Splice-unknotting operation and crosscap numbers

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Tohoku Knot Seminar

October 17, 2020

Main Result 1
[Takimura-I.] (IJM2020), [Kindred] (IJM2020)

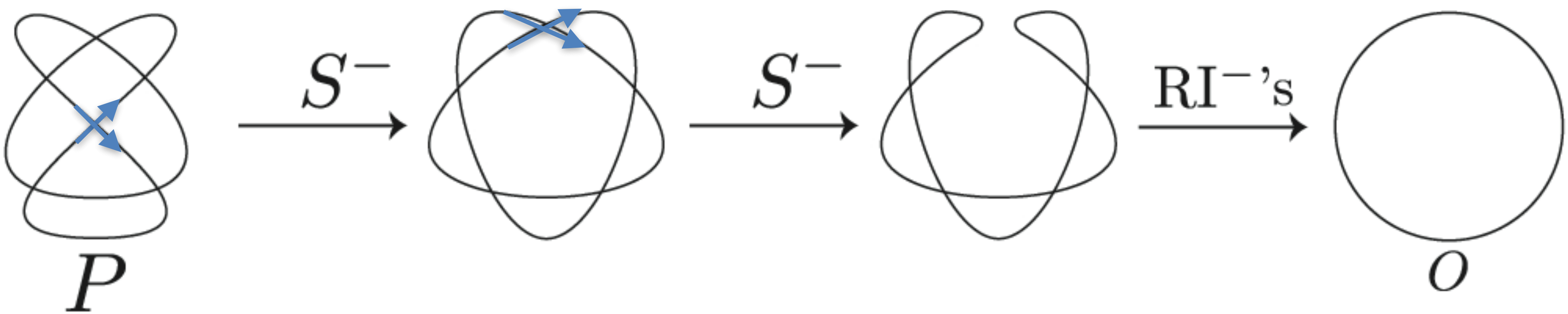
Let $C(K)$ be the crosscap number of K.

For any prime alternating knot K,

$$C(K) = u^{-}(K).$$

Recalling definition: $u^-(P)$, $u^-(K)$

- $u^-(P)$ is the minimum number of necessary splices of type S^- among any sequences of S^- and RI^- to obtain O. $u^-(K) := \min_P u^-(P)$.



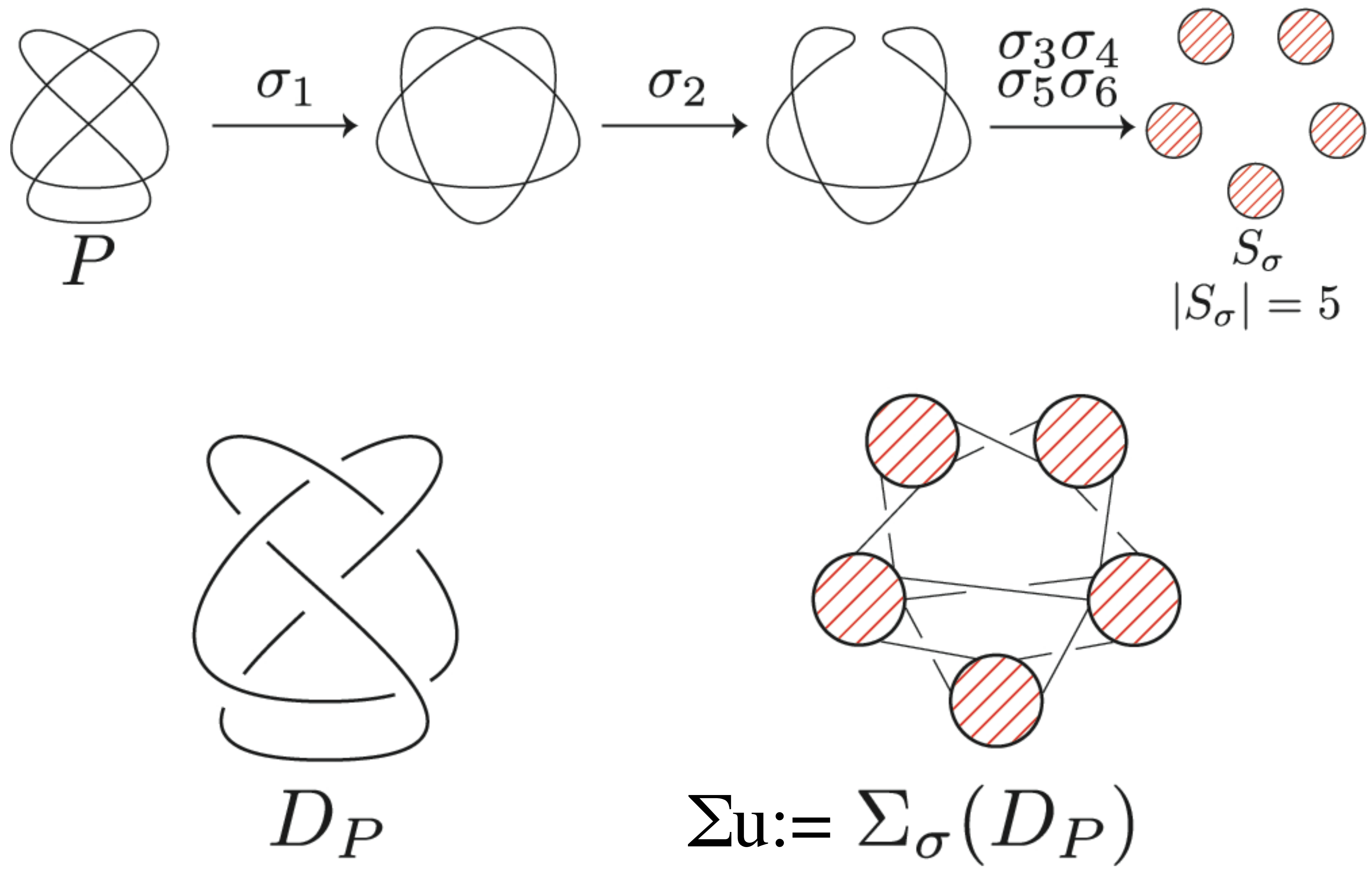
Plan of proof for $u^-(D) \leqq C(K)$

We will compare

Σu : a non-orientable state surface realizing $u^-(D)$

with

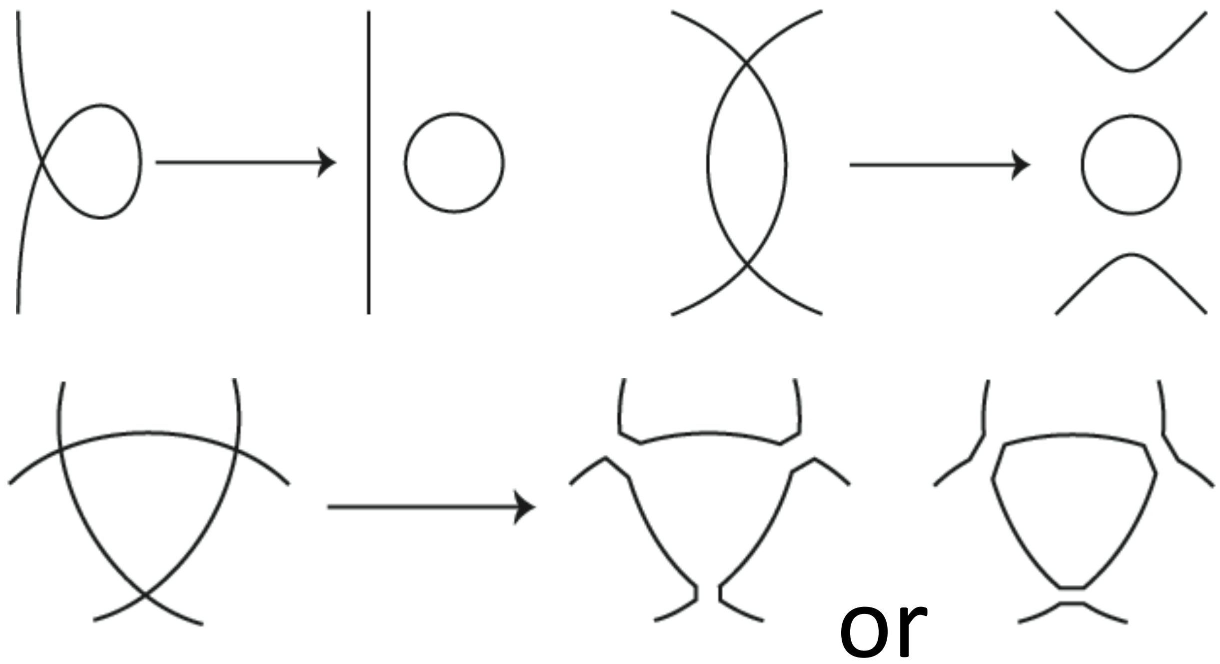
Σ_{AK} : a surface realizing $C(K)$ or $g(K)$
(Adams-Kindred).


