On a cobordism realizing crossing change on sl(2) tangle homology and a categorified Vassiliev skein relation

Joint work with Jun Yoshida (U. Tokyo)

Noboru Ito (NIT, Ibaraki College)

EKOOK Seminar (2020.9.9)

(Slide is made using Yoshida's sources w/ small modification.)

Information.

Recoding & slide of Yoshida's talk can be seen in web:
L. H. Kauffman's web, "Quantum Topology Seminar".

Question.

What is *crossing change* on Khovanov homology?

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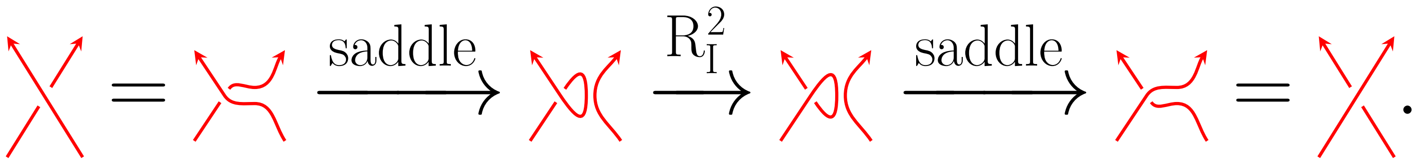
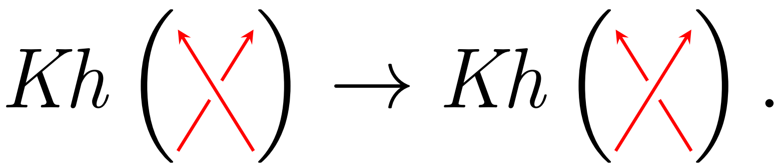
Traditionally, using *link cobordisms*, we have



It will induce a map



However, the degree does not preserve even if the crossing change does not change the knot.

Modified Question.

How to realize *crossing change* on Khovanov homology preserving the degree of Jones polynomial toward categorified Vassiliev theory?

Modified Question / Answer

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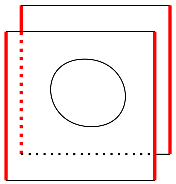
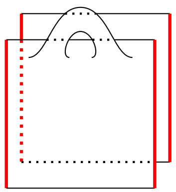
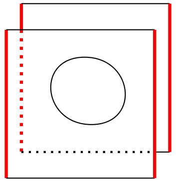
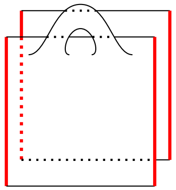
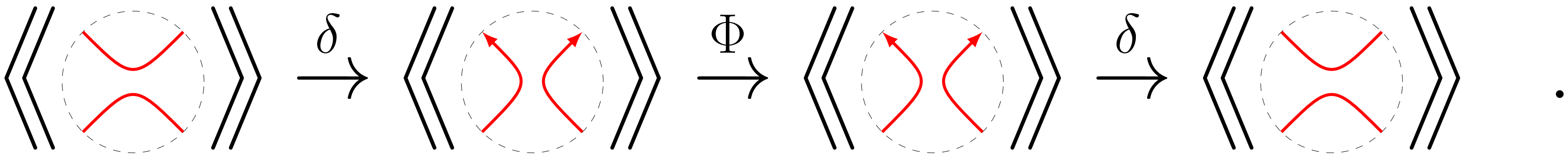
Answer: $\Phi \coloneqq$  $-$ .

