

On a conjecture by Goussarov-Polyak-Viro for virtual knots and Gauss diagram formulas

Noboru Ito (NIT, Ibaraki College)

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J.w.w. Yuka Kotorii (Hiroshima U.)

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Definition (Vassiliev invariant of finite degree)

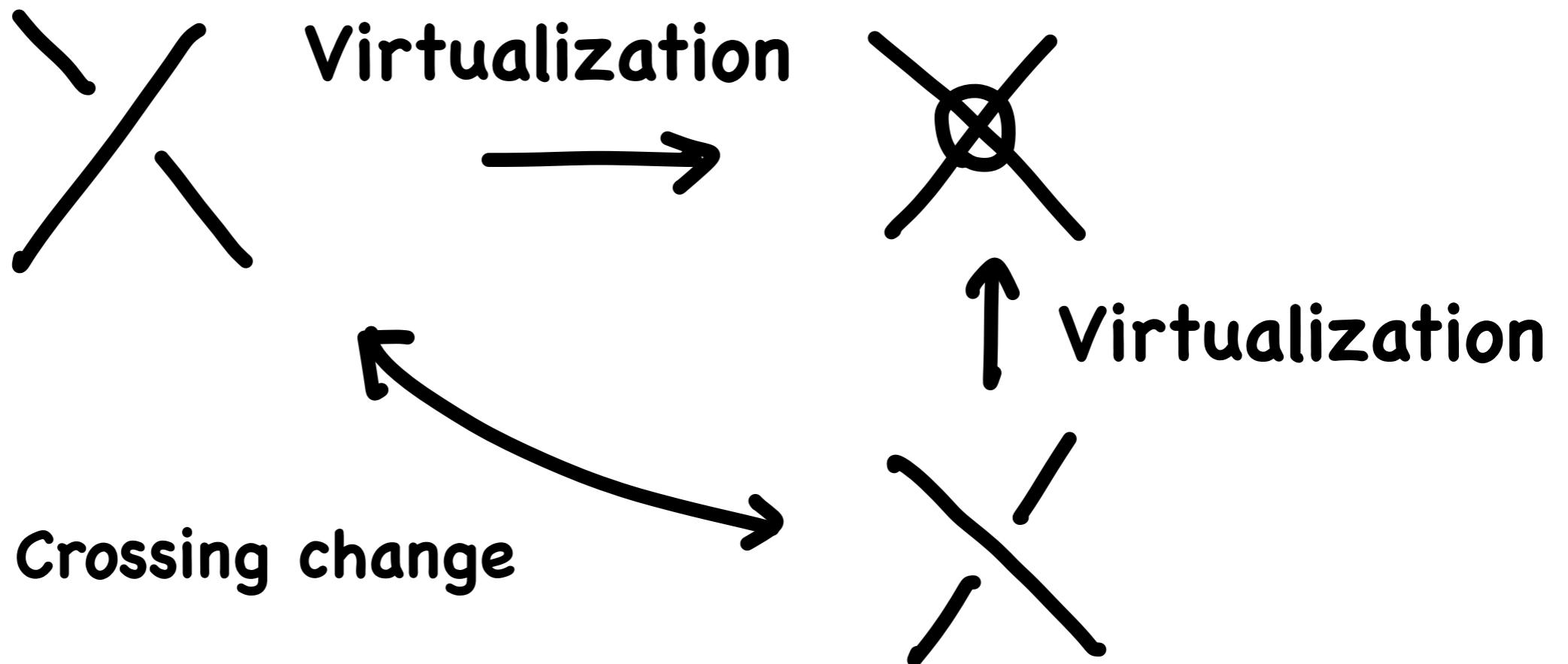
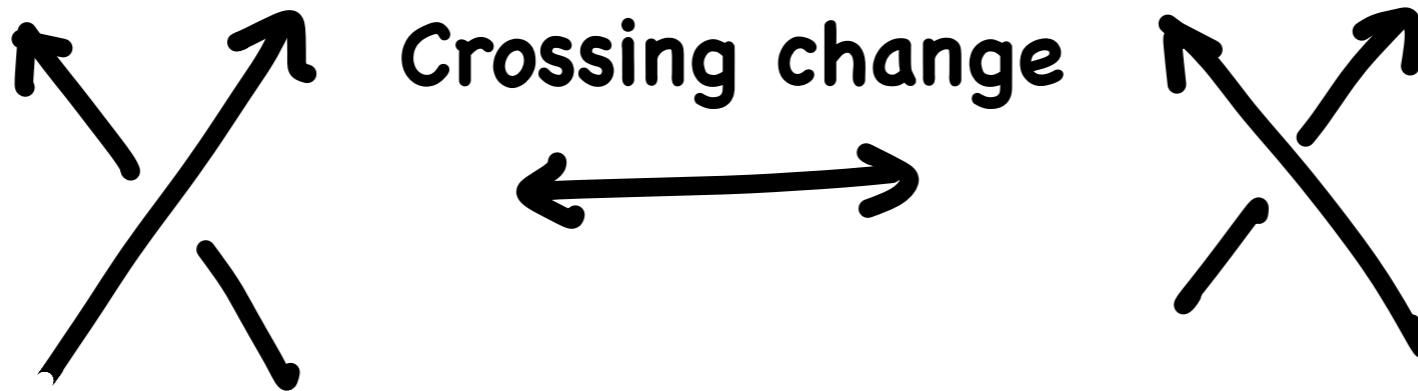
For a knot invariant v which values in an Abelian group, we extend it by

$$v(\text{X}) = v(\text{--X}) - v(\text{X--}).$$

If $\underbrace{v(\text{X} \text{ } \text{X} \text{ } \text{--} \text{X})}_{n+1} = 0,$

v is an invariant of (finite) degree n .

idea (Goussarov-Polyak-Viro)



$$v\left(\begin{array}{c} \nearrow \\ \searrow \end{array}\right) = v\left(\begin{array}{c} \nearrow \\ \nearrow \end{array}\right) - v\left(\begin{array}{c} \nearrow \\ \searrow \end{array}\right)$$

is switched to the relation:

$$v\left(\begin{array}{c} \circ \\ \times \end{array}\right) = v\left(\begin{array}{c} \times \\ \times \end{array}\right) - v\left(\begin{array}{c} \times \\ \times \end{array}\right).$$

If $v\left(\underbrace{\begin{array}{c} \times \\ \times \end{array} \cdots \begin{array}{c} \times \\ \times \end{array}}_{n+1}\right) = 0,$

v is an invariant of GPV-order n .