P
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
S_σ
$|S_\sigma| = 5$
D_P
$\Sigma u := \Sigma_\sigma(D_P)$

Construction of Σ_{AK}

Find m-gon of the smallest m and splice as follows

(P has a 3-gon if $3 \leqq$ m Eliahou-Harary-Kauffman, 2008)



Notation 1. Σ_{AK} *gives a sequence of splices* $(\sigma_i)_{i=1}^{n(D)}$ *:*

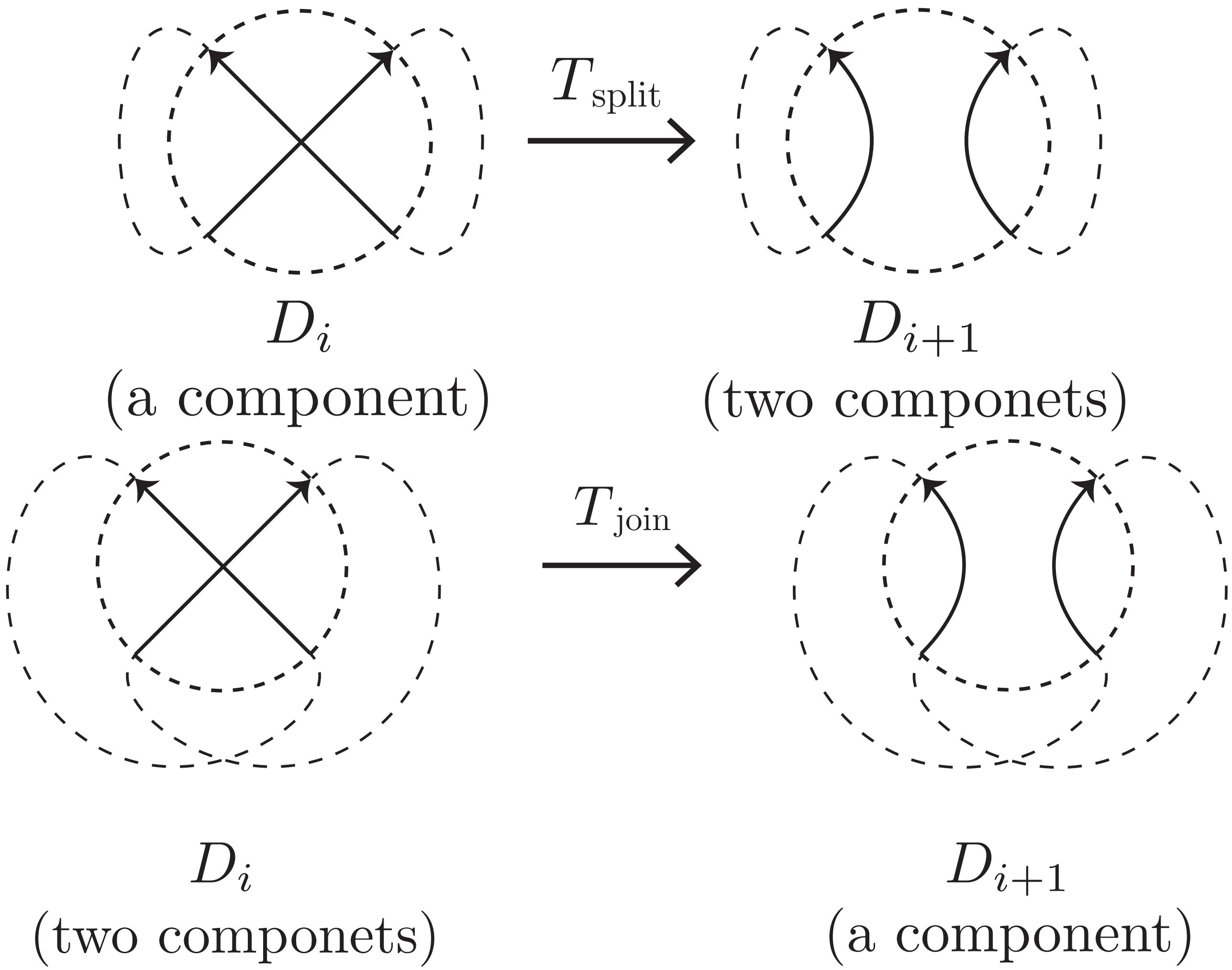
$$D = D_0 \stackrel{\sigma_1}{\to} D_1 \stackrel{\sigma_2}{\to} D_2 \stackrel{\sigma_3}{\to} \cdots \stackrel{\sigma_{n(D)}}{\to} D_{n(D)}$$

Each orientation of D_i *is of* σ_i. ($=$ *ori.* S^-, S^-_{join}, T_{split}, T_{join}).
It induces

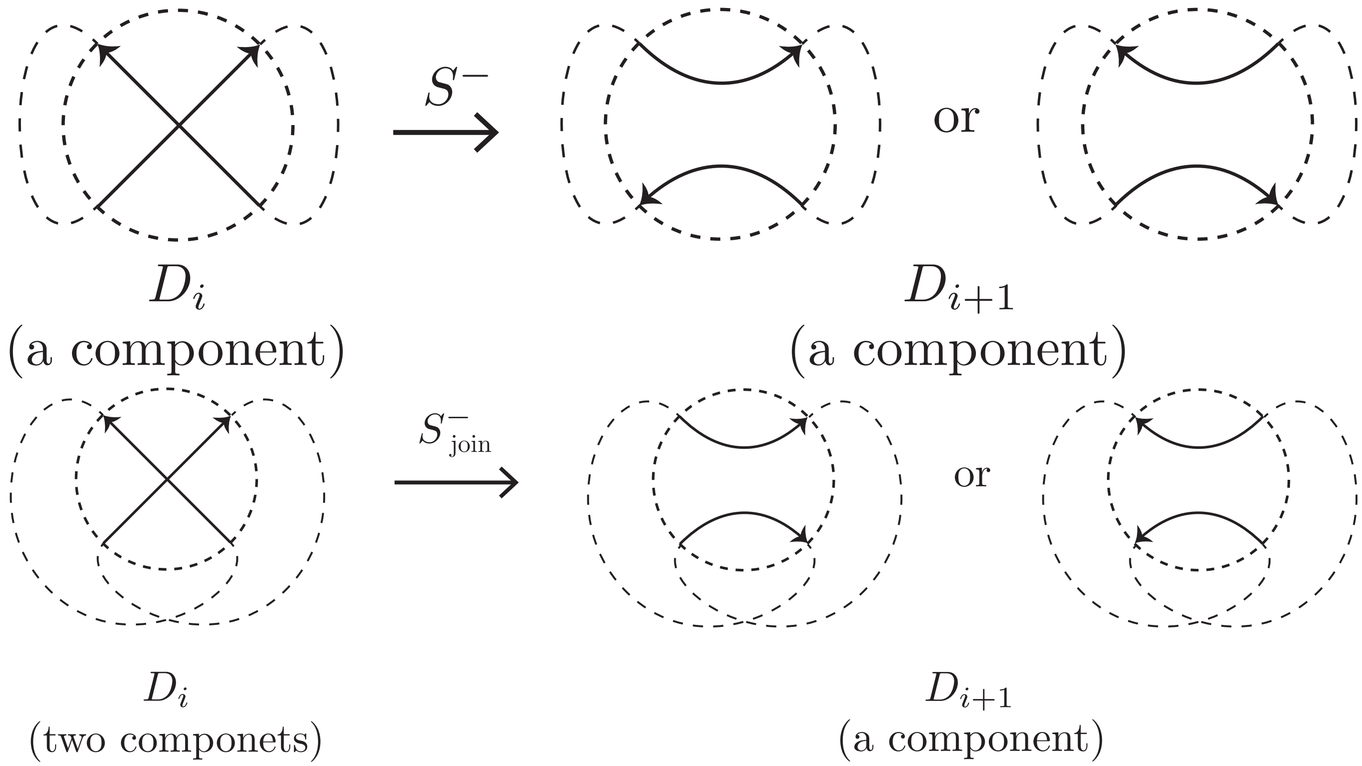
$$CD_D = CD_0 \stackrel{\sigma_1}{\to} CD_1 \stackrel{\sigma_2}{\to} CD_2 \stackrel{\sigma_3}{\to} \cdots \stackrel{\sigma_{n(D)}}{\to} CD_{n(D)}.$$

(CD_D is a Gauss diagram of D; it will be defined.)