Modified Question / Answer

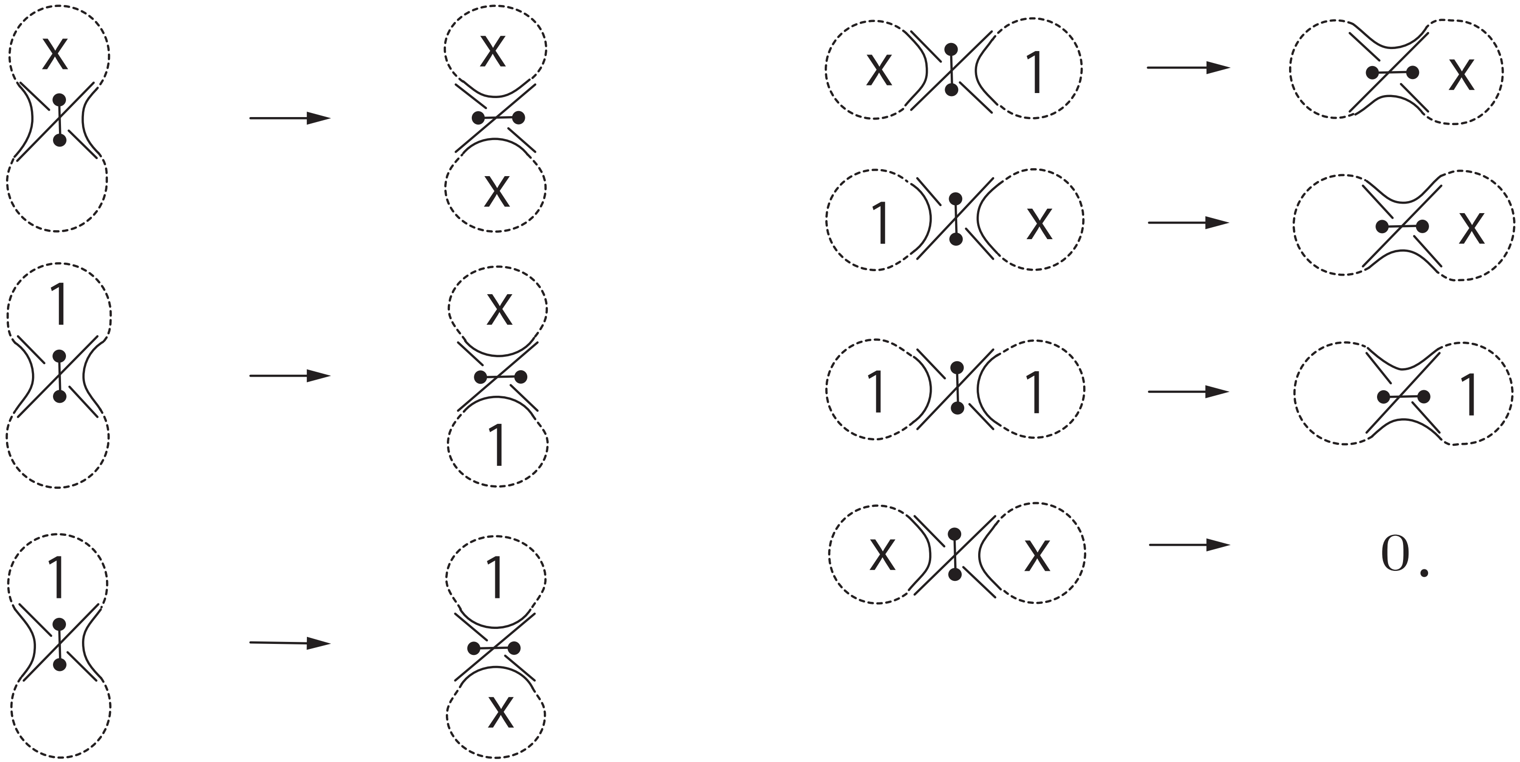
How to realize *crossing change* on Khovanov homology preserving the degree of Jones polynomial toward categorified Vassiliev theory?

Answer: $\Phi :=$  $-$ ; **Note:**

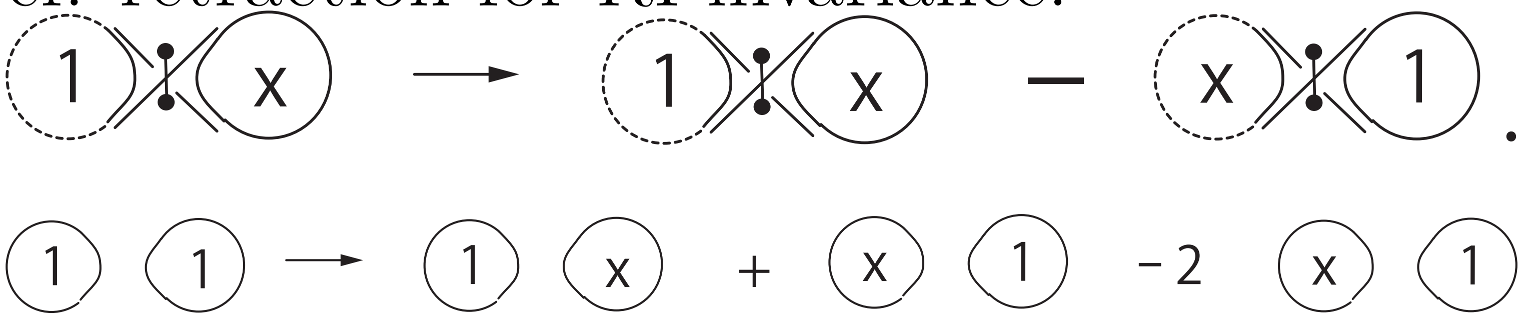
Lemma 1. *The following is a* 0-sequence*; i.e. the compositions of adjacent two morphisms vanish:*



