Splice-unknotting operation and crosscap numbers
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Main Result 1

Let $C(K)$ be the crosscap number of $K$.

For any prime alternating knot $K$,

$$C(K) = u^-(K).$$
Recalling definition: $u^-(P), u^-(K)$

- $u^-(P)$ is the minimum number of necessary splices of type $S^-$ among any sequences of $S^-$ and $RI^-$ to obtain $O$. $u^-(K) := \min_P u^-(P)$. 
Plan of proof for $u^-(D) \leq C(K)$

We will compare

$\Sigma u : \text{a non-orientable state surface realizing } u^-(D)$

with

$\Sigma_{AK} : \text{a surface realizing } C(K) \text{ or } g(K)$
(Adams-Kindred).
\[ P \xrightarrow{\sigma_1} \xrightarrow{\sigma_2} \xrightarrow{\sigma_3 \sigma_4 \sigma_5 \sigma_6} S_\sigma \quad |S_\sigma| = 5 \]

\[ D_P \]

\[ \Sigma u := \Sigma_\sigma (D_P) \]
Construction of $\Sigma_{AK}$

Find m-gon of the smallest m and splice as follows
(P has a 3-gon if $3 \leq m$)

Eliahou-Harary-Kauffman, 2008

or
Notation 1. $\Sigma_{AK}$ gives a sequence of splices $(\sigma_i)_{i=1}^{n(D)}$:

$$D = D_0 \xrightarrow{\sigma_1} D_1 \xrightarrow{\sigma_2} D_2 \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n(D)}} D_{n(D)}$$

Each orientation of $D_i$ is of $\sigma_i$. ($= \text{ori. } S^{-}, S_{\text{join}}^{-}, T_{\text{split}}, T_{\text{join}}$).

It induces

$$CD_D = CD_0 \xrightarrow{\sigma_1} CD_1 \xrightarrow{\sigma_2} CD_2 \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n(D)}} CD_{n(D)}.$$  

($CD_D$ is a Gauss diagram of $D$; it will be defined.)
Notation 2. \textit{Oriented} $T_{\text{split}}, T_{\text{join}}$. Seifert splices.

\[D_i\]
(a component)

\[D_{i+1}\]
(two components)

\[D_i\]
(two components)

\[D_{i+1}\]
(a component)
• *Oriented RI*. 1st Reidemeister move.

• *Oriented \(S^-\), \(S^\text{join}_-\). Target orientation must be chosen.
Property $S^-$

Claim

Key Lemma

Main Result 1
Property $S^-$

Claim

Key Lemma

Main Result 1

$S^-$

$S^- T T$

$S^- T T$

to $S^- S^- S^-$
Definition 1. Let $D$ be a knot diagram whose projection is $P$. Then there is a generic immersion $g : S^1 \to S^2$ such that $g(S^1) = P$. It is denoted by $CD_D$. 
Property $S^{-}$. The behavior of $S^{-}$ in $CD_D$ is as follows. (The difference of cyclic Gauss words is presented as: $c p_1 p_2 \cdots p_{2i} c p_{2i+1} \cdots p_{2n} \rightarrow p_{2i} p_{2i-1} \cdots p_1 p_{2i+1} p_{2i+2} \cdots p_{2n}$.)

Diagram: $F_1 \xrightarrow{S^{-}} F_2$
e.g.
Property $S^-$

Claim

Key Lemma

Main Result 1
Property $S^-$

Claim

Key Lemma

Main Result 1
Claim. Suppose that \((\sigma_i)_{i=1}^{n(D)}\) satisfies \(\sigma_1 = S^-, \sigma_2 = T_{split}, \) and \(\sigma_3 = T_{join}\). Then, the three chords in \(CD_D\) corresponding to \(\sigma_1, \sigma_2, \) and \(\sigma_3\) are as in

![Diagrams](image-url)
Observation 1

Component-preserving successive “T T” should have a chord intersection.
Property $S^-$  

