

On a cobordism realizing crossing change on
 $\mathfrak{sl}(2)$ tangle homology and a categorified
Vassiliev skein relation

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(Slide is made using Yoshida's sources w/ small modification.)

Information.

Recoding & slide of Yoshida's talk can be seen in web:
L. H. Kauffman's web, "Quantum Topology Seminar".

Question.

What is *crossing change* on Khovanov homology?

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Traditionally, using *link cobordisms*, we have

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \searrow \end{array} \xrightarrow{\text{saddle}} \begin{array}{c} \nearrow \quad \nearrow \\ \nwarrow \quad \nwarrow \end{array} \xrightarrow{R_1^2} \begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \searrow \end{array} \xrightarrow{\text{saddle}} \begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}.$$

It will induce a map

$$Kh \left(\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right) \rightarrow Kh \left(\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right).$$

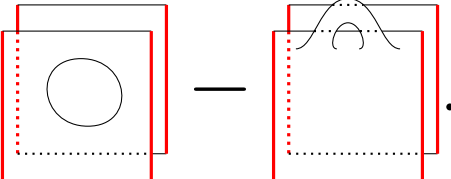
However, the degree does not preserve even if the crossing change does not change the knot.

Modified Question.

How to realize *crossing change* on Khovanov homology preserving the degree of Jones polynomial toward categorified Vassiliev theory?

Modified Question / Answer

How to realize *crossing change* on Khovanov homology preserving the degree of Jones polynomial toward categorified Vassiliev theory?

Answer: $\Phi :=$  $.$

The diagram shows two square boxes, each with a solid top and bottom edge and a dotted left and right edge. The left box contains a circle. The right box contains a crossing (two strands intersecting). A minus sign is placed between the two boxes.

Modified Question / Answer

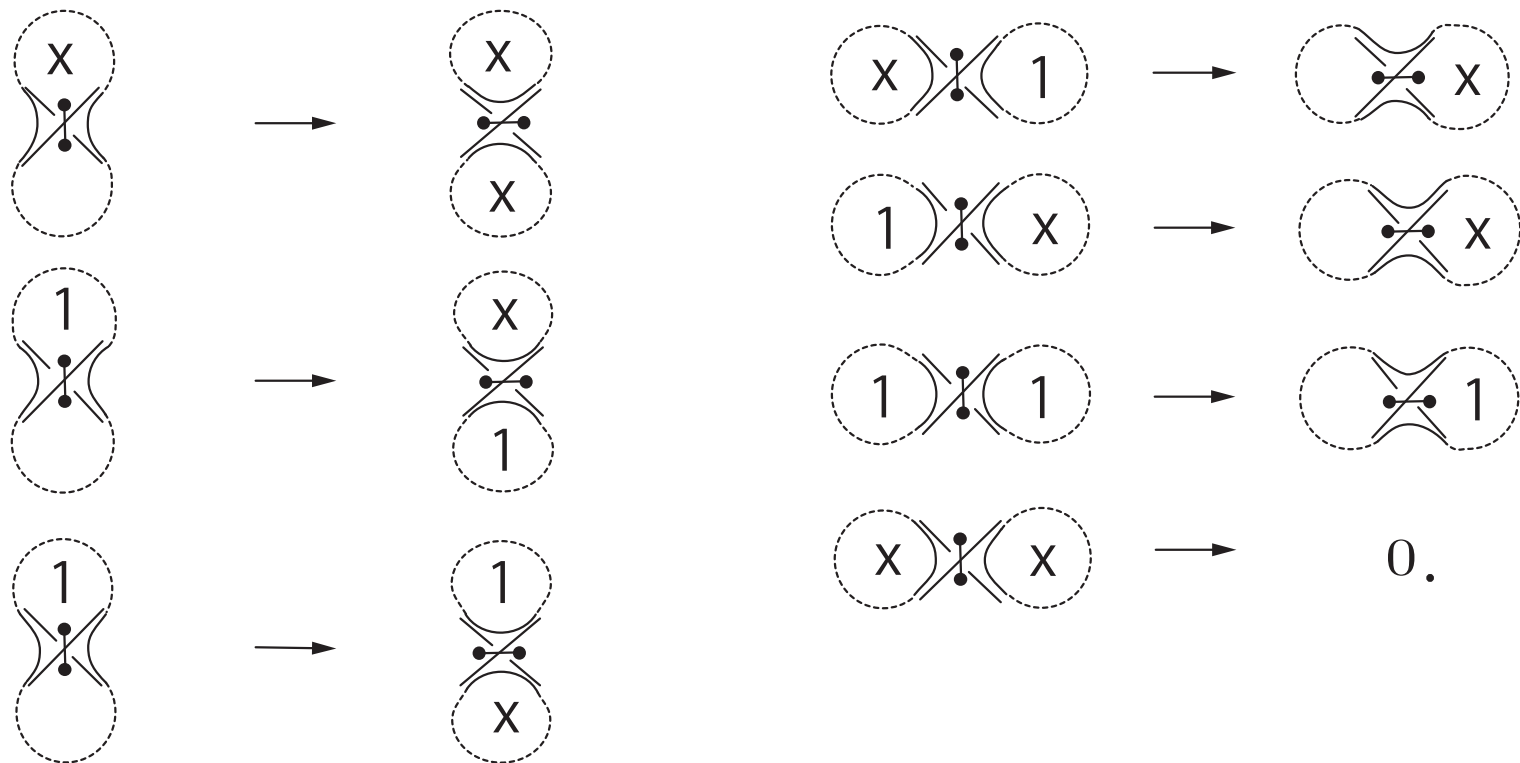
How to realize *crossing change* on Khovanov homology preserving the degree of Jones polynomial toward categorified Vassiliev theory?