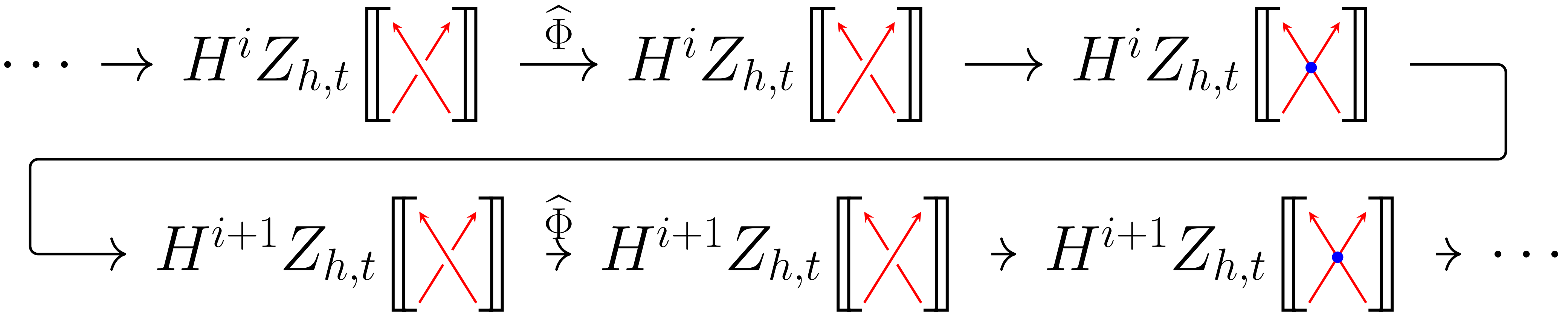
cf. TQFT $Z_{0,0}$: $\mathrm{Cob}_2 \to \mathrm{Mod}_k$; $Z_{0,0}(S^1) = A$: Frobenius:



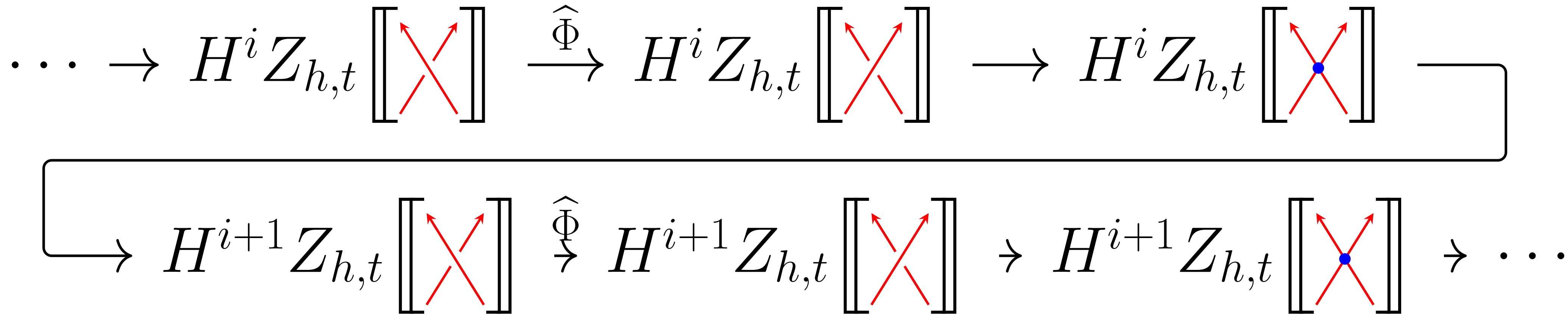
cf. retraction for RI invariance:

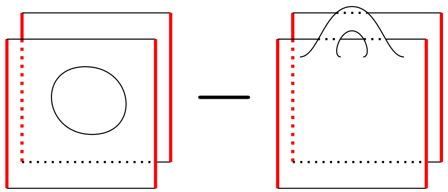


Corollary 1 (Yoshida, I. Categorified Vassiliev skein relation)**.** *For every* $h, t \in k$, *there is a* long exact sequence



Corollary 1 (Yoshida, I. Categorified Vassiliev skein relation)**.** For every $h, t \in k$, there is a *long exact sequence*



$\Phi :=$ .

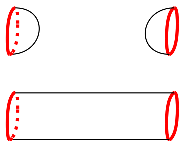
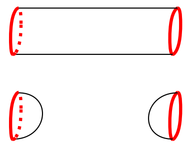
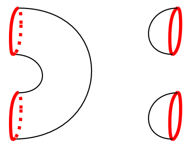
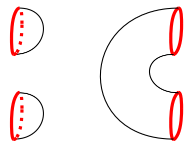
Definition 1. Define $\mathcal{C}ob_2^{\ell}(Y_0, Y_1)$ to be the *k-linear additive category* generated by

- objects are (oriented) 1-*cobordisms* $W : Y_0 \to Y_1$;
- morphisms are (diffeo. classes of) 2-*cobordisms with corners* (aka. 2-*bordisms*).

The morphisms are subject to the following relations:

S-**relation** $S \amalg S^2 \sim 0$ for $S : W_0 \to W_1$;

T-**relation** $S \amalg T^2 \sim 2 \cdot S$ for $S : W_0 \to W_1$;

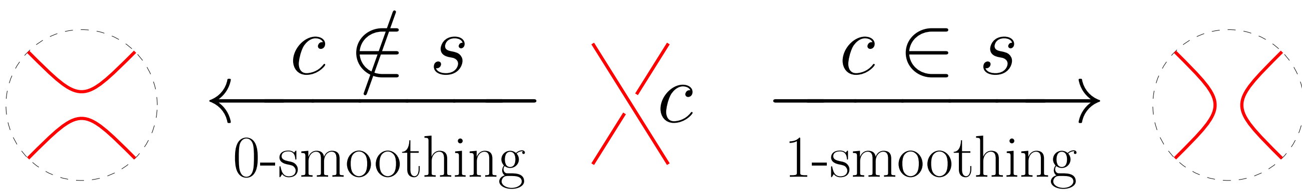
$4Tu$-**relation**  $+$  $-$  $-$  $= 0.$

Definition 2. • $c(D)$: the set of *crossings* in D.