Notation 2. • *Oriented* T_{split}, T_{join}. *Seifert splices.*



- *Oriented* RI^-. *1st Reidemeister move.*
- *Oriented* S^-, S^-_{join}. *Target orientation must be chosen.*



Property S^-

S^-

$\downarrow$

Claim

$\downarrow$

Key Lemma

$\downarrow$

Main Result 1

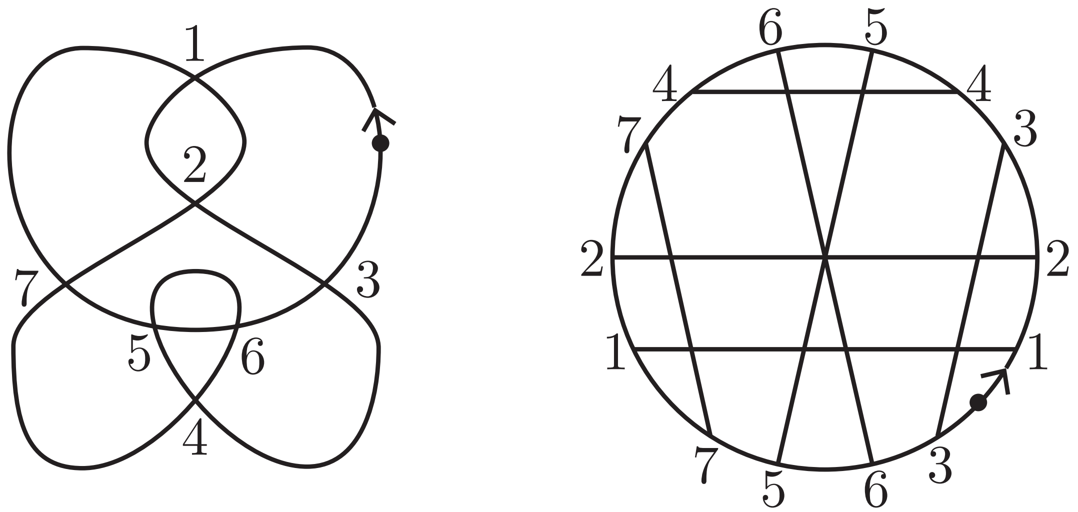
Property S^- — S^-

Claim — $S^- \; T \; T$

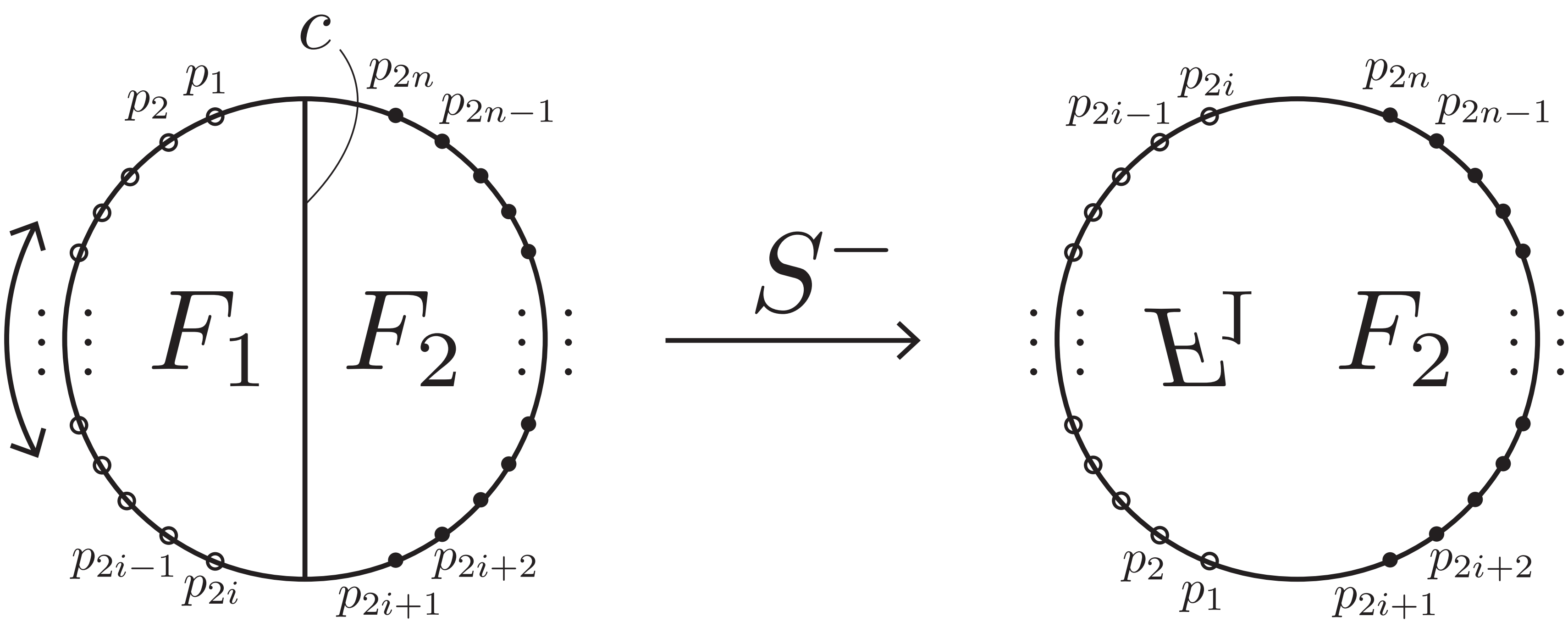
Key Lemma — $S^- \; T \; T$ to $S^-S^-S^-$

Main Result 1

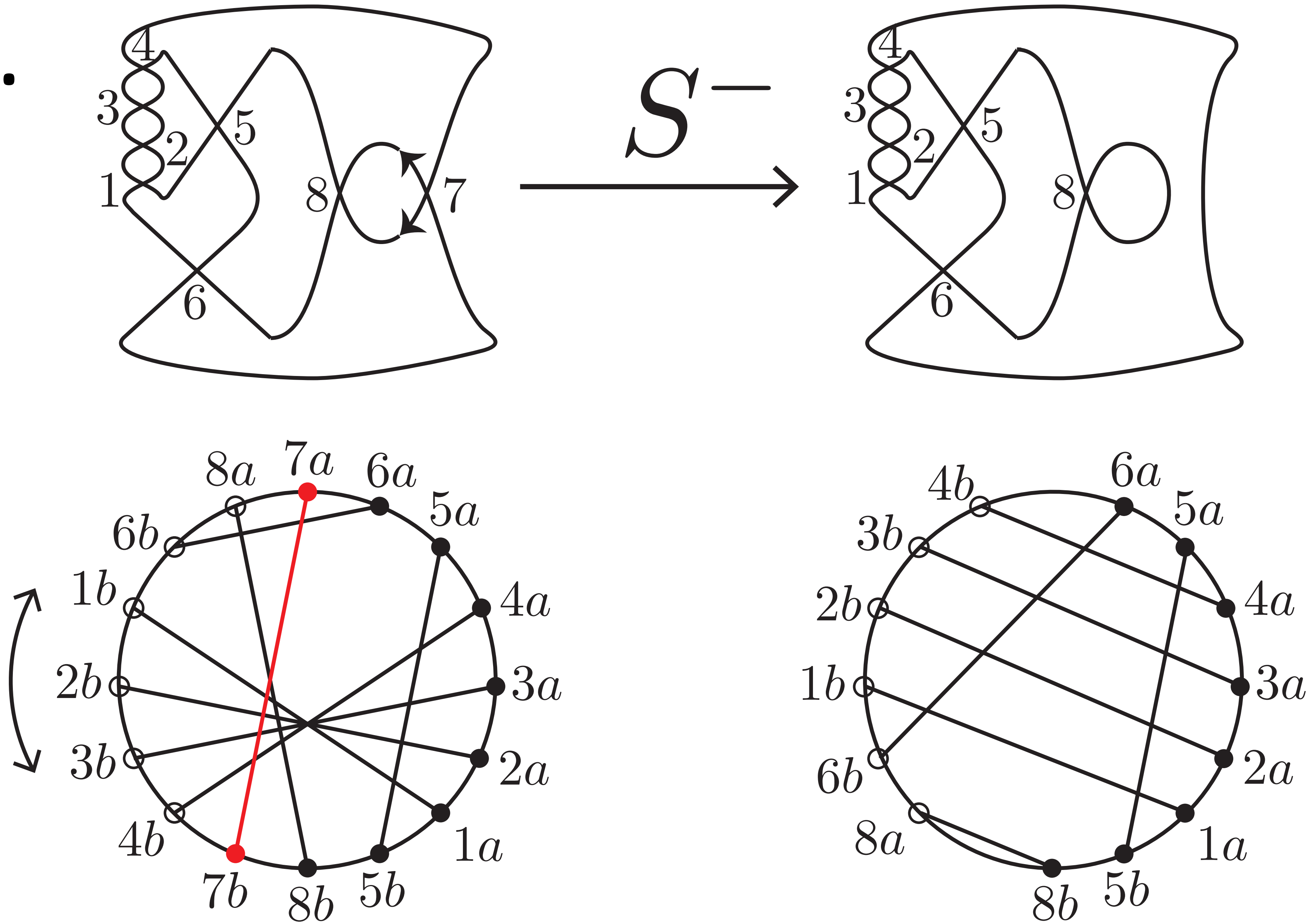
Definition 1. Let D be a knot diagram whose projection is P. Then there is a generic immersion $g : S^1 \to S^2$ such that $g(S^1) = P$. *It is denoted by CD_D.*



Property S⁻. *The behavior of S^- in CD_D is as follows.* (*The difference of cyclic Gauss words is presented as:* $cp_1p_2\dots p_{2i}cp_{2i+1}\dots p_{2n} \longrightarrow p_{2i}p_{2i-1}\dots p_1p_{2i+1}p_{2i+2}\dots p_{2n}.$)



e.g.