Virtualization



has advantage to construct a universal invariant.

I will explain it from the next page.

Def. (universal invariant) [Goussarov-Polyak-Viro]

Let \mathcal{K} be the set of virtual knots and

P, G abelian groups. Let I (v , resp.) be a P -
(G -, resp.)valued virtual knot invariant.

We call I a universal invariant of degree n

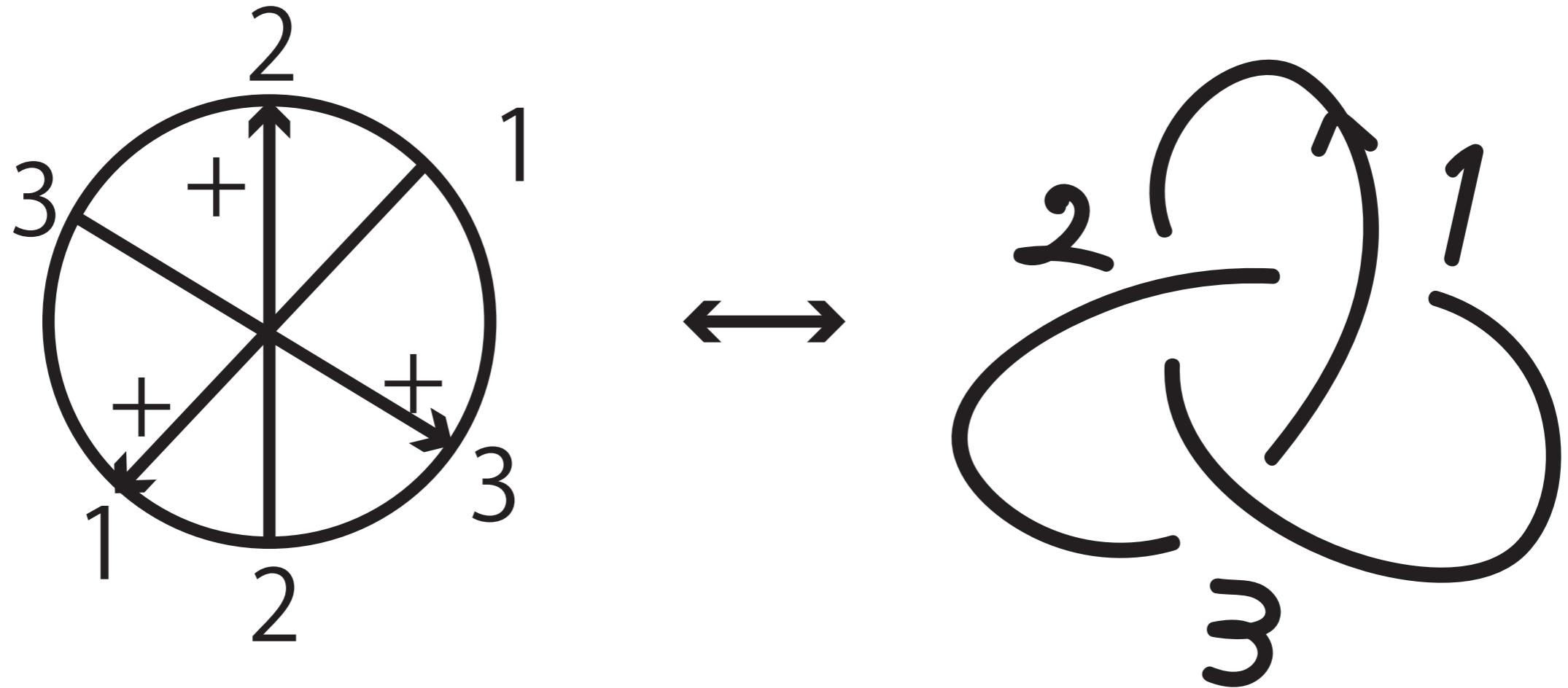
if for any v of degree n , there exists π
such that

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{I} & P \\ & \searrow v & \downarrow \pi \\ & & G \end{array}$$