- Each subset $s \subset c(D)$ is called a *state* on D.

  $\rightsquigarrow$ $|s|$: the *cardinality*.

- For each state $s \subset c(D)$, define $D_s \subset \mathbb{R} \times [0,1]$ by the following *smoothing* on each crossing:



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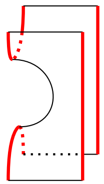
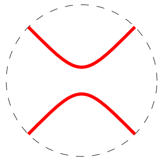
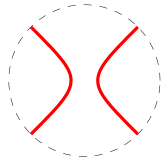
Definition 3. D: a *tangle diag.*; s : a state, $c \in c(D) \setminus s$.

- Define $\delta_c : D_s \to D_{s \cup \{c\}} \in \mathcal{C}ob^{\ell}_2(\partial_0 D, \partial_1 D)$:

$$\delta_c =$$  $:$  $\to$  .

- Define a chain complex $\langle\!\langle D \rangle\!\rangle$ in $\mathcal{C}ob^{\ell}_2(\partial_0 D, \partial_1 D)$ by

$$\langle\!\langle D \rangle\!\rangle^i \coloneqq \bigoplus_{\substack{c \subset c(D) \\ |c| = i}} D_s \otimes E_s \ , \quad d \coloneqq \sum \delta_c \otimes (\wedge c) \quad .$$

- $[\![D]\!]^i \coloneqq \langle\!\langle D \rangle\!\rangle^{i+n_-} \ , \quad d_{[\![D]\!]} = (-1)^{n_-} d_{\langle\!\langle D \rangle\!\rangle} \quad .$

Known Theorem 1 (Khovanov2000)**.** *A bigraded* chain complex $C^{*,\star}(D)$ *(of abelian groups) for each link diagram* D *so that*

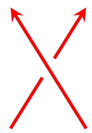
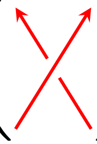
$$Kh^{i,j}(D) \coloneqq H^i(C^{*,j}(D))$$

is invariant under Reidemeister moves.

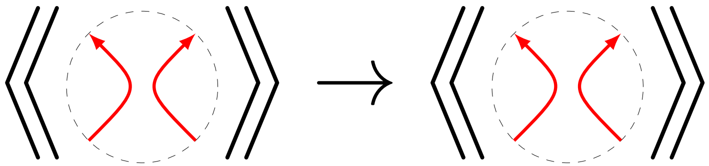
This is nowadays called *Khovanov homology.*

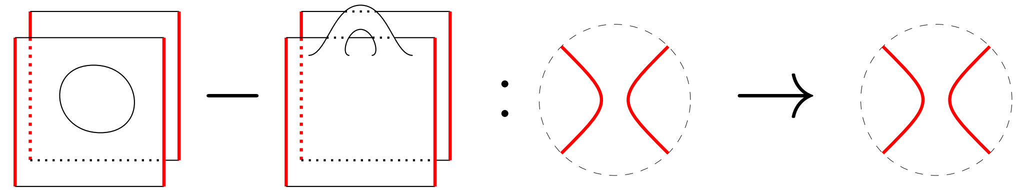
Known Theorem 2 (Bar-Natan2005)**.** *The complex* $[\![D]\!]$ *is* invariant *under* Reidemeister moves *up to* chain homotopy equivalences.

Theorem 1 (Yoshida, I. , arXiv:2005.12664)**.** *There is a non-trivial map*

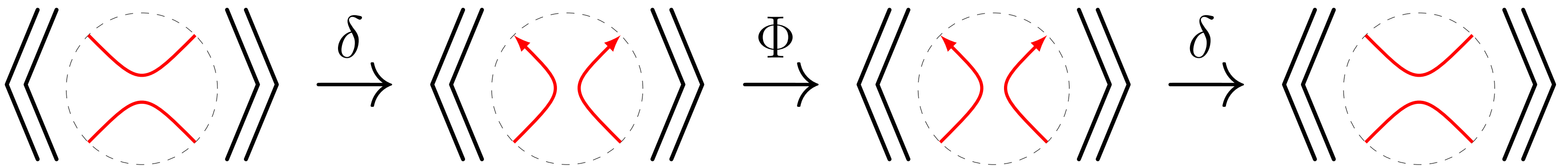
$\widehat{\Phi} : Kh \Big($  $\Big) \to Kh \Big($  $\Big)$