Answer: $\Phi := \boxed{\bigcirc} - \boxed{\text{cap}}$; **Note:**

Lemma 1. *The following is a 0-sequence; i.e. the compositions of adjacent two morphisms vanish:*

$$\langle\langle \text{X} \rangle\rangle \xrightarrow{\delta} \langle\langle \text{cap} \rangle\rangle \xrightarrow{\Phi} \langle\langle \text{cup} \rangle\rangle \xrightarrow{\delta} \langle\langle \text{X} \rangle\rangle \quad .$$

cf. TQFT $Z_{0,0}: \text{Cob}_2 \rightarrow \text{Mod}_k; Z_{0,0}(S^1) = A$: Frobenius:



cf. retraction for RI invariance:



Corollary 1 (Yoshida, I. Categorified Vassiliev skein relation). *For every $h, t \in k$, there is a long exact sequence*

$$\begin{array}{c} \cdots \rightarrow H^i Z_{h,t} \left[\begin{array}{|l} \nearrow \\ \nwarrow \end{array} \right] \xrightarrow{\widehat{\Phi}} H^i Z_{h,t} \left[\begin{array}{|l} \nearrow \\ \nwarrow \end{array} \right] \longrightarrow H^i Z_{h,t} \left[\begin{array}{|l} \nearrow \\ \nwarrow \end{array} \right] \\ \downarrow \\ \longrightarrow H^{i+1} Z_{h,t} \left[\begin{array}{|l} \nearrow \\ \nwarrow \end{array} \right] \xrightarrow{\widehat{\Phi}} H^{i+1} Z_{h,t} \left[\begin{array}{|l} \nearrow \\ \nwarrow \end{array} \right] \rightharpoonup H^{i+1} Z_{h,t} \left[\begin{array}{|l} \nearrow \\ \nwarrow \end{array} \right] \rightharpoonup \cdots \end{array}$$

Corollary 1 (Yoshida, I. Categorized Vassiliev skein relation). For every $h, t \in k$, there is a *long exact sequence*

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^i Z_{h,t} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right] & \xrightarrow{\widehat{\Phi}} & H^i Z_{h,t} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right] & \rightarrow & H^i Z_{h,t} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \bullet \end{array} \right] \right] \\ & & & & & & \downarrow \\ & & & & & & \rightarrow H^{i+1} Z_{h,t} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right] \xrightarrow{\widehat{\Phi}} H^{i+1} Z_{h,t} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right] \rightarrow H^{i+1} Z_{h,t} \left[\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \bullet \end{array} \right] \right] \rightarrow \cdots \end{array}$$

$$\textcircled{1} \textcircled{1} \rightarrow \textcircled{1} \textcircled{x} + \textcircled{x} \textcircled{1} - 2 \textcircled{x} \textcircled{1}$$

$$\Phi := \left[\begin{array}{c} \text{red box} \\ \text{circle} \end{array} \right] - \left[\begin{array}{c} \text{red box} \\ \text{cup} \end{array} \right].$$