Claim  

Key Lemma  

Main Result 1
Key Lemma. Let $D$ be a prime (alternating or non-alternating) knot diagram with exactly $n(D) (> 1)$ crossings with $\sigma_i \neq \text{RI}^- (\forall i)$. Suppose that $\sigma_1 = S^-$ and that $(\sigma_i)_{i=2}^{n(D)}$ includes at least one $T_{\text{join}}$, $S^-_{\text{join}}$, or $S^-$. Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ such that $\sigma'_1 = S^-$ and $\sigma'_2 = S^-$, and $\sigma'_i \neq \text{RI}^- (\forall i)$. 
**Key Lemma.** Let $D$ be a prime (alternating or non-alternating) knot diagram with exactly $n(D) > 1$ crossings with $\sigma_i \neq RI^- (\forall i)$. Suppose that $\sigma_1 = S^-$ and that $(\sigma_i)_{i=2}^{n(D)}$ includes at least one $T_{\text{join}}$, $S^-_{\text{join}}$, or $S^-$. Then it is possible to re-index the same set of splices as $(\sigma'_i)_{i=1}^{n(D)}$ such that $\sigma'_1 = S^-$ and $\sigma'_2 = S^-$, and $\sigma'_i \neq RI^- (\forall i)$.

**Roughly speaking,** suppose that AK-sequence starts from one $S^-$. If “join” or more $S^-$ appears in the seq., $S^- \rightarrow S^- S^- ...$ by reordering.
Key Lemma

Property $S^{-}$

Claim

Main Result 1
Proof of Key Lemma

Case (1): \((\sigma_i)_{i=2}^{n(D)}\) includes at least one \(S^-\) or \(S_{\text{join}}^-\). 
\(S^- T \cdots TS^- \cdots\), or \(S^- T \cdots TS_{\text{join}}^- \cdots\). Moving \(\sigma_m (= S^- \text{ or } S_{\text{join}}^-)\) to \(\sigma'_2\),

\[
\begin{array}{c}
S^- \xrightarrow{c} D_1 \xrightarrow{T} \cdots \xrightarrow{T} D_{m-1} \xrightarrow{\sigma_m} \\
\end{array}
\]

we obtain \(S^- S^- \cdots\).
Observation 1’

Component-preserving pair “T S” should have a chord intersection.
Case (2): \( (\sigma_i)^{\frac{n(D)}{i=2}} \) includes no splice \( S^- \) and no splice \( S_{\text{join}}^- \), but includes a splice \( T_{\text{join}} \).

We have reordering:

\[
S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \rightarrow S^- T_{\text{split}} T_{\text{join}} T \cdots T.
\]

- **Case:** either (X) or (X’) is included:
  By property of \( S^- \), reordering \( 123 \rightarrow 231 \) or \( 321 \) obtains a sequence \( S^- S^- S^- \ldots \)

- **Case:** there is no (X) and no (X’), but (Y) appears:
Observation 1

Component-preserving pair “T T” should have a chord intersection.
Proof of Key Lemma

Case (2): \((\sigma_i)^{n(D)}\in_{i=2}\) includes no splice \(S^-\) and no splice \(S_{\text{join}}^-\), but includes a splice \(T_{\text{join}}\).

We have reordering:

\[S^-T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \rightarrow S^-T_{\text{split}} T_{\text{join}} T \cdots T.\]

- Case: either (X) or (X') is included:
  By property of \(S^-\), reordering \(123 \rightarrow 231\) or \(321\) obtains a sequence \(S^- S^- S^- \cdots\).

- Case: there is no (X) and no (X'), but (Y) appears:

It’s the highest point of the proof, we’ll go to the next slide!
Reordering: $123 \rightarrow 321$ or $231$
Case (2): \((\sigma_i)^{n(D)}\) includes no splice \(S^-\) and no splice \(S'_{\text{join}}\), but includes a splice \(T_{\text{join}}\).

We have reordering:

\[
S^-T_{\text{split}} \cdots T_{\text{split}}T_{\text{join}}T \cdots T \rightarrow S^-T_{\text{split}}T_{\text{join}}T \cdots T.
\]

- Case: either \((X)\) or \((X')\) is included:

By property of \(S^-\), reordering \(123 \rightarrow 231\) or \(321\) obtains a sequence \(S^-S^-S^-\ldots\) It’s the highest point of the proof, we’ll go to the next slide!

- Case: there is no \((X)\) and no \((X')\), but \((Y)\) appears:
Proof of Key Lemma

Case (2): \( (\sigma_i)^{n(D)} \) includes no splice \( S^- \) and no splice \( S^-_{\text{join}} \), but includes a splice \( T_{\text{join}} \).

We have reordering:

\[
S^- T_{\text{split}} \cdots T_{\text{split}} T_{\text{join}} T \cdots T \rightarrow S^- T_{\text{split}} T_{\text{join}} T \cdots T.
\]

• Case: either \( (X) \) or \( (X') \) is included:

By property of \( S^- \), reordering \( 123 \rightarrow 231 \) or \( 321 \) obtains a sequence \( S^- S^- S^- \ldots \).