Definition

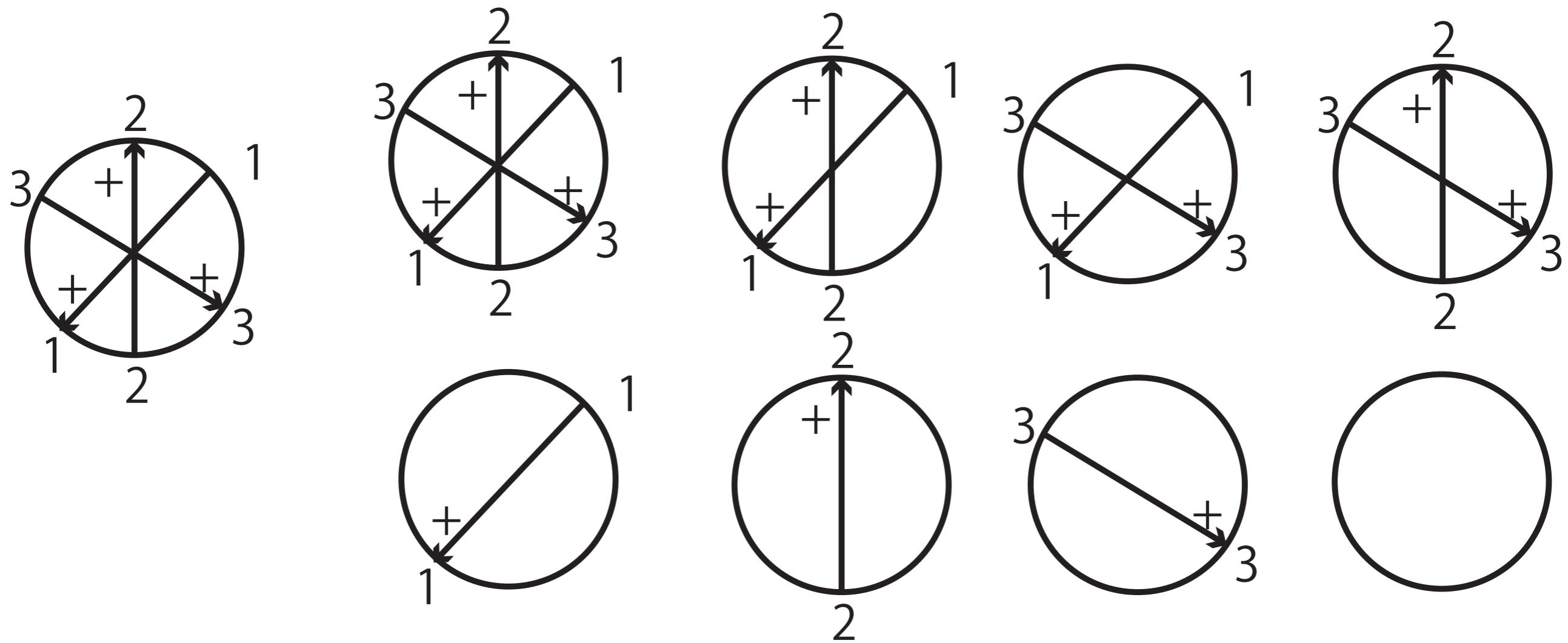
$$I(D) := \sum_{D' \subset D} i(D') .$$

D : Gauss diagram.



Definition $I(D) := \sum_{D' \subset D} i(D')$.

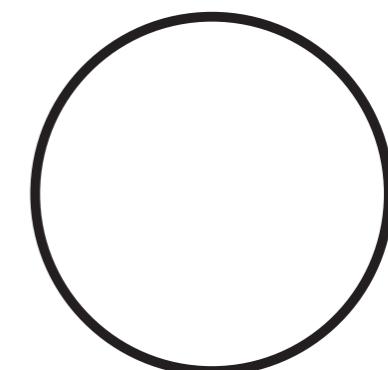
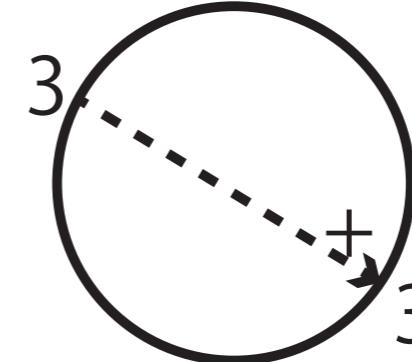
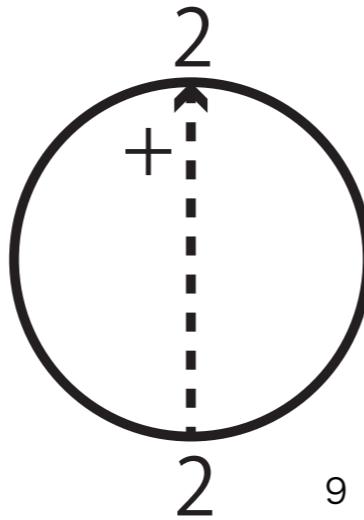
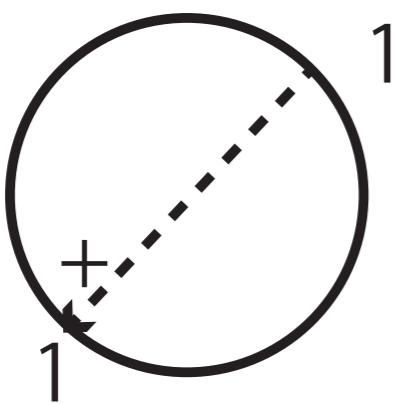
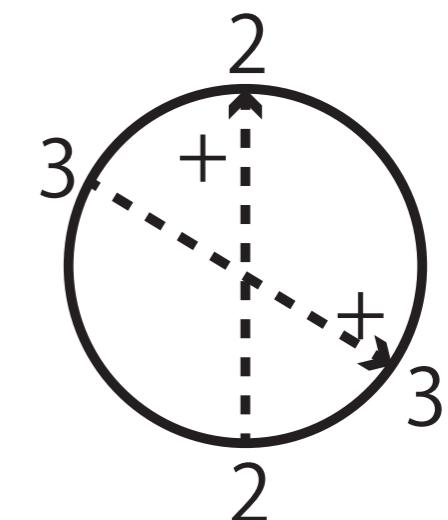
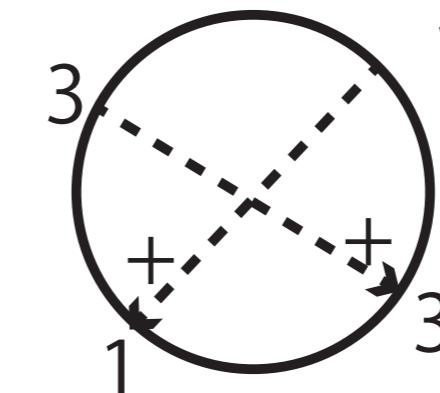
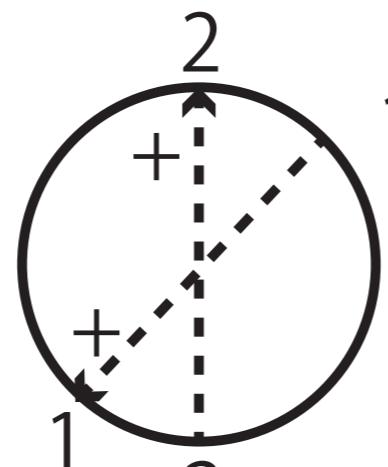
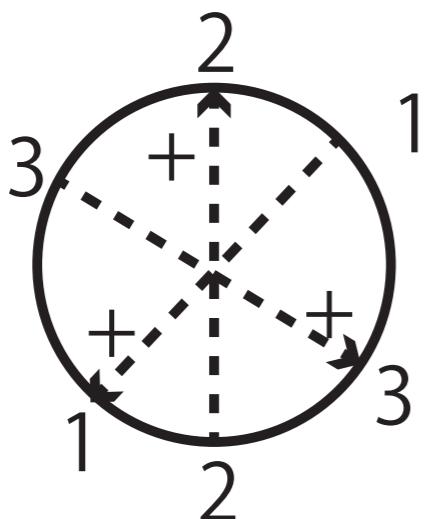
$D' \subset D$: taking sub-diagrams