4
3
2
5
1
8
7
6
S^-
4
3
2
5
1
8
6
8a
7a
6a
5a
6b
1b
4a
2b
3a
3b
2a
4b
1a
7b
8b
5b
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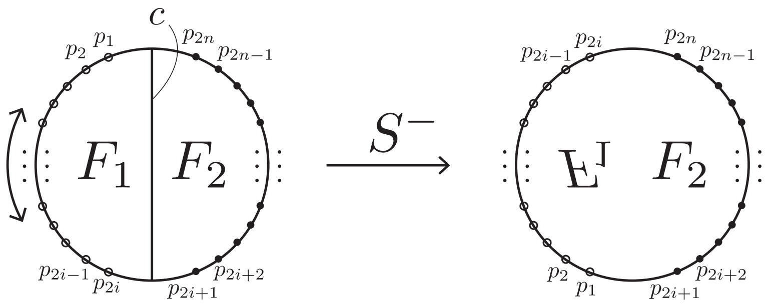
8a
1a
8b
5b



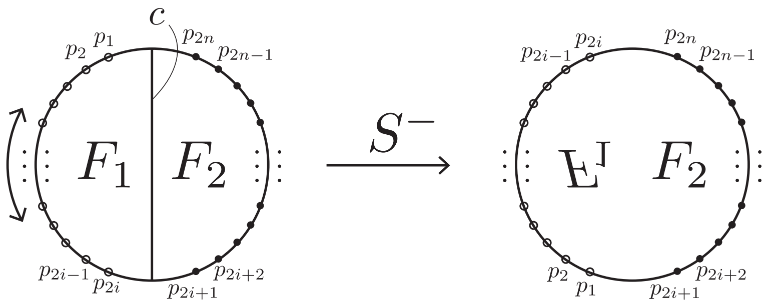
Property S^-

S^-

Claim

Key Lemma

Main Result 1



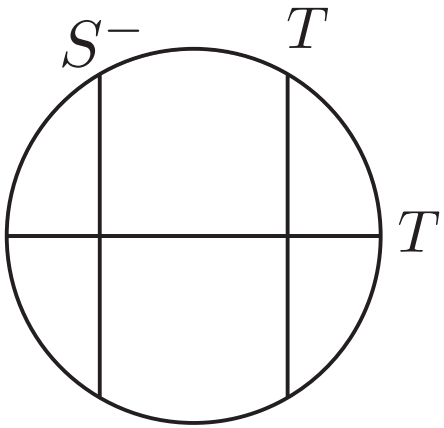
Property S⁻ **S⁻**

Claim

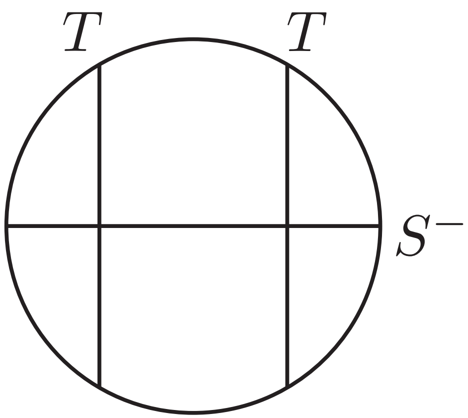
Key Lemma

Main Result 1

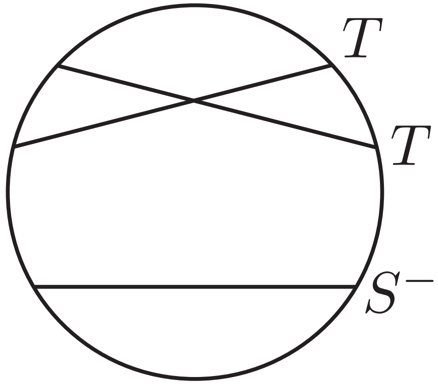
Claim. *Suppose that* $(\sigma_i)_{i=1}^{n(D)}$ *satisfies* $\sigma_1 = S^-$, $\sigma_2 = T_{\mathrm{split}}$, *and* $\sigma_3 = T_{\mathrm{join}}$. *Then, the three chords in* CD_D *corresponding to* σ_1, σ_2, *and* σ_3 *are as in*



(X)



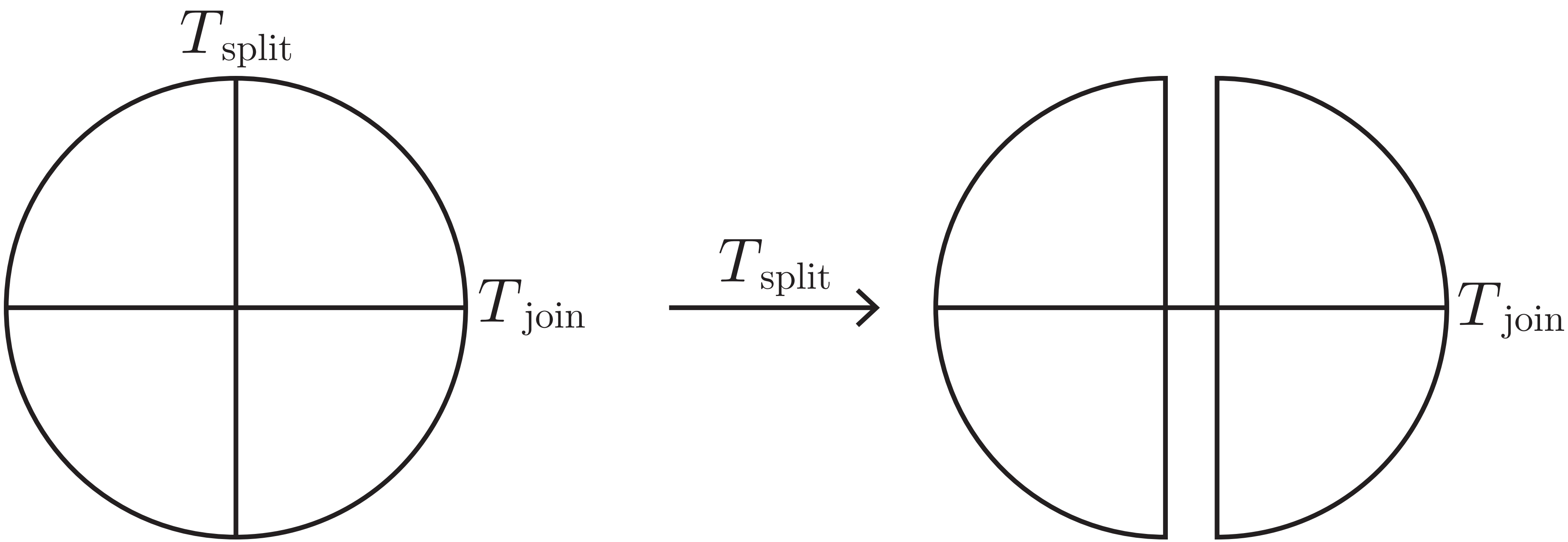
(X')

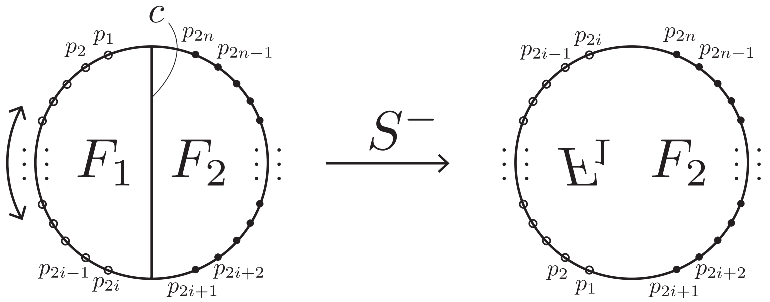


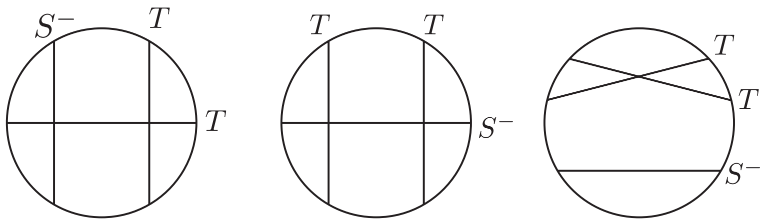
(Y)

Observation 1

<u>Component-preserving successive “T T” should have a chord intersection.</u>







(X) (X') (Y)

Property S^-

S^-

Claim

S^- T T

Key Lemma

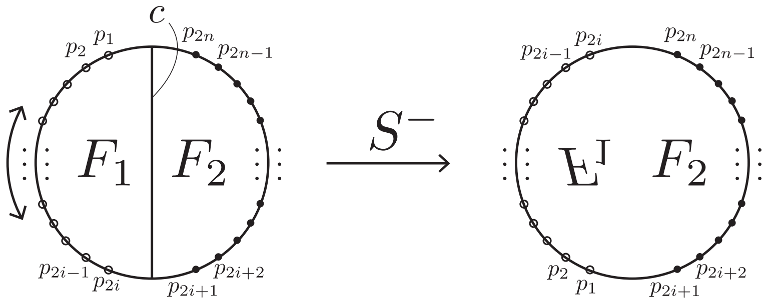
Main Result 1

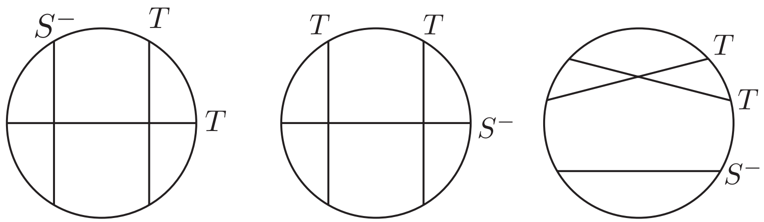
Key Lemma . *Let* D *be a prime (alternating or non-alternating) knot diagram with exactly* $n(D)$ (> 1) *crossings with* $\sigma_i \neq \mathrm{RI}^-$ $(\forall i)$*. Suppose that* $\sigma_1 = S^-$ *and that* $(\sigma_i)_{i=2}^{n(D)}$ *includes at least one* T_{join}, S^-_{join}, *or* S^-*.*

Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ *such that* $\sigma'_1 = S^-$ *and* $\sigma'_2 = S^-$, *and* $\sigma'_i \neq \mathrm{RI}^-$ $(\forall i)$*.*

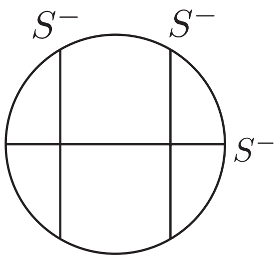
Key Lemma . *Let* D *be a prime (alternating or non-alternating) knot diagram with exactly* $n(D)$ (> 1) *crossings with* $\sigma_i \neq \mathrm{RI}^-$ $(\forall i)$*. Suppose that* $\sigma_1 = S^-$ *and that* $(\sigma_i)_{i=2}^{n(D)}$ *includes at least one* T_{join}*,* S^-_{join}*, or* S^-*.*
Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ *such that* $\sigma'_1 = S^-$ *and* $\sigma'_2 = S^-$*, and* $\sigma'_i \neq \mathrm{RI}^-$ $(\forall i)$*.*

Roughly speaking, suppose that AK-sequence starts from one S⁻. if "join" or more S⁻ appears in the seq., S⁻ … → S⁻ S⁻….by reordering.





(X) (X') (Y)



(X) (X')

Property S^- **S^-**

Claim **S^- T T**

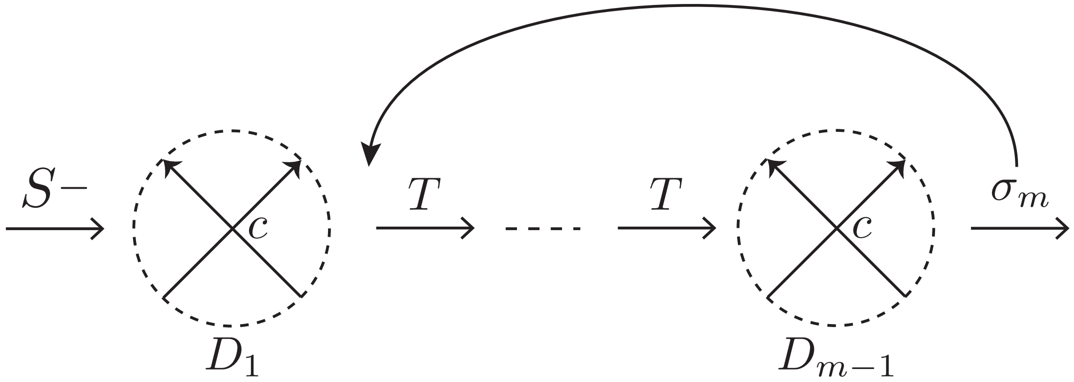
Key Lemma **S^- T T to $S^-S^-S^-$**

Main Result 1

Proof of Key Lemma

Case (1): $(\sigma_i)_{i=2}^{n(D)}$ includes at least one S^- or S^-_{join}.

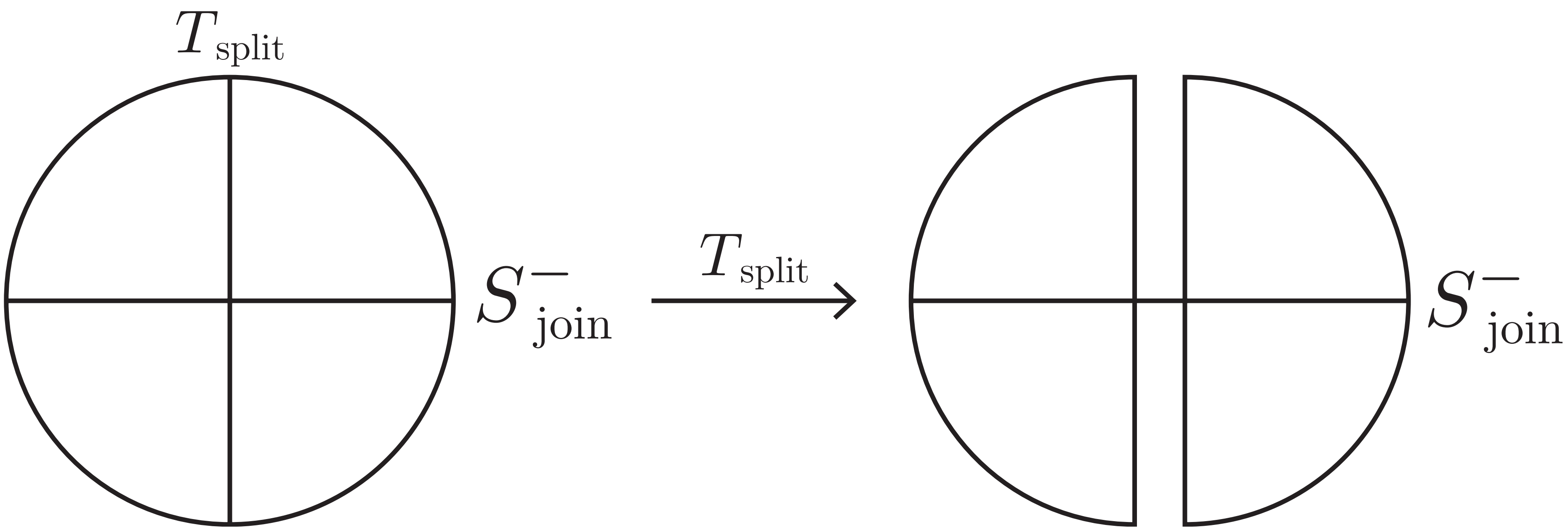
$S^-T\cdots TS^-\cdots$, or $S^-T\cdots TS^-_{\mathrm{join}}\cdots$. Moving σ_m ($= S^-$ or S^-_{join}) to σ'_2,



we obtain $S^-S^-\cdots$.

Observation 1'

<u>Component-preserving pair "T S" should have a chord intersection.</u>



Proof of Key Lemma

Case (2): $(\sigma_i)_{i=2}^{n(D)}$ includes no splice S^- and no splice S^-_{join}, but includes a splice T_{join}.

We have reordering:

$$S^- T_{\mathrm{split}} \cdots T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T \to S^- T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T.$$

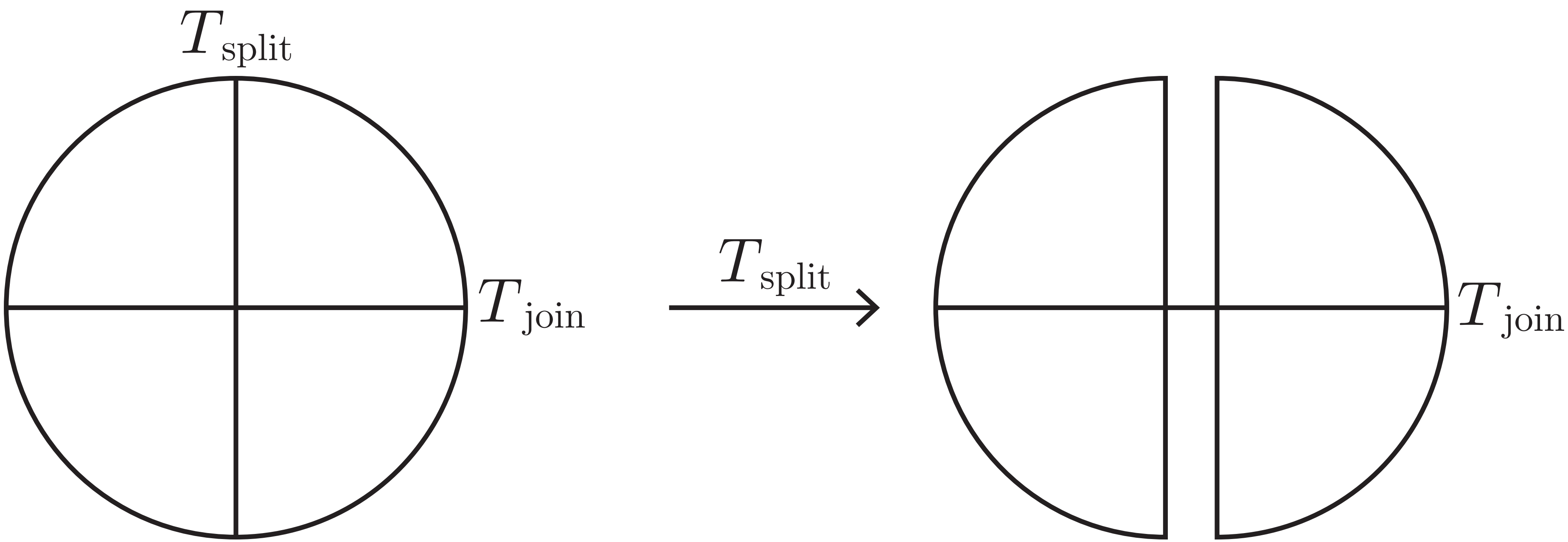
- Case: either (X) or (X') is included:

By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^-$...

