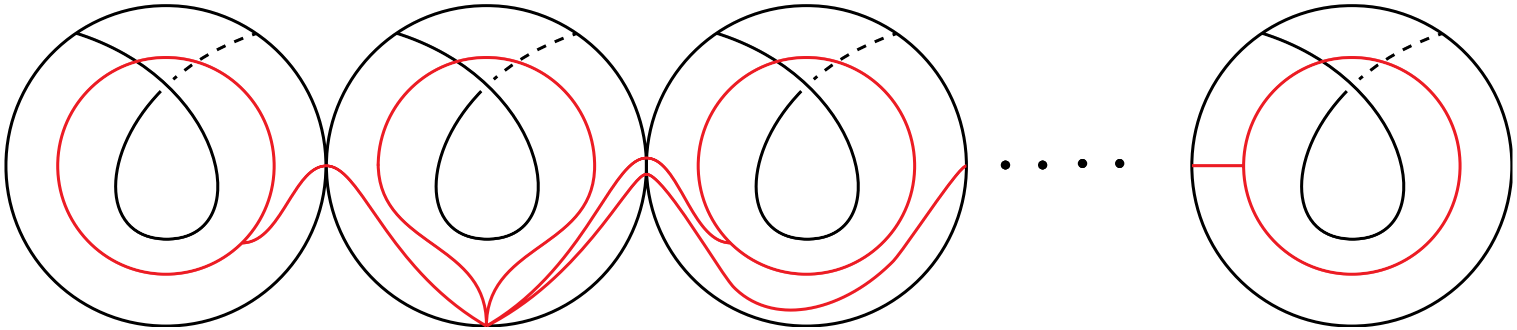
$$B(K) \leqq \Gamma(K).$$

$\Gamma(K)$ is 1st Betti num. $= \min\{C(K), 2g(K)\}$.



Proof $\Gamma(K) \leqq B(K)$ for Case $C(K) = \Gamma(K)$

$C(K)$ $(=\Gamma(K))$

$= b_1$ ()

= min {# necessary bands to obtain a disk}

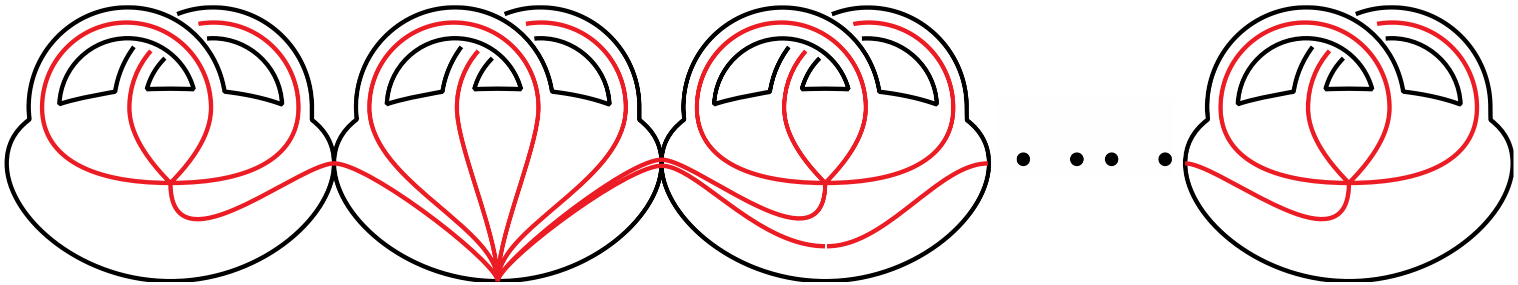
≦ min {# necessary bands to obtain a disk

from “a” state non-ori. surface of D}

= B(K).

Proof: Case $C(K) \neq \Gamma(K)$

$2g(K)$ $(=\Gamma(K))$

$= b_1$ ()

= min {# necessary bands to obtain a disk}

≦ min {# necessary bands to obtain a disk

from "a" state ori. surface of D}

$= B(K)$. □

Another expression of Main Result 3.

<u>Main Result 3 (Ito-Takimura, JKTR2020).</u>

For any alternating knot K,

B(K) = C(K) if and only if C(K) $\leqq$ 2g(K),
B(K) = C(K)-1 if and only if C(K) > 2g(K).