Definition

$$I(D) := \sum_{D' \subset D} i(D') .$$

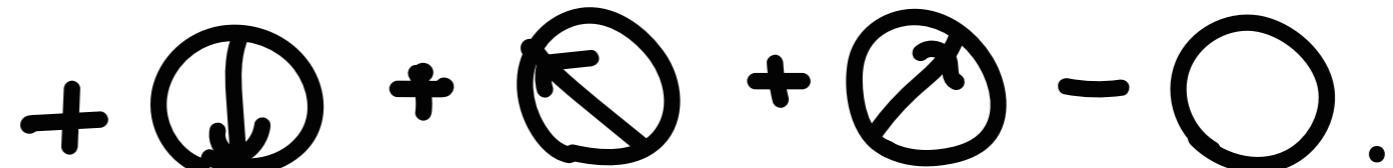
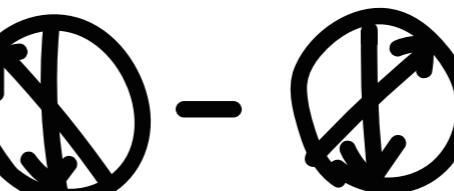
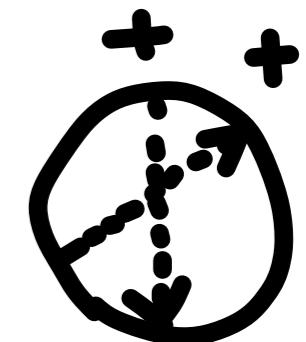
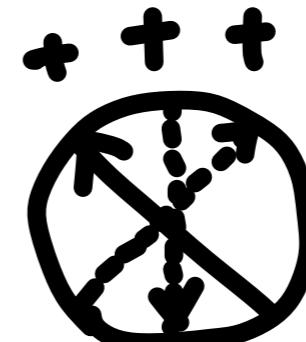
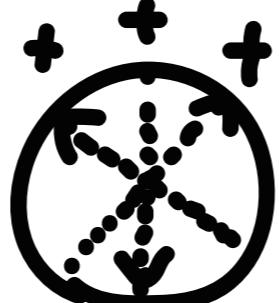
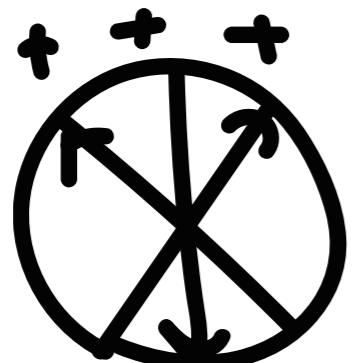
i sends sub-diagrams to arrow
diagrams (= dotted diagrams)



“Dotted arrows”