- Case: there is no (X) and no (X'), but (Y) appears:

Observation 1

<u>Component-preserving pair “T T” should have a chord intersection.</u>



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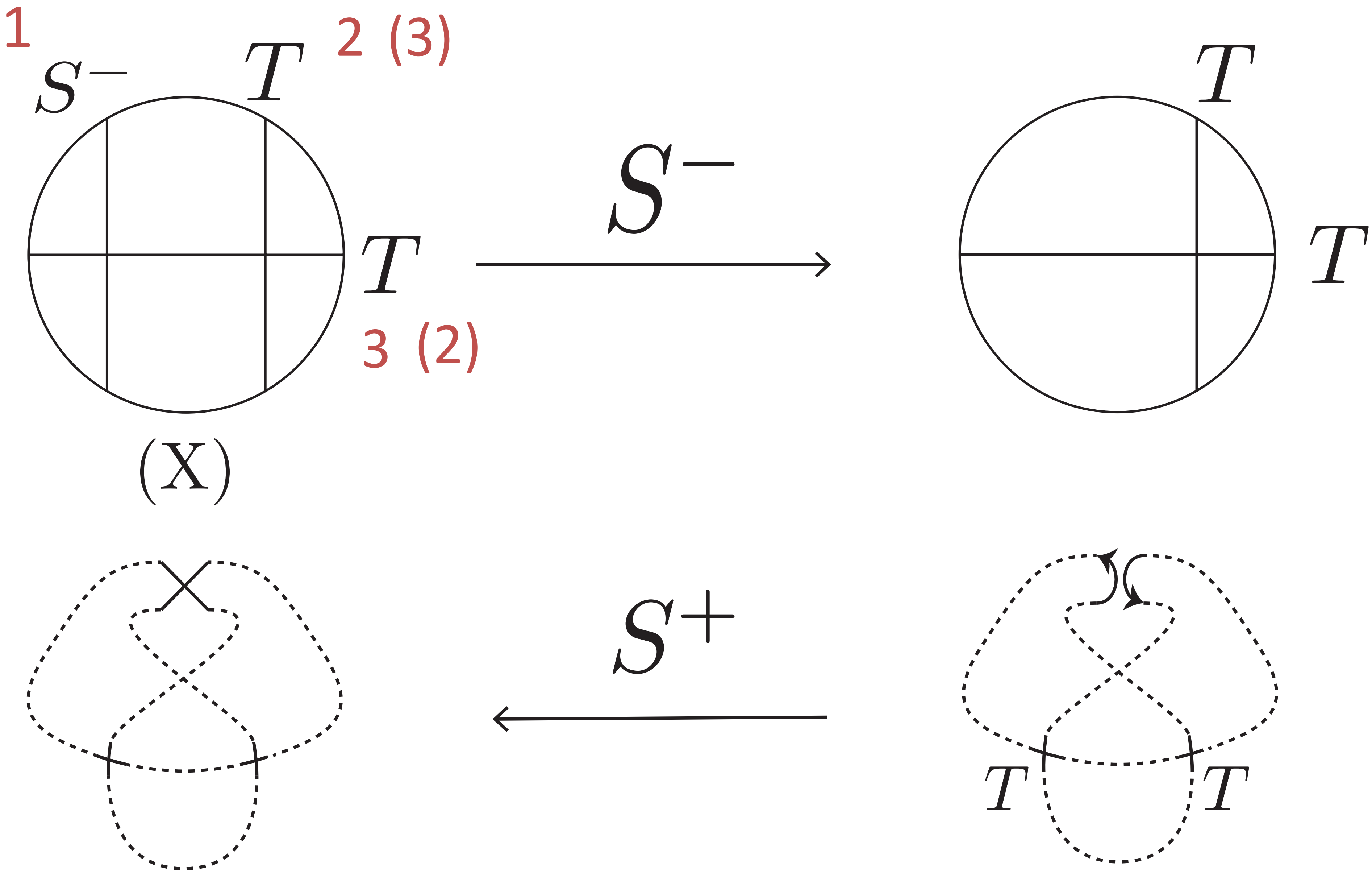
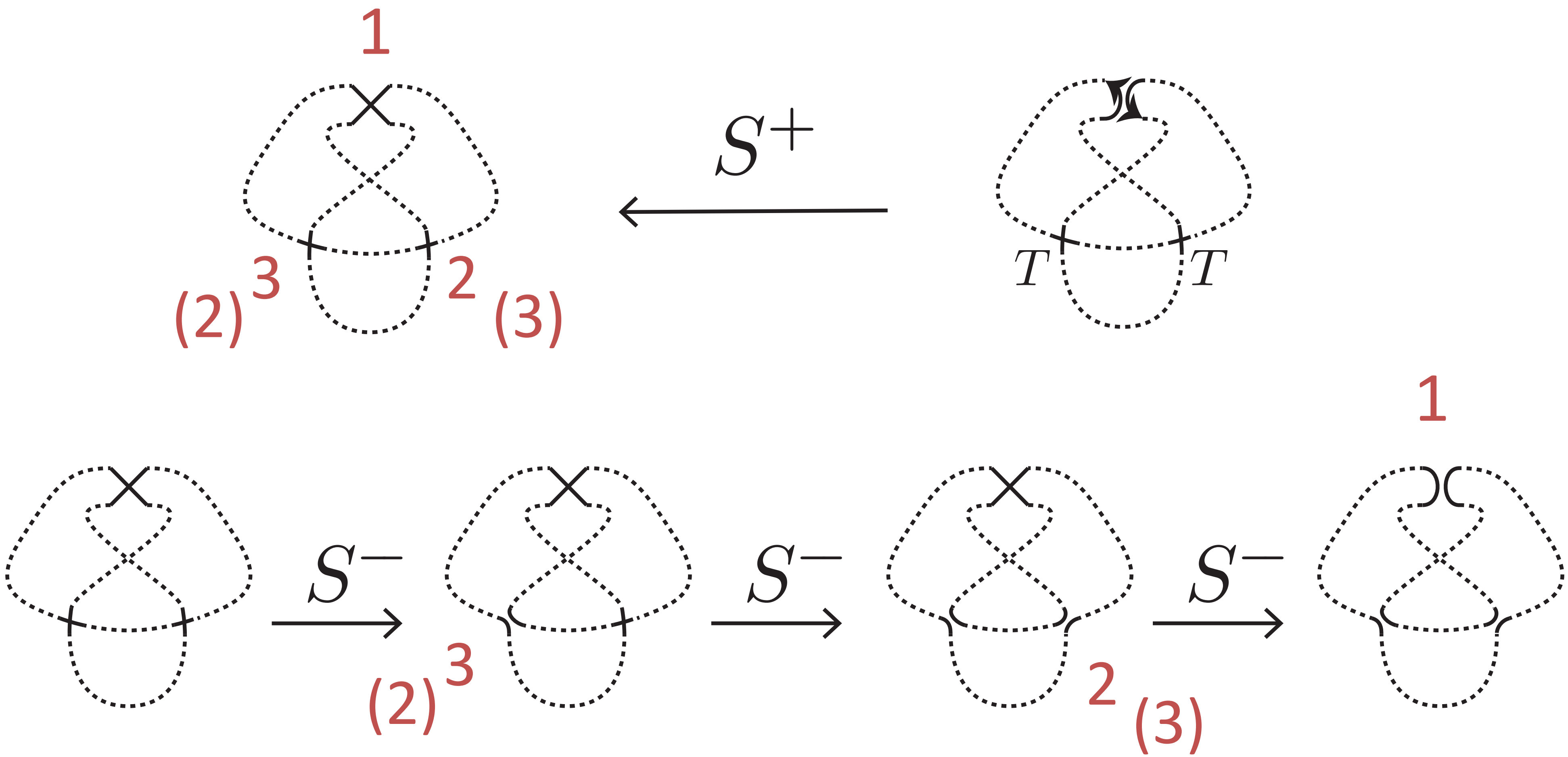
By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^- \ldots$ It's the highest point of the proof, we'll go to the next slide!

- Case: there is no (X) and no (X'), but (Y) appears:


1
S^-
T
2 (3)
T
3 (2)
(X)
S^-
T
T
S^+
T
T



Reordering: 123 —> 321 or 231

Proof of Key Lemma

Case (2): $(\sigma_i)_{i=2}^{n(D)}$ includes no splice S^- and no splice S^-_{join}, but includes a splice T_{join}.