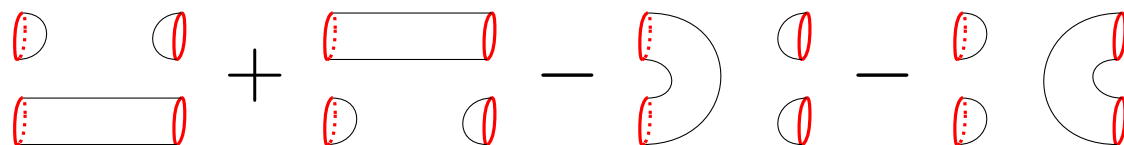
Definition 1. Define $\mathcal{Cob}_2^\ell(Y_0, Y_1)$ to be the k -linear additive category generated by

- objects are (oriented) 1-cobordisms $W : Y_0 \rightarrow Y_1$;
- morphisms are (diffeo. classes of) 2-cobordisms with corners (aka. 2-bordisms).

The morphisms are subject to the following relations:

S -relation $S \amalg S^2 \sim 0$ for $S : W_0 \rightarrow W_1$;

T -relation $S \amalg T^2 \sim 2 \cdot S$ for $S : W_0 \rightarrow W_1$;

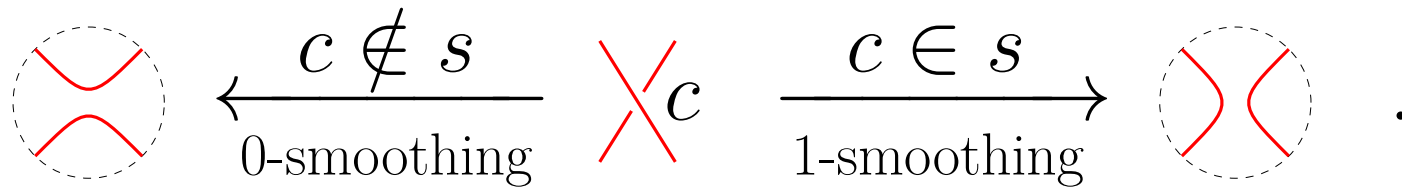
$4Tu$ -relation  $= 0$.

Definition 2. • $c(D)$: the set of *crossings* in D .

• Each subset $s \subset c(D)$ is called a *state* on D .

$\rightsquigarrow |s|$: the *cardinality*.

• For each state $s \subset c(D)$, define $D_s \subset \mathbb{R} \times [0, 1]$ by the following *smoothing* on each crossing:



Definition 3. D : a tangle diag.; s : a state, $c \in c(D) \setminus s$.

- Define $\delta_c : D_s \rightarrow D_{s \cup \{c\}} \in \mathcal{Cob}_2^\ell(\partial_0 D, \partial_1 D)$:

$$\delta_c = \text{[diagram]} : \text{[diagram]} \rightarrow \text{[diagram]} .$$

The diagram shows a mapping from a tangle with a crossing to a tangle with a crossing, where the crossing is now part of a larger structure.

- Define a chain complex $\ll D \gg$ in $\mathcal{Cob}_2^\ell(\partial_0 D, \partial_1 D)$ by

$$\ll D \gg^i := \bigoplus_{\substack{c \subset c(D) \\ |c|=i}} D_s \otimes E_s , \quad d := \sum \delta_c \otimes (\wedge c) .$$

- $\ll D \gg^i := \ll D \gg^{i+n_-}$, $d_{\ll D \gg} = (-1)^{n_-} d_{\ll D \gg}$.

Known Theorem 1 (Khovanov2000). *A bigraded chain complex $C^{*,*}(D)$ (of abelian groups) for each link diagram D so that*

$$Kh^{i,j}(D) := H^i(C^{*,j}(D))$$

is invariant under Reidemeister moves.