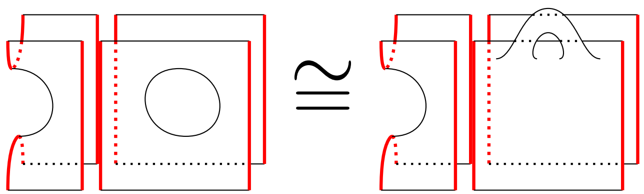
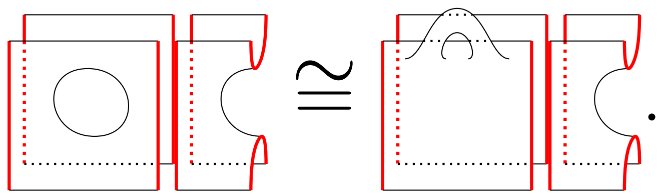
of bidegree $(0, 0)$. *Furthermore, it is* invariant *under* moves with respect to double points.

Definition 4. We define $\Phi :$  by

$\Phi :=$  .

Lemma 1. The following is a 0-*sequence*; i.e. the compositions of adjacent two morphisms vanish:

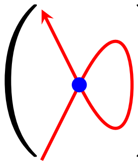


Proof:  and  . $\square$

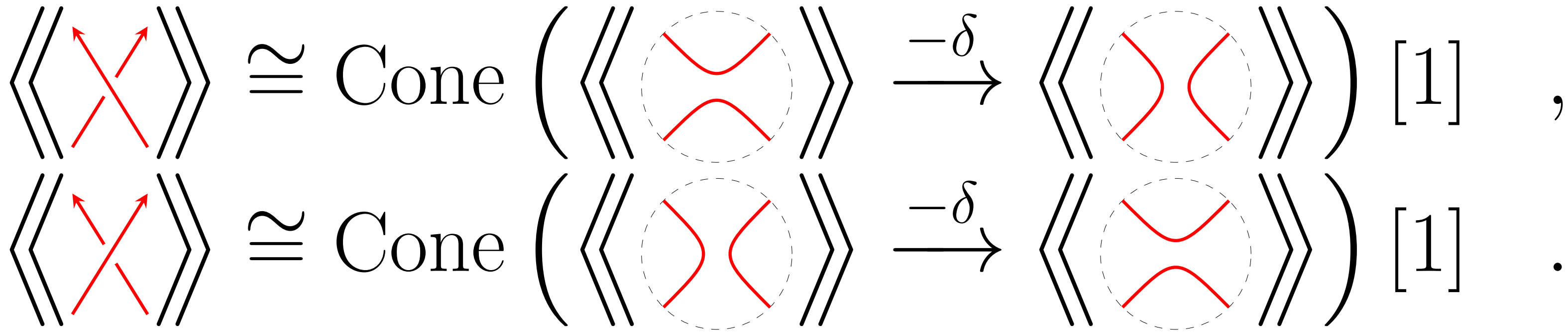
Goal

1. Construction of $\widehat{\Phi}$ *in terms of cobordisms*.

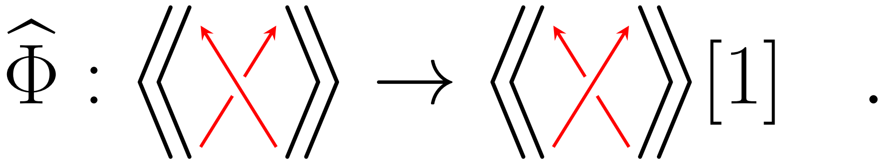
2. $\widehat{\Phi}$ extends Khovanov homology to *singular links* via a *categorified Vassiliev skein relation*.

3. A *categorified FI relation*; i.e. Kh () $= 0$ as the first formula of weight system.

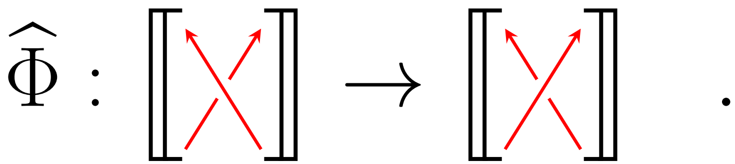
Recall that

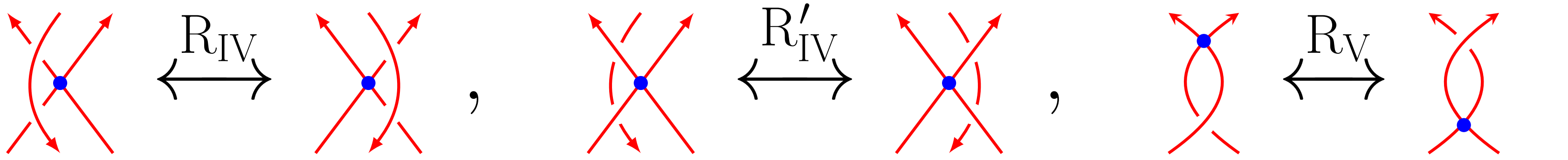


Definition 5. We define the *genus-one morphism* induced by the sequence in Lemma 1:

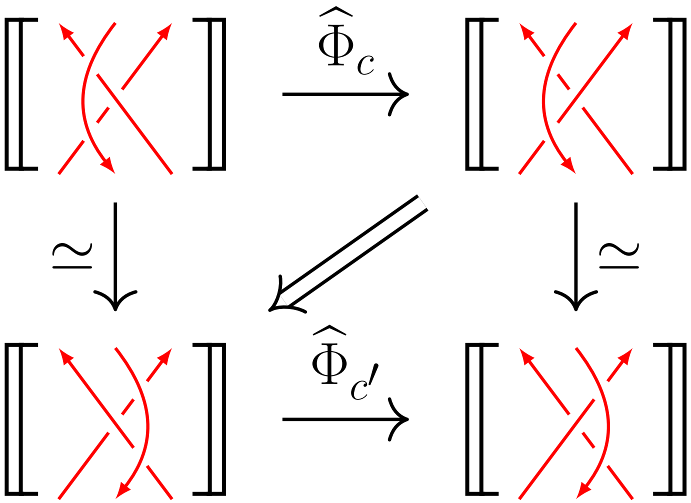


Remark 2. After the normalization of *degree* and *grading*, we get a morphism of *bidegree* $(0, 0)$:





Proposition 3. *The* genus-one morphism *is* invariant *under the moves above; i.e. there are homotopy commutative squares*



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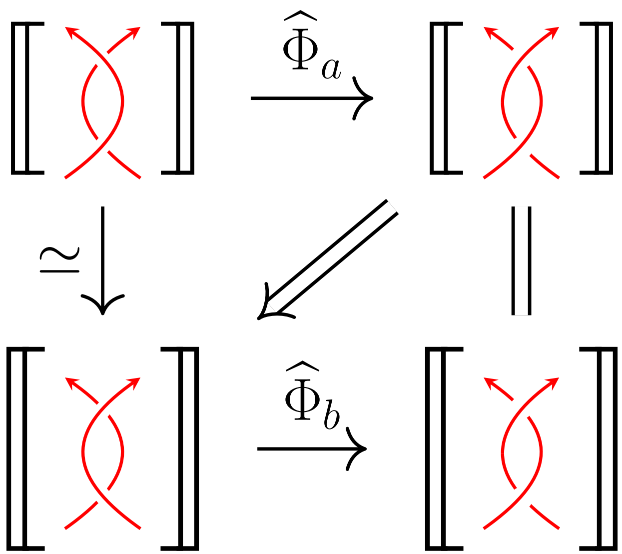


.

Proposition 4. *Suppose we are given a* chain-homotopy commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] \arrow[dl, Rightarrow, "F"'] & Z' \arrow[r, "h'"] \arrow[d] \arrow[dl, Rightarrow, "G"'] & W' \arrow[d] \arrow[dl, Rightarrow, "H"'] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & W
\end{tikzcd}$$

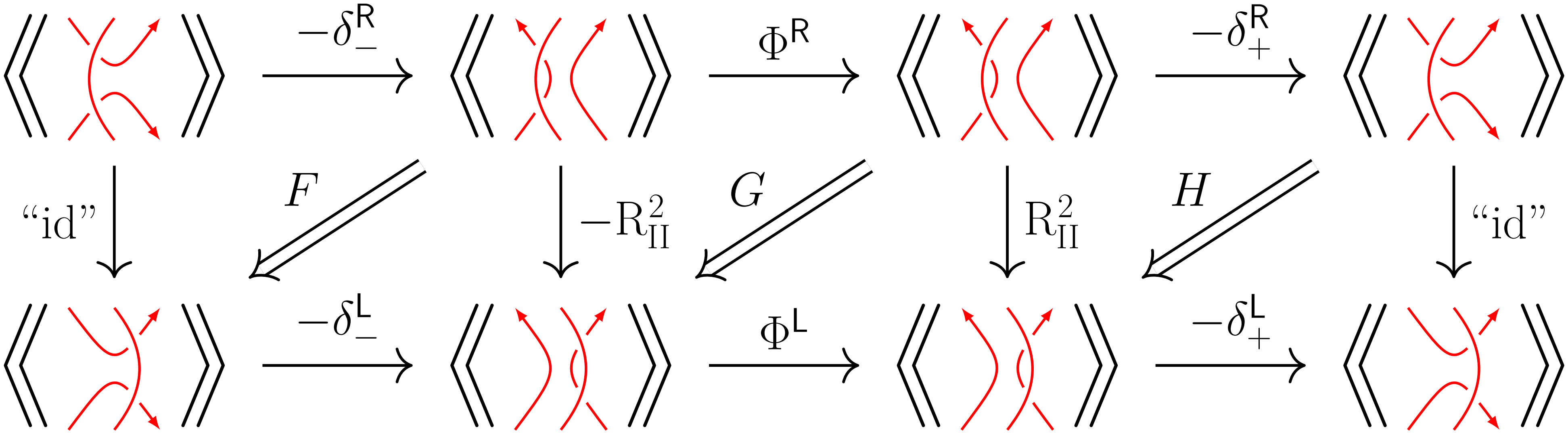
with $gf = 0$, $hg = 0$, $g'f' = 0$, $h'g' = 0$, $\Psi = \{\Psi^i : X'^i \to Z^{i-2}\}_i$, $\Xi = \{\Xi^i : Y'^i \to W^{i-2}\}_i$ *satisfying*

$$d\Psi - \Psi d = gF + Gf' \, , d\Xi - \Xi d = hG + Hg' \, , h\Psi - \Xi f' = 0 \quad .$$