We have reordering:

$$S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \to S^- T_{\text{split}} T_{\text{join}} T \cdots T.$$

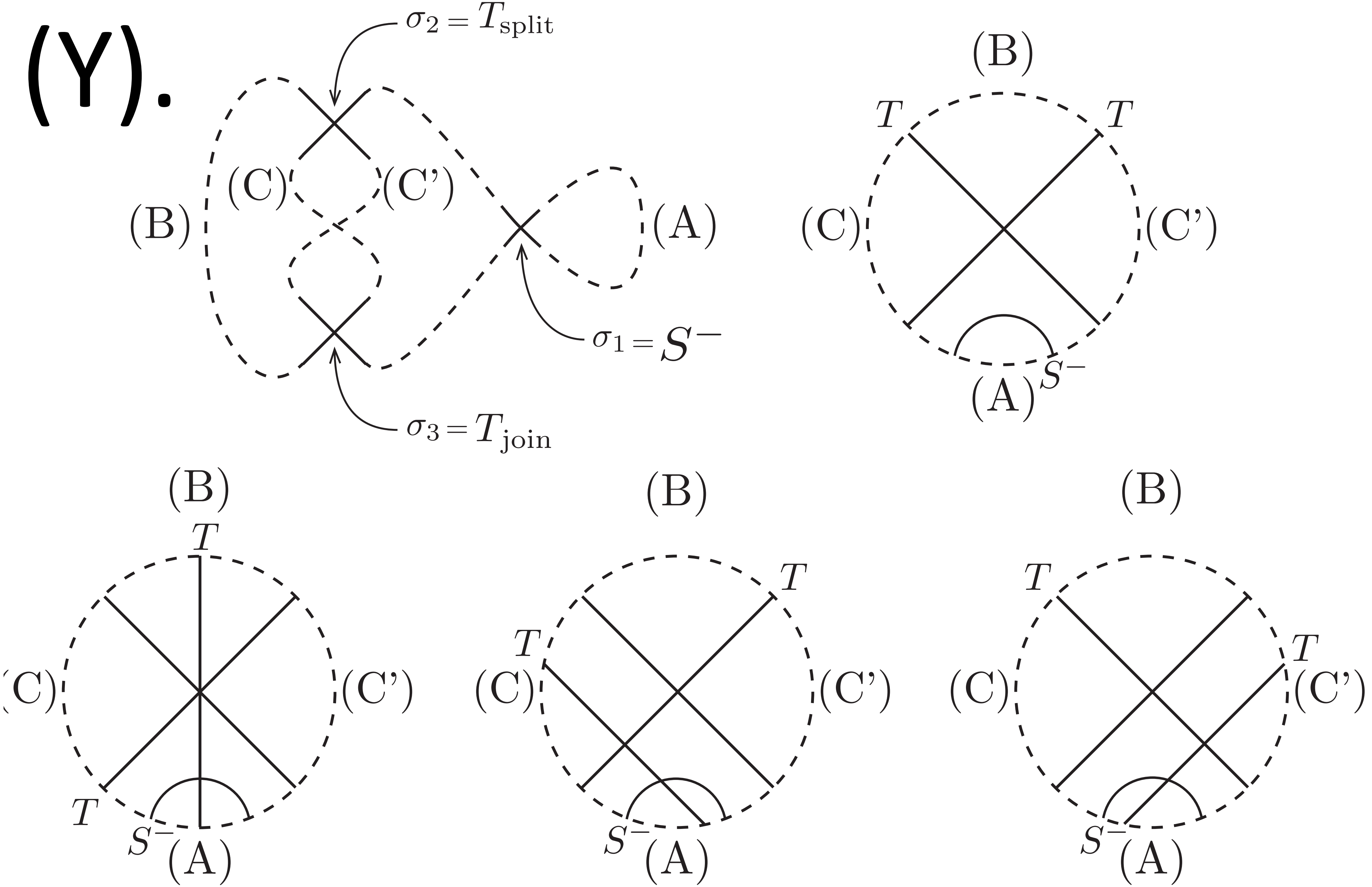
- Case: either (X) or (X') is included:

By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^- \ldots$

It's the highest point of the proof, we'll go to the next slide!

- Case: there is no (X) and no (X'), but (Y) appears:

(Y).



Proof of Key Lemma

Case (2): $(\sigma_i)_{i=2}^{n(D)}$ includes no splice S^- and no splice S^-_{join}, but includes a splice T_{join}.

We have reordering:

$$S^- T_{\mathrm{split}} \cdots T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T \to S^- T_{\mathrm{split}} T_{\mathrm{join}} T \cdots T.$$

- Case: either (X) or (X') is included:

By property of S^-, reordering $123 \to 231$ or 321 obtains a sequence $S^- S^- S^- \ldots$

- Case: there is no (X) and no (X'), but (Y) appears:

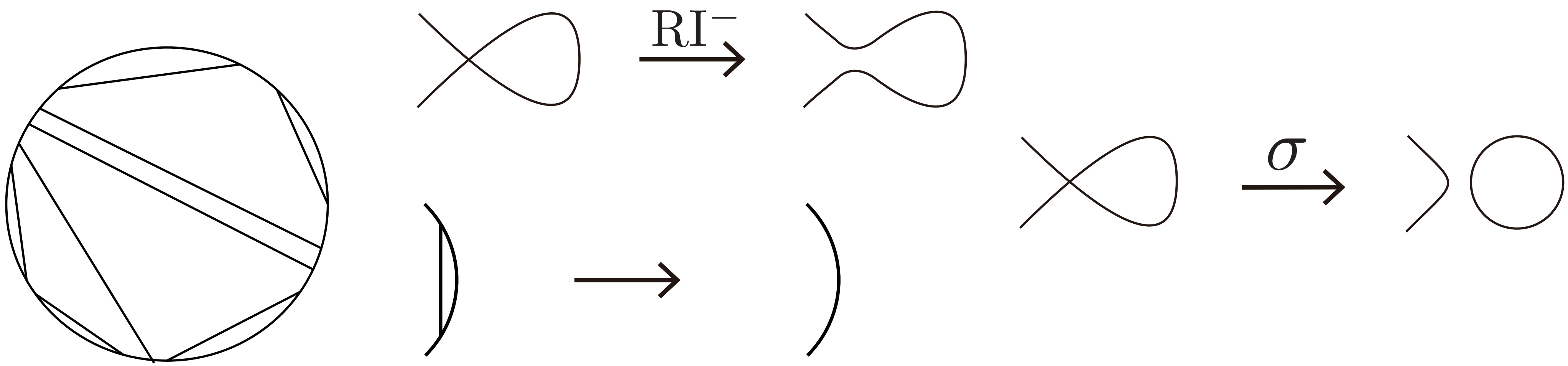
By primeness, (X) should be included $\to$ contradiction. $\square$

Applying Key Lemma the sequence of splices repeatedly, we have:

$$S^{-} \cdots S^{-} T_{\text{split}} \cdots T_{\text{split}}$$

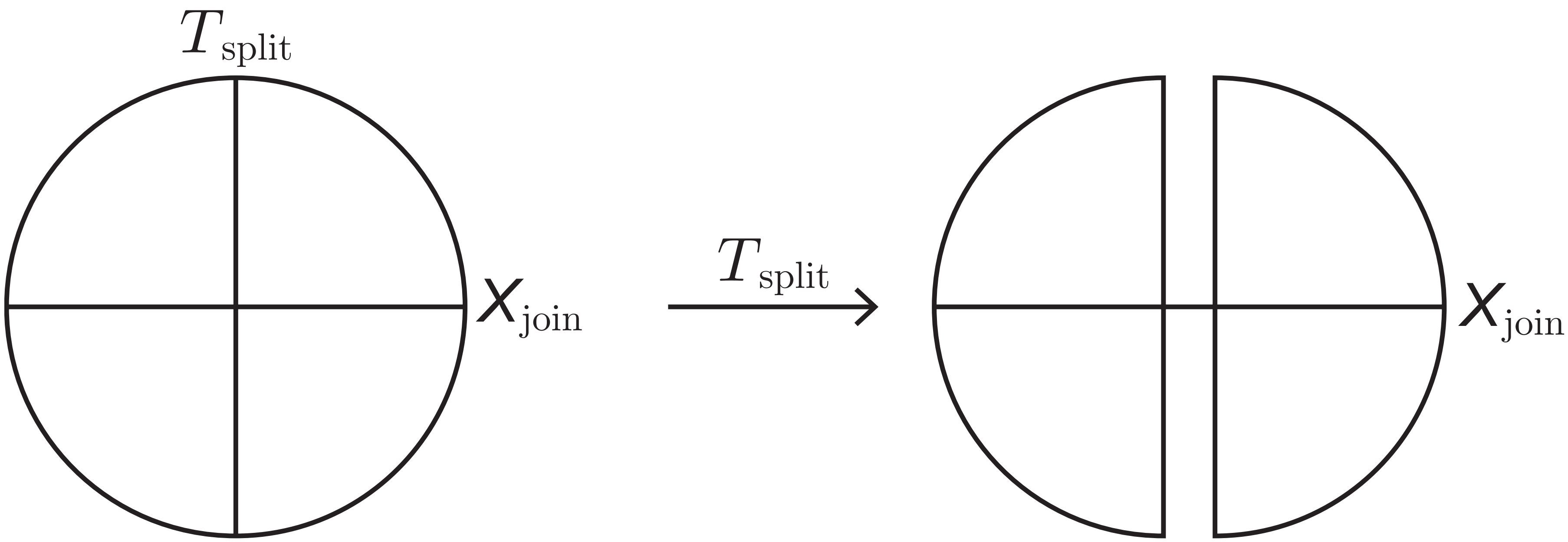
from Σ_{AK}.