This is nowadays called *Khovanov homology*.

Known Theorem 2 (Bar-Natan2005). *The complex $[[D]]$ is invariant under Reidemeister moves up to chain homotopy equivalences.*

Theorem 1 (Yoshida, I. , arXiv:2005.12664). *There is a non-trivial map*

$$\widehat{\Phi} : Kh \left(\begin{array}{c} \nearrow \nwarrow \\ \swarrow \searrow \end{array} \right) \rightarrow Kh \left(\begin{array}{c} \nwarrow \nearrow \\ \swarrow \searrow \end{array} \right)$$

of bidegree $(0, 0)$. Furthermore, it is invariant under moves with respect to double points.

Definition 4. We define $\Phi : \langle\langle \text{circle with two red arcs} \rangle\rangle \rightarrow \langle\langle \text{circle with two red arcs} \rangle\rangle$ by

$$\Phi := \left[\text{square with circle} \right] - \left[\text{square with arc} \right] : \text{circle with two red arcs} \rightarrow \text{circle with two red arcs} .$$

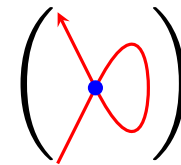
Lemma 1. The following is a *0-sequence*; i.e. the compositions of adjacent two morphisms vanish:

$$\langle\langle \text{circle with two red arcs} \rangle\rangle \xrightarrow{\delta} \langle\langle \text{circle with two red arcs} \rangle\rangle \xrightarrow{\Phi} \langle\langle \text{circle with two red arcs} \rangle\rangle \xrightarrow{\delta} \langle\langle \text{circle with two red arcs} \rangle\rangle .$$

Proof: $\left[\text{square with circle} \right] \cong \left[\text{square with arc} \right]$ and $\left[\text{square with circle} \right] \cong \left[\text{square with arc} \right] . \quad \square$

Goal

1. Construction of $\hat{\Phi}$ *in terms of cobordisms*.
2. $\hat{\Phi}$ extends Khovanov homology to *singular links* via a *categorified Vassiliev skein relation*.
3. A *categorified FI relation*; i.e. $Kh \left(\text{diagram} \right) = 0$ as the first formula of weight system.



Recall that

$$\begin{aligned} \langle\langle \text{X} \rangle\rangle &\cong \text{Cone} \left(\langle\langle \text{X} \rangle\rangle \xrightarrow{-\delta} \langle\langle \text{X} \rangle\rangle \right) [1] \quad , \\ \langle\langle \text{X} \rangle\rangle &\cong \text{Cone} \left(\langle\langle \text{X} \rangle\rangle \xrightarrow{-\delta} \langle\langle \text{X} \rangle\rangle \right) [1] \quad . \end{aligned}$$

Definition 5. We define the *genus-one morphism* induced by the sequence in Lemma 1:

$$\widehat{\Phi} : \langle\langle \text{X} \rangle\rangle \rightarrow \langle\langle \text{X} \rangle\rangle [1] \quad .$$

Remark 2. After the normalization of *degree* and *grading*, we get a morphism of *bidegree* $(0, 0)$:

$$\widehat{\Phi} : \llbracket \text{X} \rrbracket \rightarrow \llbracket \text{X} \rrbracket \quad .$$

$$\begin{array}{c} \text{Diagram 1} \end{array} \xleftrightarrow{R_{IV}} \begin{array}{c} \text{Diagram 2} \end{array}, \quad \begin{array}{c} \text{Diagram 3} \end{array} \xleftrightarrow{R'_{IV}} \begin{array}{c} \text{Diagram 4} \end{array}, \quad \begin{array}{c} \text{Diagram 5} \end{array} \xleftrightarrow{R_V} \begin{array}{c} \text{Diagram 6} \end{array}.$$