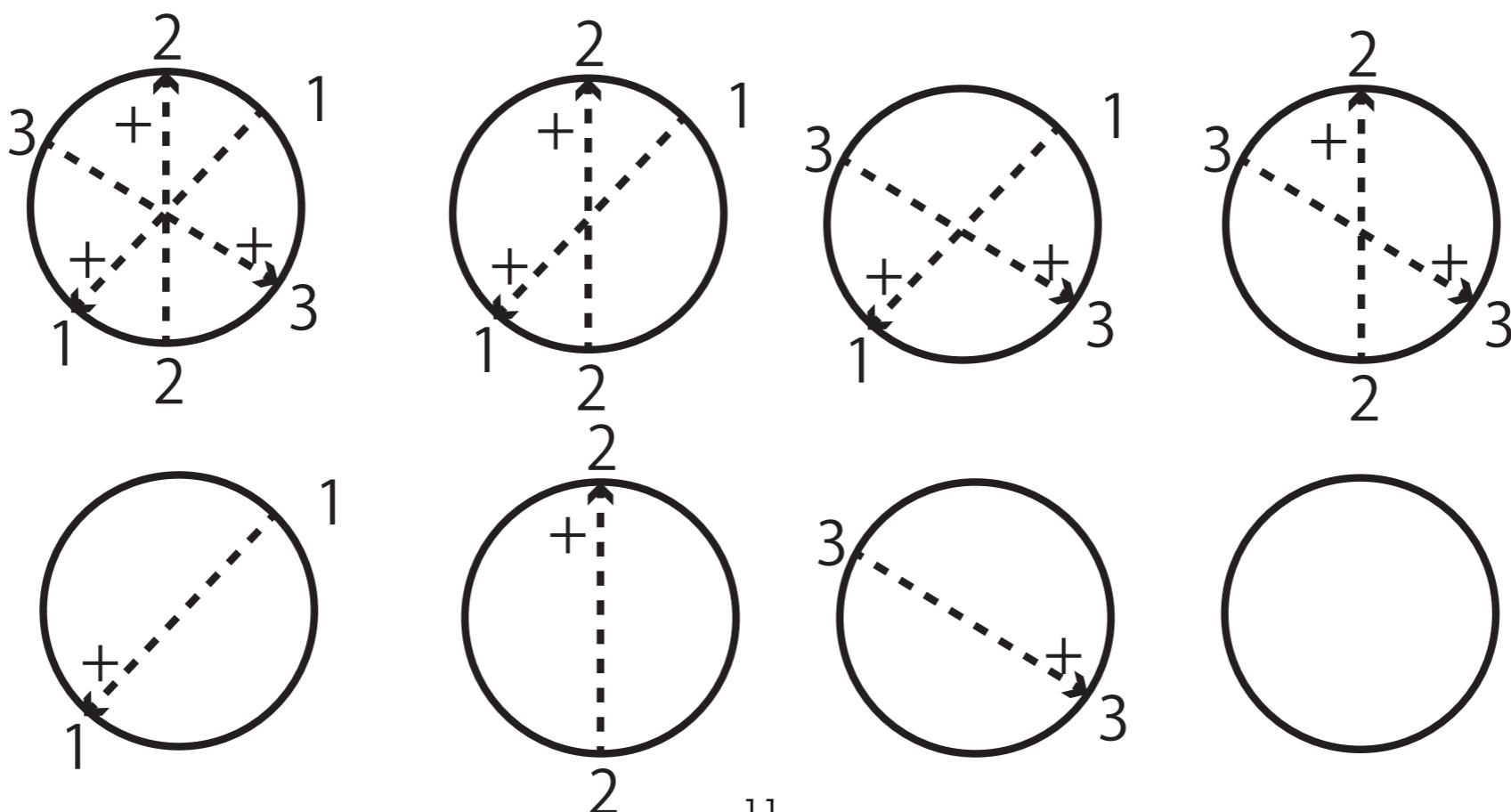
example



Universal invariant $I_n(D) := \sum_{D' \subset D} i_n(D')$.
(Truncated)

i_n sends sub-diagrams to arrow
diagrams (= dotted diagrams)

D' with $n + 1$ arrows $\mapsto 0$.



Map I

$$I(D) := \sum_{D' \subset D} i(D') .$$

Target space divided by

$$\text{Diagram } = 0, \quad \text{Diagram}_{-\varepsilon} + \text{Diagram}_{\varepsilon} + \text{Diagram}_{-\varepsilon} = 0,$$

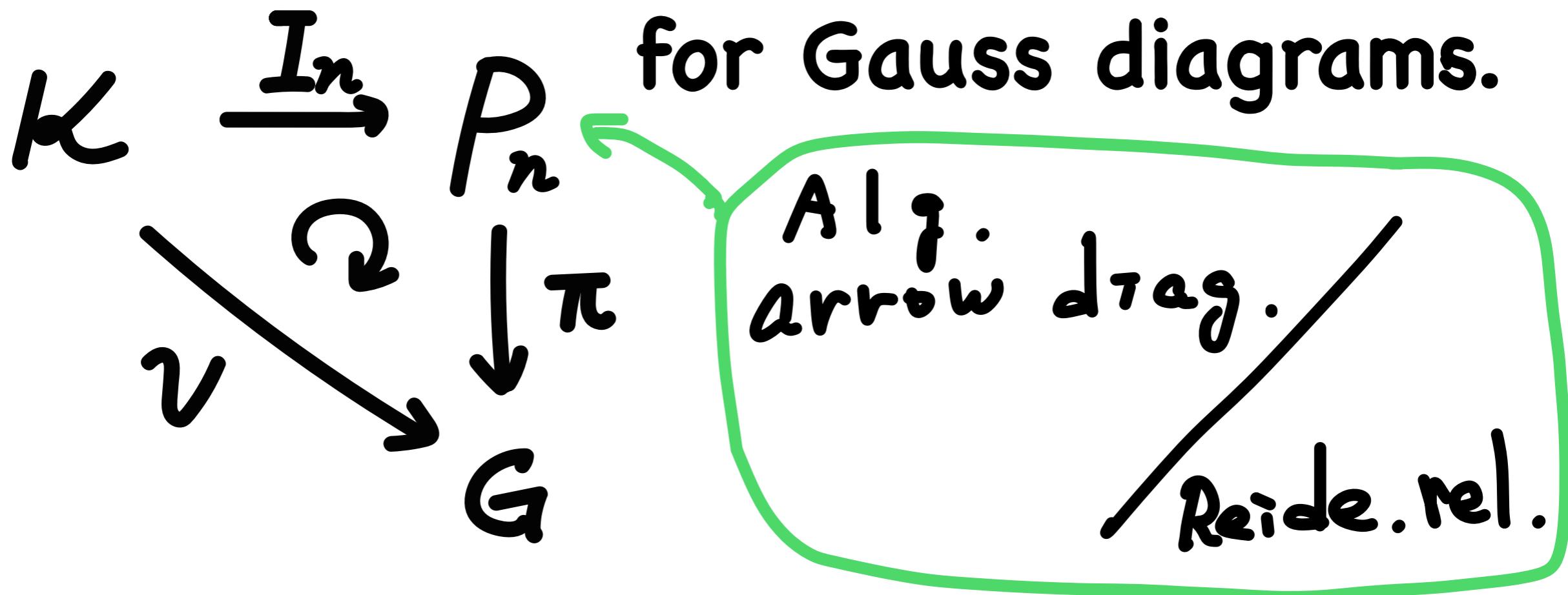
$$\text{Diagram}_{-\varepsilon} + \text{Diagram}_{\varepsilon} + \text{Diagram}_{-\varepsilon} + \text{Diagram}_{\varepsilon} = \text{Diagram}_{-\varepsilon} + \text{Diagram}_{\varepsilon} + \text{Diagram}_{-\varepsilon} + \text{Diagram}_{\varepsilon}.$$

Theorem (Goussarov-Polyak-Viro)

$I : \mathcal{K} \rightarrow \mathcal{A}/rel.$ is complete inv.