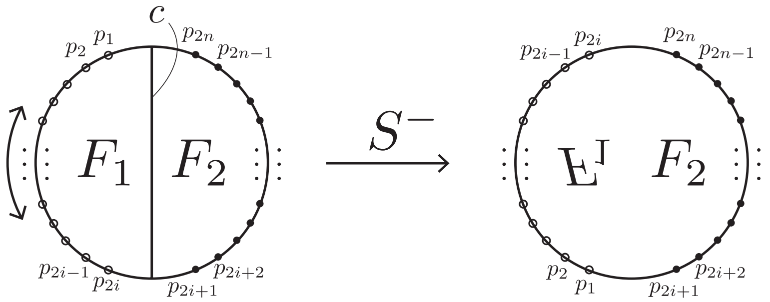
Here, in this seq., every T_{split} splits a monogon since there is no chord intersection after $S^{-} S^{-} \dots S^{-}$ applies.

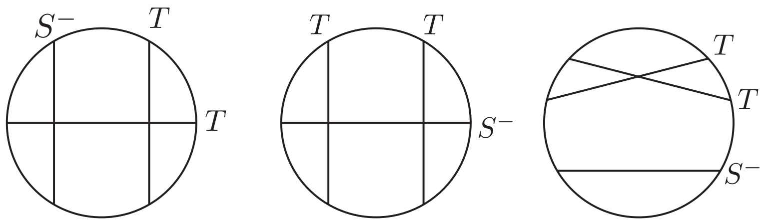


Observation 1’’

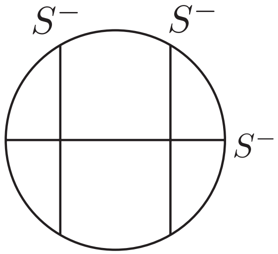
Any component-preserving pair “T X” should have a chord intersection.







(X) (X') (Y)



(X) (X')

$u^-(D) = C(K)$

Property S^-

Claim

Key Lemma

Main Result 1

S^-

S^- T T…

S^- T T…

to $S^-S^-S^-$

Finalizing Proof of Main Result 1 (lower bound)

Case Σ_{AK} is a non-orientable surface with the maximal Euler characteristic χ. (**Note**: the seq. has S^-; any $\sigma_i \neq \mathrm{RI}^-$.) Thus, by **Key Lemma**, this seq. realizes $u^-(D)$ by reordering.

$$S^- S^- \dots S^- T_{\mathrm{split}} T_{\mathrm{split}} \dots T_{\mathrm{split}}.$$

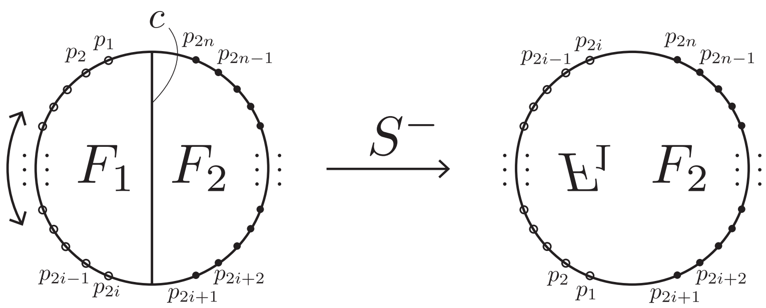
The reordering process implies Observation 2.

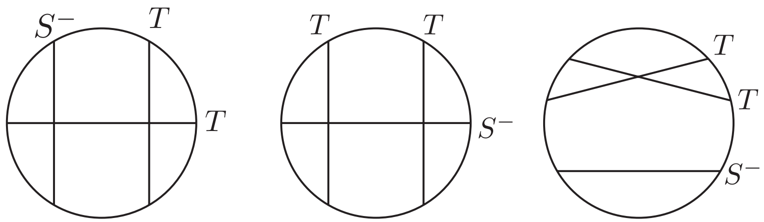
Observation 2. *Each reordering may cause:*

$$T_{\text{split}}, T_{\text{join}} \leftrightarrow S^-, S^- \quad \textit{or} \quad T_{\text{split}}, S^-_{\text{join}} \leftrightarrow S^-, S^-.$$

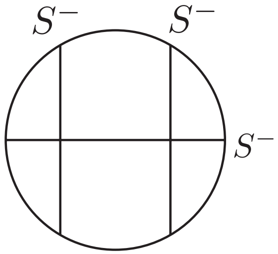
Thus,

$$\begin{aligned} 1 - u^-(D) &= 1 - \sharp\{S^- \text{in seq.}\} \\ &= 1 - 2\sharp T_{\text{join}} - 2\sharp S^-_{\text{join}} - \sharp S^- \\ &= 1 + (\sharp T_{\text{split}} - \sharp T_{\text{join}} - \sharp S^-_{\text{join}}) - n(D) \\ &= \chi(\Sigma_{AK}) = 1 - C(K). \end{aligned}$$





(X) (X') (Y)



(X) (X')

Property S^- — **S^-**

↓

Claim — **S^- T T…**

↓

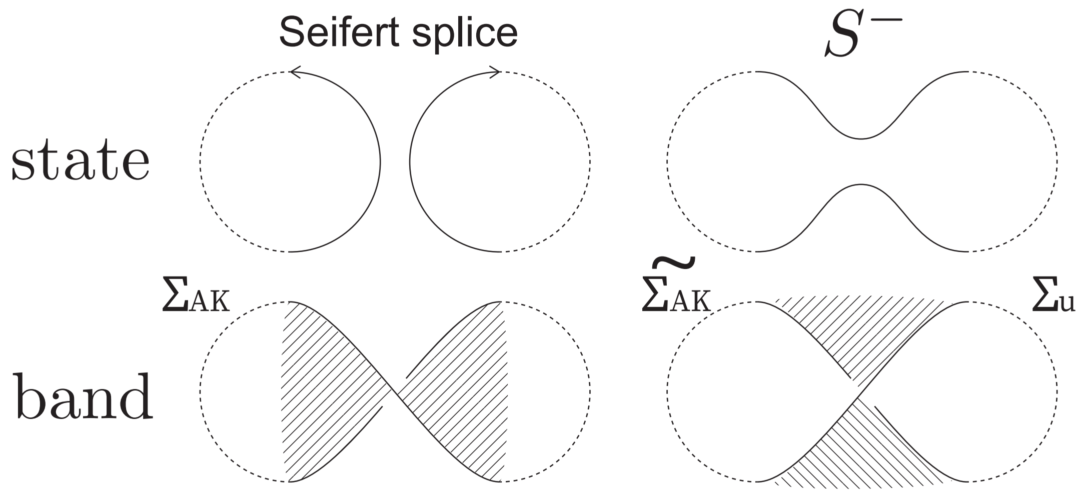
Key Lemma — **S^- T T… to $S^-S^-S^-$**

↓

$u^-(D) = C(K)$ — **Main Result 1**

Finalizing Proof of Main Result 1 (lower bound)

Case Σ_{AK} is a orientable surface with the maximal Euler characteristic. **Note**: $2g(K) < C(K) \Leftrightarrow C(K) = 2g(K) + 1$. It **returns to the non-orientable case** since χ $(= 1 - 2g(K))$ is changed into $1 - (2g(K) + 1)$ $(= 1 - C(K))$ by the replacement:



Then for any prime alternating knot diagram D,

$$u^{-}(K) \leq \min_{D} u^{-}(D) = C(K).$$

Recalling that $C(K) \leq u^{-}(K)$, it completes the proof. $\square$

Ito-Takimura, 2018, arXiv: 2008.11061

By the argument of this proof, we have:

Main Result 2 [Takimura-I. IJM2020]

For any knot K, if there exists a state realizing the maximal Euler characteristic,

$$u^{-}(K) = C(K).$$

Next target

- Categorification of C(K). Can we relate sl(2) homology to crosscap? (cf. HFK determines orientable genera.) Comment by Prof. J.S Carter.
- Can we have more volume bounds ?

Thank you for your attention!

文献など(敬称略)

Clark (1978) 定義の導入とC(K)=1の決定 [IJMS]

<80年代に Hatcher-Thurston [Invent. Math, 1985] Hacher-Oertel [Topology, 1989] >

村上斉-安原 (1995) 加法性が成り立つ必要十分条件 (特にC(K)=2g(K)+1のKにおいては加法性が崩れる) [PJM]

別所 (1996) 結び目補空間によるクロスキャップの計算およびC(K)=2g(K)+1の無限列 [阪大修士論文]

寺垣内 (2004) torus knot のcrosscap numberの決定 [Topology Appl.]

寺垣内-平澤 (2006) 2-bridge knotのcrosscap numberの決定 [Topology]

市原-水嶋 (2006) many pretzel knot のcrosscap number の決定 [Topology Appl.]

小沢 (2011) state surfaceの導入 [J. Aust. Math.Soc.]

Adams-Kindred (2013) alternating knotのcrosscap numberの理論的な決定 [AGT]

Kalfagianni-Lee (2016) (colored) Jones polynomialとの関係 [Advanced Math]

Takimura-I. (2018) u-(P)の導入, crosscap two alternating knotsの決定[IJM] (cf. 2016, Ichihara-Masai [CAG])

Takimura-I. (2019) crosscap n alternating knots via band surgery [JKTR]

Kindred (2020) crosscap n alternating knots by splice-unknotting number Takimura-I. u- and fylpes [IJM]

Takimura-I. (2020) crosscap n alternating knots by u- and Jones polynomial [IJM]