Proposition 3. *The genus-one morphism is invariant under the moves above; i.e. there are homotopy commutative squares*

$$\begin{array}{ccc}
 \left[\begin{array}{c} \text{Diagram 1} \end{array} \right] & \xrightarrow{\hat{\Phi}_c} & \left[\begin{array}{c} \text{Diagram 2} \end{array} \right] \\
 \downarrow \cong & \swarrow & \downarrow \cong \\
 \left[\begin{array}{c} \text{Diagram 3} \end{array} \right] & \xrightarrow{\hat{\Phi}_{c'}} & \left[\begin{array}{c} \text{Diagram 4} \end{array} \right]
 \end{array}, \quad
 \begin{array}{ccc}
 \left[\begin{array}{c} \text{Diagram 5} \end{array} \right] & \xrightarrow{\hat{\Phi}_a} & \left[\begin{array}{c} \text{Diagram 6} \end{array} \right] \\
 \downarrow \cong & \swarrow & \parallel \\
 \left[\begin{array}{c} \text{Diagram 7} \end{array} \right] & \xrightarrow{\hat{\Phi}_b} & \left[\begin{array}{c} \text{Diagram 8} \end{array} \right]
 \end{array}.$$

Proposition 4. *Suppose we are given a chain-homotopy commutative diagram*

$$\begin{array}{ccccccc}
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & W' \\
 \downarrow & \swarrow F & \downarrow & \swarrow G & \downarrow & \swarrow H & \downarrow \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & W
 \end{array}$$

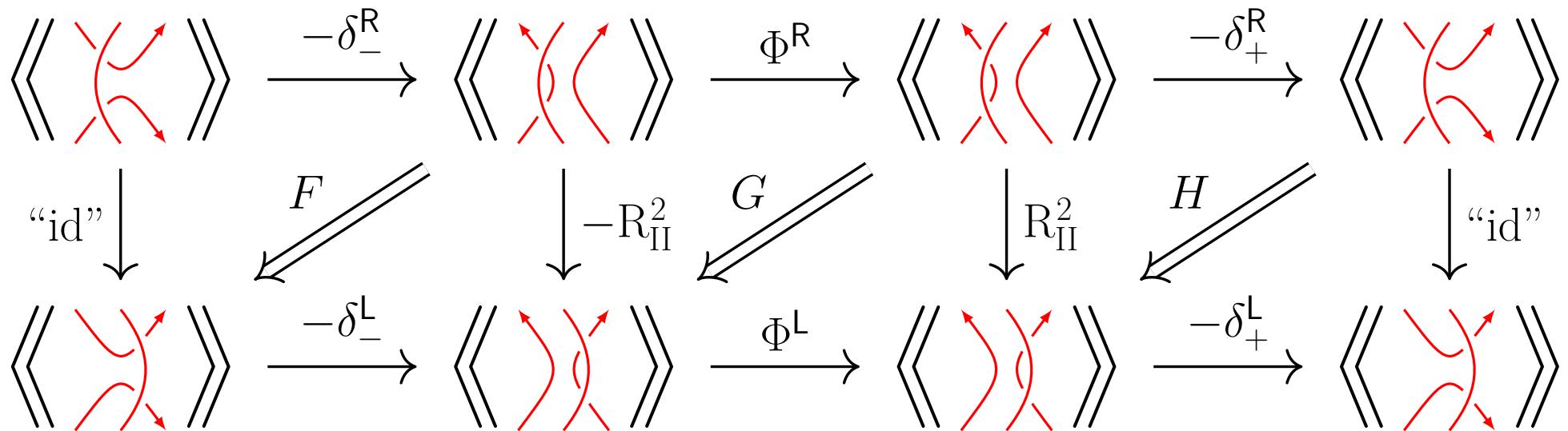
with $gf = 0$, $hg = 0$, $g'f' = 0$, $h'g' = 0$, $\Psi = \{\Psi^i : X'^i \rightarrow Z^{i-2}\}_i$, $\Xi = \{\Xi^i : Y'^i \rightarrow W^{i-2}\}_i$ satisfying

$$d\Psi - \Psi d = gF + Gf', \quad d\Xi - \Xi d = hG + Hg', \quad h\Psi - \Xi f' = 0 \quad .$$

Then, there is a chain-homotopy commutative square

$$\begin{array}{ccc}
 \text{Cone}(f') & \longrightarrow & \text{Cone}(h')[1] \\
 F_* \downarrow & \swarrow & \downarrow H_* \\
 \text{Cone}(f) & \longrightarrow & \text{Cone}(h)[1]
 \end{array} .$$