Applications

- Relationship with Jones polynomials
- Relationship with hyperbolic volume bounds
- $u^{-}(K)$ is flype invariant

Corollary 1. *Let* $V_K(q) = a_n q^n + a_{n+1} q^{n+1} + \cdots + a_{m-1} q^{m-1} + a_m q^m$ *be the Jones polynomial of a knot* K*. If* K *is a prime alternating knot, then*

Tait Conj. Kauffman, Murasugi, Thistlethwaite (1987)

Dasbach-Lin (2007)

$$C(K) = u^-(K) \leq \begin{cases} \min\{\lfloor \frac{m-n}{2} \rfloor, |a_{n+1}| + |a_{m-1}|\} \\ \textit{if } C(K) = \Gamma(K), \\ \min\{\lfloor \frac{m-n}{2} \rfloor, |a_{n+1}| + |a_{m-1}| + 1\} \\ \textit{if } C(K) \neq \Gamma(K). \end{cases}$$

Rmk. $C(K) \leq \min\{\lfloor n(K)/2 \rfloor, t+1\}$.   t: twisted number

Murakami-Yasuhara (1997)

Kalfagianni-Lee (2016)

Notation. • *D be a knot diagram of a hyperbolic link K.*

- *$t(D)$: the twist number of D (Lackenby, 2004),*
- *v_3 the volume of a regular hyperbolic ideal tetrahedron,*
- *v_8 the volume of a regular hyperbolic ideal octahedron.*
- *$\mathrm{vol}(S^3 \setminus K)$: the volume of knot compliment.*

Agol-Strom-Thurston

Corollary 2. *Let K be a prime alternating hyperbolic knot.*

$$v_8(u^-(K)-3)/2 \leq \mathrm{vol}(S^3 \setminus K) \leq 10v_3(3u^-(K)-4).$$

Futer-Kalfagianni-Purcell

Corollary 3. *Let K be a hyperbolic knot that is the closure of a positive braid with at least three crossings in each twist region.*

$$\mathrm{vol}(S^3 \setminus K) \leq 10v_3(3u^-(K)-4).$$

Futer-Kalfagianni-Purcell

Corollary 4. *Let K be a prime Montesinos hyperbolic knot.*

$$\mathrm{vol}(S^3 \setminus K) \leq 6v_8(u^-(K)-1).$$

Corollary 5.

$$C(K) \leq u^-(K) \leq \left\lfloor \frac{n(K)}{2} \right\rfloor$$

(the left inequality holds even if K is non-prime)

when K is a prime (alternating or non-alternating) knot K.

Corollary 6. *Let K be a prime (alternating or non-alternating) knot and for the twist number t, suppose $t \geq 2$. Then, $C(K) \leq u^{-}(K) \leq \min\{t, \left\lfloor \frac{n(K)}{2} \right\rfloor\}$ if the diagram has a non-orientable state surface whose Euler characteristic is at least as large as that of the diagram's Seifert state surface, $C(K) \leq u^{-}(K) \leq \min\{t+1, \left\lfloor \frac{n(K)}{2} \right\rfloor\}$ otherwise.*

Corollary 7. *If D and D' are prime reduced (alternating or non-alternating) knot diagrams that are related by flypes,* $u^-(D) = u^-(D')$.

Tait Flyping Conj. Menasco-Thistlethwaite

Corollary 8. *Let K be a non-alternating knot having the same prime reduced knot projection as that of an alternating knot diagram of an alternating knot K^{alt}. Then,*

$$C(K) \leq u^{-}(K) \leq u^{-}(K^{alt}) = C(K^{alt}).$$

Next target

- Categorification of C(K). Can we relate sl(2) homology to crosscap? (cf. HFK determines orientable genera.) This relates to the comment by Prof. J.S Carter in this seminar.
- Can we have more refined/new volume bounds ?

Next target

Thank you for your attention!

- Categorification of C(K). Can we relate sl(2) homology to crosscap? (cf. HFK determines orientable genera.) This relates to the comment by Prof. J.S Carter in this seminar.
- Can we have more refined/new volume bounds ?