• Case: there is no \( (X) \) and no \( (X') \), but \( (Y) \) appears:

By primeness, \( (X) \) should be included \( \rightarrow \) contradiction. □
Applying Key Lemma the sequence of splices repeatedly, we have:

$$S^- \cdots S^- T_{\text{split}} \cdots T_{\text{split}}$$

from $\Sigma_{AK}$.

Here, in this seq., every $T_{\text{split}}$ splits a monogon since there is no chord intersection after $S^- S^- \ldots S^-$ applies.
Observation 1”

Any component-preserving pair “T X” should have a chord intersection.
Property $S^{-}$

Claim

Key Lemma

Main Result 1

$u^{-}(D) = C(K)$

$S^{-}$

$S^{-} T T T...$

$S^{-} T T T...$

to $S^{-} S^{-} S^{-}$
Finalizing Proof of Main Result 1 (lower bound)

Case $\Sigma_{AK}$ is a non-orientable surface with the maximal Euler characteristic $\chi$. (Note: the seq. has $S^-; \text{any } \sigma_i \neq RI^-.$)

Thus, by Key Lemma, this seq. realizes $u^-(D)$ by reordering.

$$S^- S^- \ldots S^- T_{split} T_{split} \ldots T_{split}.$$ 

The reordering process implies Observation 2.
Observation 2. Each reordering may cause:

\[ T_{\text{split}}, T_{\text{join}} \leftrightarrow S^-, S^- \quad \text{or} \quad T_{\text{split}}, S_{\text{join}}^- \leftrightarrow S^-, S^- . \]

Thus,

\[
1 - u^-(D) = 1 - \#\{S^- \text{in seq.}\} \\
= 1 - 2\#T_{\text{join}} - 2\#S_{\text{join}}^- - \#S^- \\
= 1 + (\#T_{\text{split}} - \#T_{\text{join}} - \#S_{\text{join}}^-) - n(D) \\
= \chi(\Sigma_{AK}) = 1 - C(K). 
\]
Claim

Key Lemma

Property $S^-$

Main Result 1

$u^{-}(D) = C(K)$

$S^-$

$S^{-} T T T...$

$S^{-} T T T...$

to $S^{-} S^{-} S^{-}$
Finalizing Proof of Main Result 1 (lower bound)

**Case** $\Sigma_{AK}$ is a **orientable** surface with the maximal Euler characteristic. **Note:** $2g(K) < C(K) \Leftrightarrow C(K) = 2g(K) + 1$.

It **returns to the non-orientable case** since $\chi \ (= 1 - 2g(K))$ is changed into $1 - (2g(K) + 1) \ (= 1 - C(K))$ by the replacement:

![Diagram of Seifert splice](image_url)
Then for any prime alternating knot diagram $D$,\[ u^-(K) \leq \min_D u^-(D) = C(K). \]

Recalling that $C(K) \leq u^-(K)$, it completes the proof. \[\square\]
By the argument of this proof, we have:

**Main Result 2** [Takimura-I. IJM2020]

For any knot $K$, if there exists a state realizing the maximal Euler characteristic,

$$u^-(K) = C(K).$$
Next target

• Categorification of $C(K)$. Can we relate $sl(2)$ homology to crosscap? (cf. $HFK$ determines orientable genera.)
Comment by Prof. J.S Carter.

• Can we have more volume bounds?
文献など(敬称略)

Clark (1978) 定義の導入とC(K)=1の決定 [IJMS]


村上斎-安原 (1995) 加法性が成立立つ必要十分条件 (特にC(K)=2g(K)+1のKにおいては加法性が崩れる) [PJM]

別所 (1996) 結び目補空間によるクロスキャップの計算およびC(K)=2g(K)+1の無限列 [阪大修士論文]


寺垣内-平澤 (2006) 2-bridge knotのcrosscap numberの決定 [Topology]

市原-水嶋 (2006) many pretzel knot のcrosscap number の決定 [Topology Appl.]


Adams-Kindred (2013) alternating knotのcrosscap numberの理論的な決定 [AGT]


Kindred (2020) crosscap n alternating knots by splice-unknotting number Takimura-I. u- and fylpes [IJM]

Takimura-I. (2020) crosscap n alternating knots by u- and Jones polynomial [IJM]

Thank you for your attention!