Goussarov-Polyak-Viro's Theorem 2.E

$I_n(D) := \sum_{D' \subset D} i_n(D')$ gives a universal invariant of degree n



Brief review

Goussarov's Theorem 3.A

Let G be an abelian groups and let ν be a G -valued invariant of degree n of long knots. Then there exists a function

$\pi : \mathcal{A} \rightarrow G$ such that $\nu = \pi \circ I$ and such that π vanishes on any arrow diagram with more than n arrows.

$$\begin{array}{ccc} \kappa & \xrightarrow{I} & P \\ & \searrow \nu & \downarrow \pi \\ & & G \end{array} .$$

Brief review

Corollary 3.B of Goussarov's Theorem

Any integer-valued finite-type invariant of degree n of long knots can be presented as

$$\langle A, D \rangle := (A, I(D)),$$

where A is a linear sum of arrow diagrams on a line with at most n .

Brief review

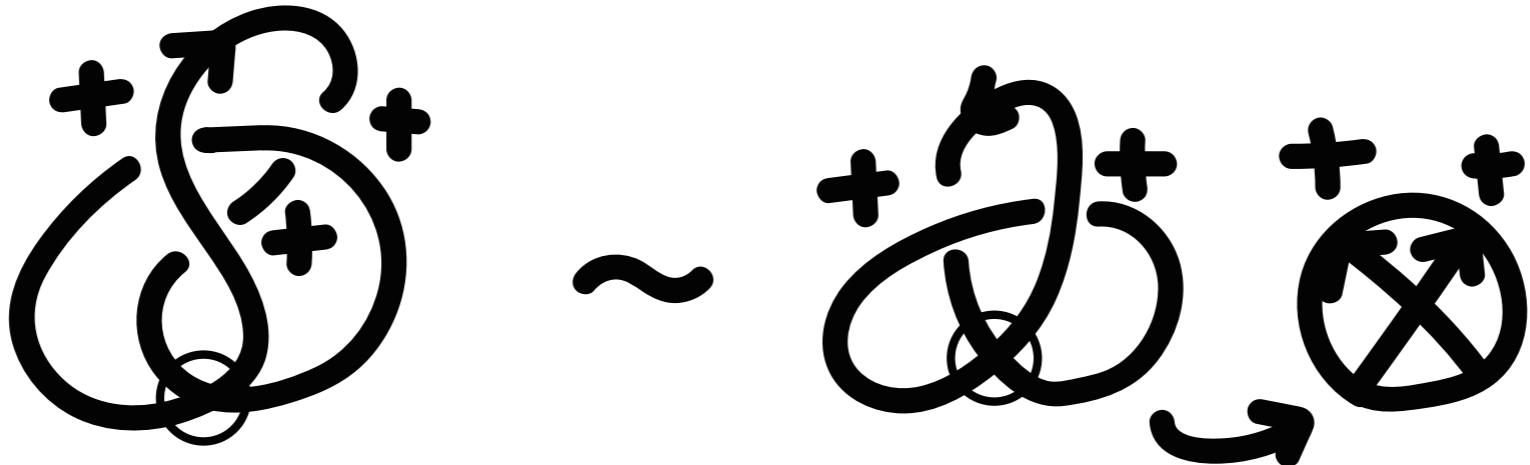
Goussarov-Polyak-Viro's Conjecture 3.C

Every finite-type invariant of classical knots can be extended to a finite-type invariant of long virtual knots.