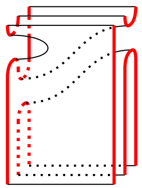
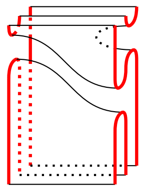
Then, there is a chain-homotopy commutative square

$$\begin{tikzcd}
\mathrm{Cone}(f') \arrow[r] \arrow[d, "F_*"'] & \mathrm{Cone}(h')[1] \arrow[d, "H_*"] \arrow[dl, Rightarrow] \\
\mathrm{Cone}(f) \arrow[r] & \mathrm{Cone}(h)[1]
\end{tikzcd} \quad .$$

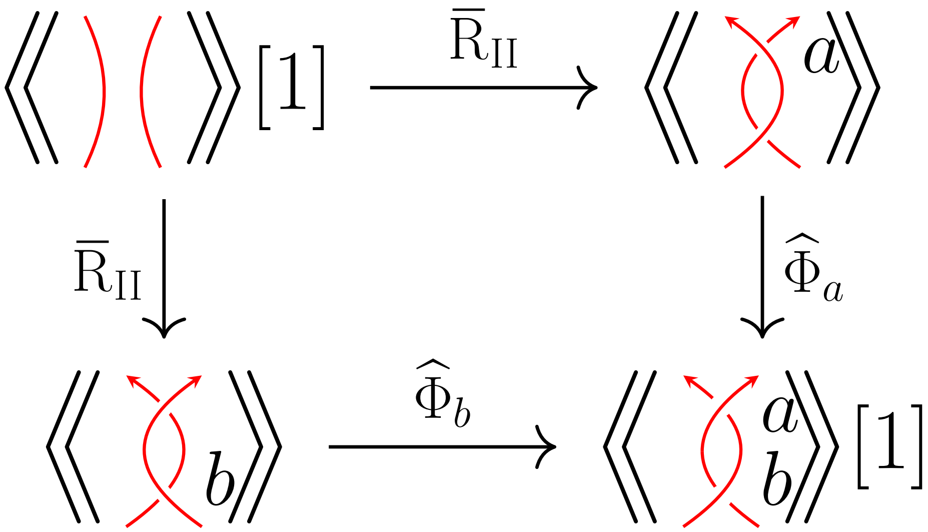
Invariance under $\mathrm{R_{IV}}$ **.** Apply Proposition 4 to



with

$\Psi =$  $\otimes\, \breve{b}\breve{a}$, $\Xi =$  $\otimes\, \breve{b}\breve{a}$.

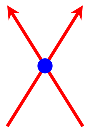
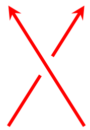
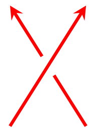
Invariance under $\mathrm{R_V}$. The direct computation shows the following (strictly) square commutes:



,

here $\overline{\mathrm{R}}_{\mathrm{II}}$ is the chain-homotopy equivalence for 2nd RII.

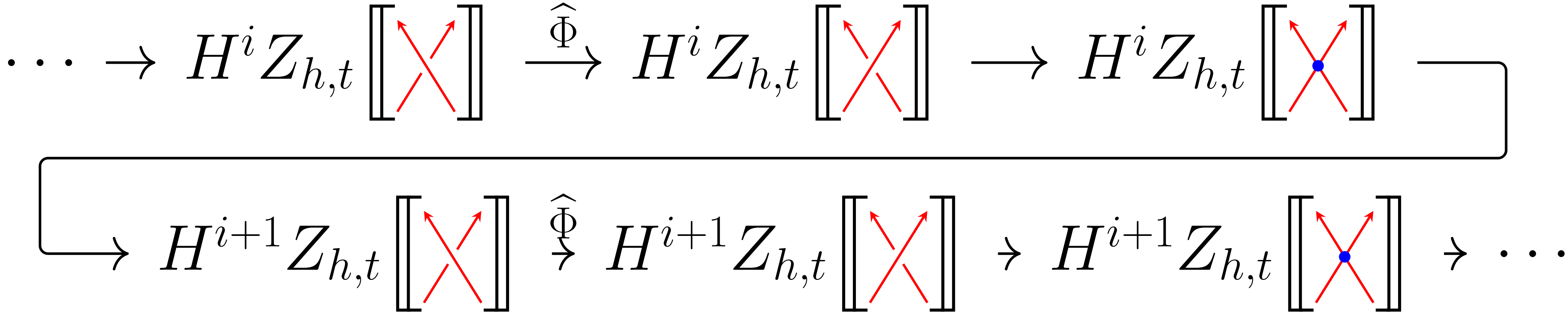
Theorem 5 (Yoshida, I. [ItoYoshida2020]). *For every* singular tangle diagram D*, there exists a complex* $[\![D]\!]$ *in* $\mathcal{C}ob_2^{\ell}(\partial_0 D, \partial_1 D)$ *having an isomorphism*