Invariance under R_{IV} . Apply Proposition 4 to



with

$$\Psi = \text{[diagram]} \otimes \breve{b}\breve{a} \ , \quad \Xi = \text{[diagram]} \otimes \breve{b}\breve{a} \ .$$

Invariance under R_V . The direct computation shows the following (strictly) square commutes:

$$\begin{array}{ccc}
 \langle\langle \rangle \rangle \langle \rangle [1] & \xrightarrow{\bar{R}_{II}} & \langle\langle \rangle \rangle \langle \rangle^a \\
 \bar{R}_{II} \downarrow & & \downarrow \hat{\Phi}_a \\
 \langle\langle \rangle \rangle \langle \rangle^b & \xrightarrow{\hat{\Phi}_b} & \langle\langle \rangle \rangle \langle \rangle^a_b [1]
 \end{array} ,$$

here \bar{R}_{II} is the chain-homotopy equivalence for 2nd RII.

Theorem 5 (Yoshida, I. [ItoYoshida2020]). *For every singular tangle diagram D , there exists a complex $[[D]]$ in $\mathcal{Cob}_2^\ell(\partial_0 D, \partial_1 D)$ having an isomorphism*

$$\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \cong \text{Cone} \left(\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \xrightarrow{\hat{\Phi}} \left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right) \quad ;$$

$[[D]]$ is invariant under the moves of singular tangle diagrams.

\rightsquigarrow Applying the $TQFT$ $Z_{h,t}$, we obtain extensions of sl_2 homologies, e.g. Kh (with arbitrary coeff., $h = t = 0$), Lee (\mathbb{Q} , $h = 0, t = 1$), BN ($\mathbb{Z}/2$, $t = 0$), etc. to singular link diagrams.

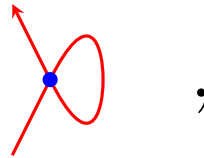
Corollary 1 (Yoshida, I. Categorified Vassiliev skein relation). For every $h, t \in k$, there is a *long exact sequence*

[illegible]

Its decategorification is the Vassiliev skein relation:

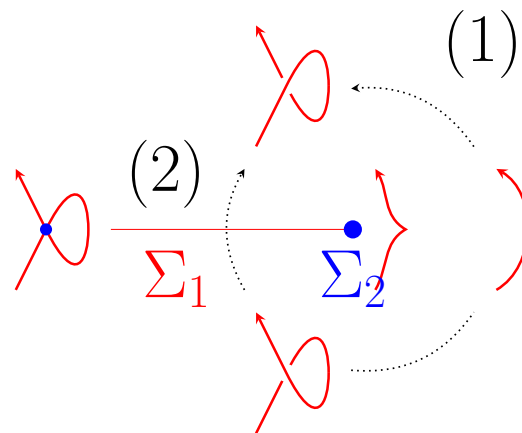
$$\chi \left(H^* Z_{h,t} \left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right) - \chi \left(H^* Z_{h,t} \left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right) + \chi \left(H^* Z_{h,t} \left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] \right) = 0.$$

Theorem 6 (Yoshida, I. The FI relation). *If a singular tangle diagram D contains a local tangle of the form*



then $\llbracket D \rrbracket$ is contractible; i.e. the identity is null-homotopic.

cf. the FI relation is obtained by comparing the two paths:



Proof.

The triangle below commutes:

$$\begin{array}{ccc}
 & \begin{array}{c} \text{[[D]]} \\ \bar{R}_I^- \swarrow \quad \searrow \bar{R}_I^+ \end{array} & \\
 \begin{array}{c} \text{[[X]]} \end{array} & \xrightarrow{\hat{\Phi}} & \begin{array}{c} \text{[[X]]} \end{array}
 \end{array}$$

because using *categorified Vassiliev skein rel.*, we have

$$\text{[[D]]} \simeq 0 \iff \hat{\Phi} : \text{[[X]]} \rightarrow \text{[[X]]} : \text{homotopy-equivalence.}$$

□

Next Targets

Kauffman's questions at Quantum Topology Seminar:

- How about categorifying weight system ?
- How about sl_n ?