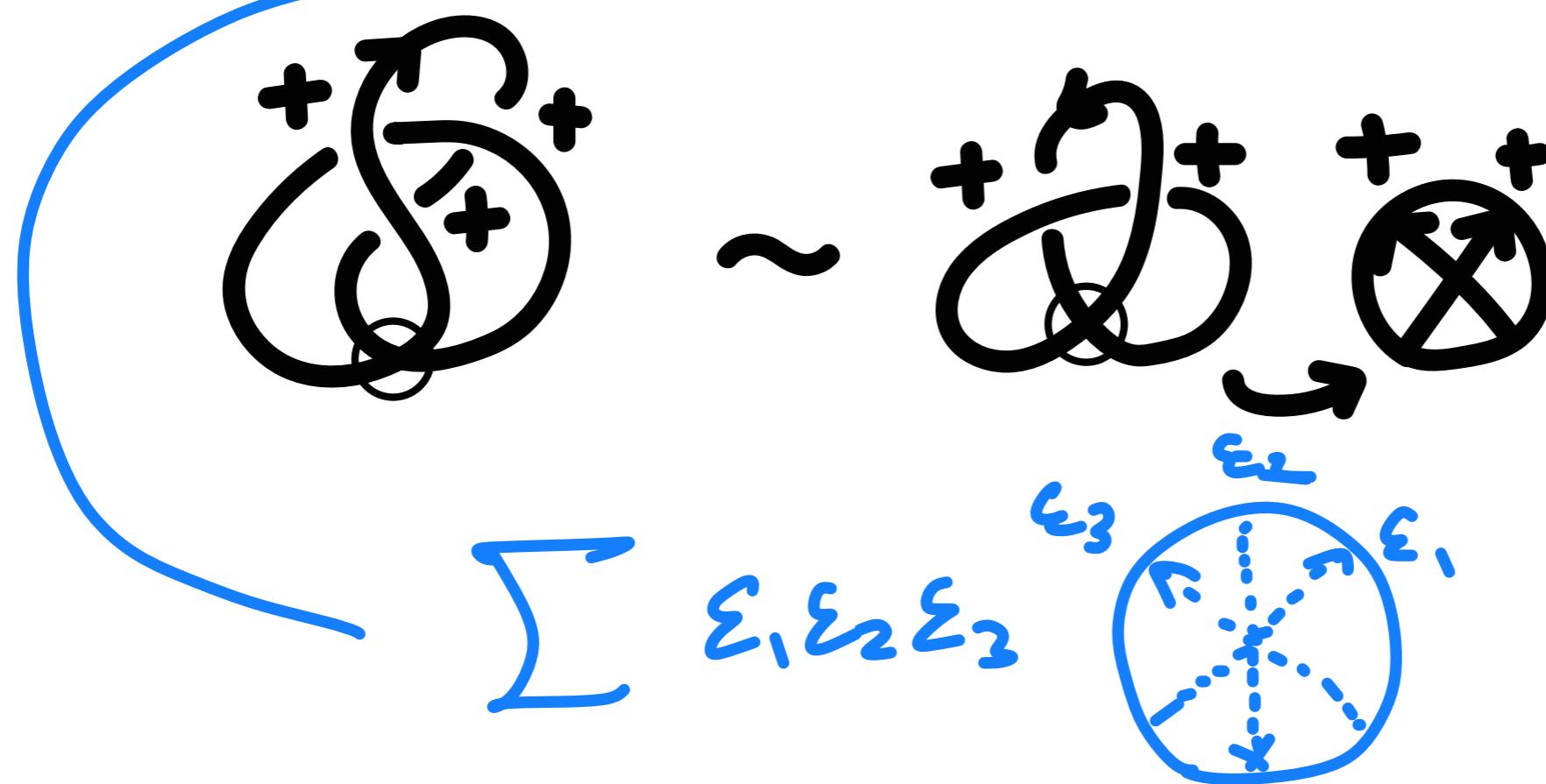
Brief review

What's problem ?

Polyak-Viro formula is not a virtual knot invariant.

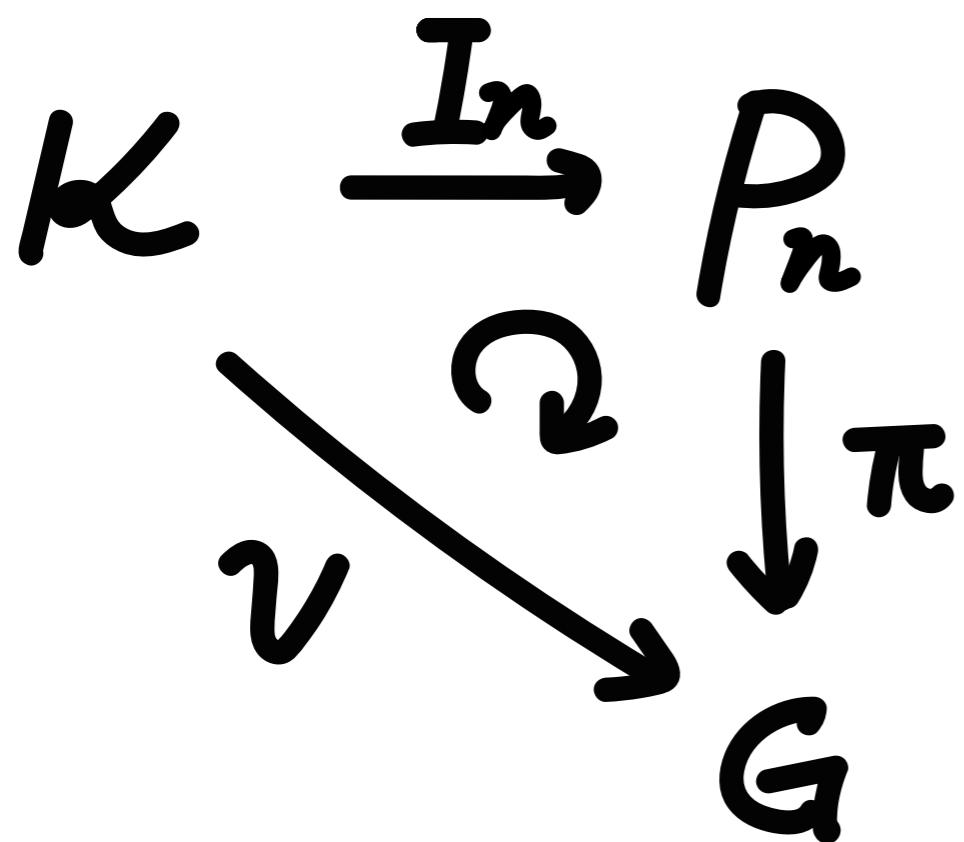
$$\mathcal{N}_{PV} = \left\langle 2 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array}, \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} + \right\rangle$$


What's problem ?