$[\![$  $]\!] \cong \mathrm{Cone}\Big($ $[\![$  $]\!] \xrightarrow{\widehat{\Phi}}$ $[\![$  $]\!]$ $\Big)$;

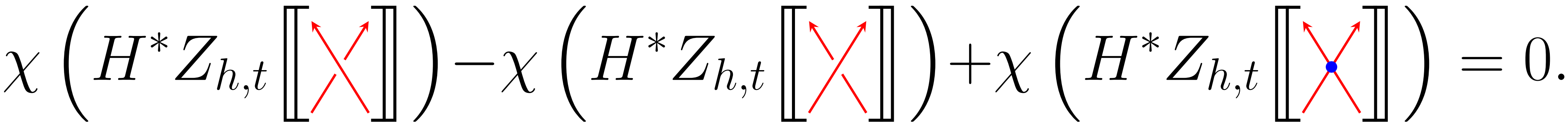
$[\![D]\!]$ *is* invariant *under the moves of singular tangle diagrams.*

$\rightsquigarrow$ Applying the *TQFT* $Z_{h,t}$, we obtain extensions of sl_2 homologies, e.g. *Kh* (with arbitrary coeff., $h = t = 0$), *Lee* ($\mathbb{Q}$, $h = 0, t = 1$), *BN* ($\mathbb{Z}/2$, $t = 0$), etc. to singular link diagrams.

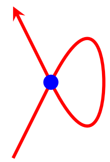
Corollary 1 (Yoshida, I. Categorified Vassiliev skein relation)**.** For every $h, t \in k$, there is a *long exact sequence*



Its decategorification is the Vassiliev skein relation:



Theorem 6 (Yoshida, I. The FI relation)**.** *If a singular tangle diagram D contains a local tangle of the form*



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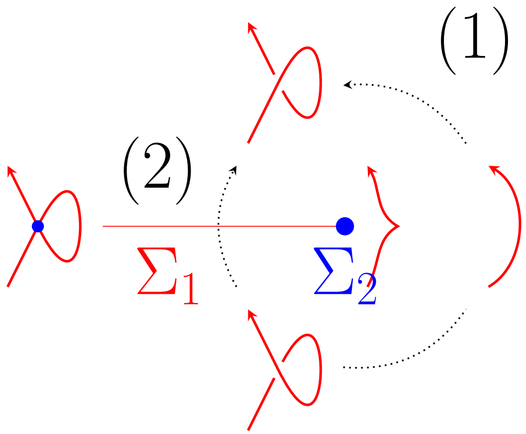
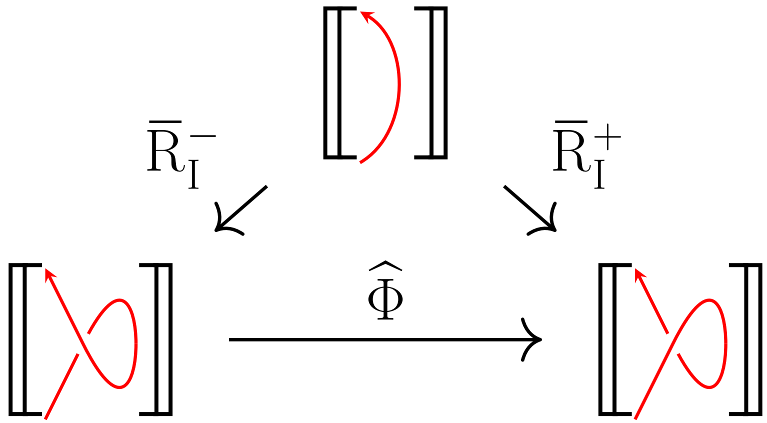
then $[\![D]\!]$ *is* contractible*; i.e. the identity is null-homotopic.*

cf. the FI relation is obtained by comparing the two paths:

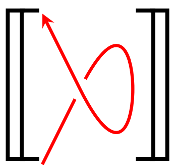
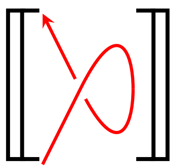


Proof.

The triangle below commutes:



because using *categorified Vassiliev skein rel.*, we have

$[\![D]\!] \simeq 0 \iff \widehat{\Phi}$:  $\to$  : homotopy-equivalence.

$\square$

Next Targets

Kauffman's questions at Quantum Topology Seminar:

- How about categorifying weight system ?
- How about sl_n ?