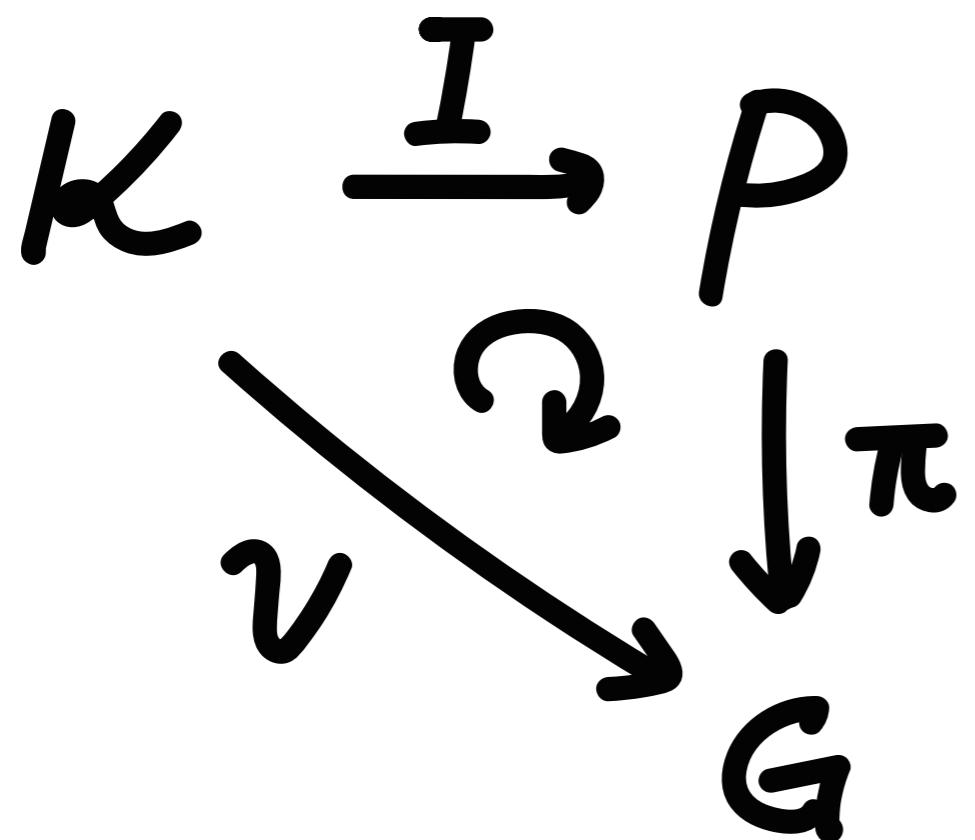
Goussarov-Polyak-Viro's Theorem 2.E

$I_n(D) := \sum_{D' \subset D} i_n(D')$ gives a universal invariant of degree n for Gauss diagram.

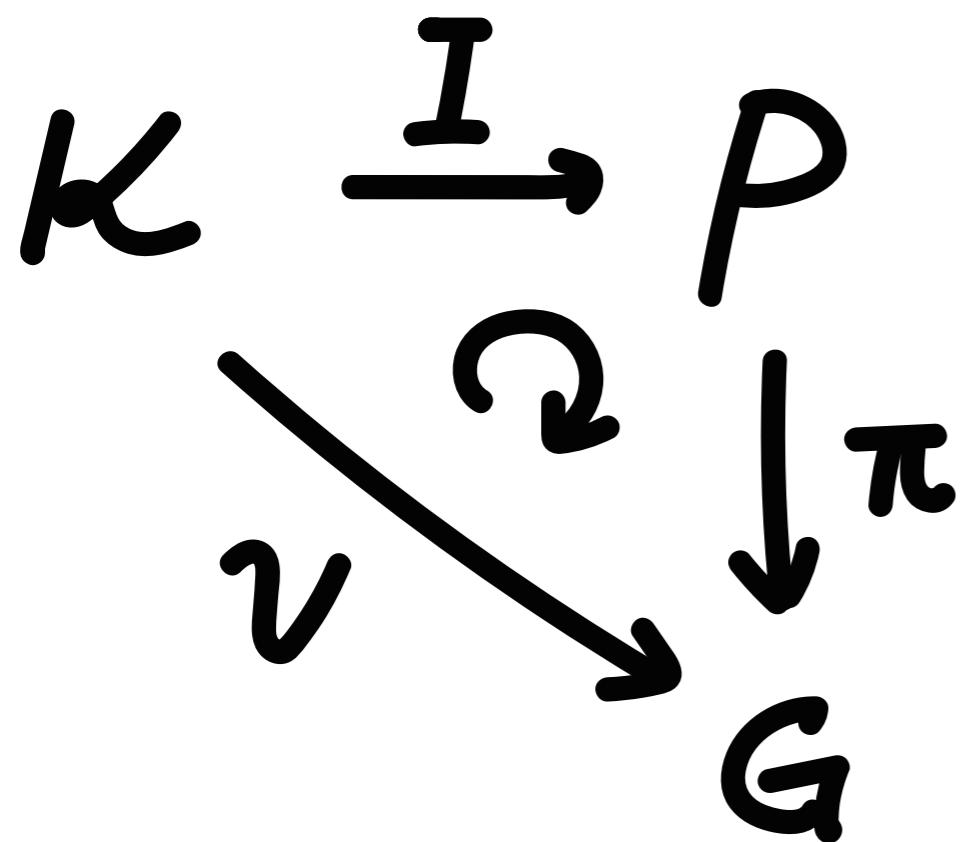


This module gives universal for virtual, not for classical.

What is a module P
fitting Polyak-Viro ν_3 ?



What is a module P fitting Polyak-Viro ν_3 ?



We find P_3 implying
Polyak-Viro formula.
(Kotorii-I.
arXiv:1905.01418)

Also we checked a known formula
from the paper Goussarov-Polyak-Viro

\mathcal{P}_3 (an example of P) as follows:

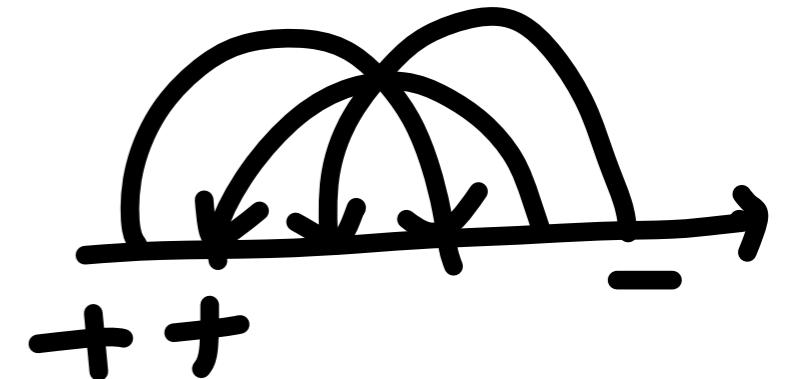
$$\begin{aligned} & \left\langle \text{Diagram 1} \right\rangle + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \\ & + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \\ & + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} - \frac{\text{Diagram 12}}{+} \\ & + \frac{\text{Diagram 13}}{++} - \frac{\text{Diagram 14}}{+-}, \cdot \rangle. \end{aligned}$$

This is not a virtual knot invariant.

e.g.



1st - term



2

2

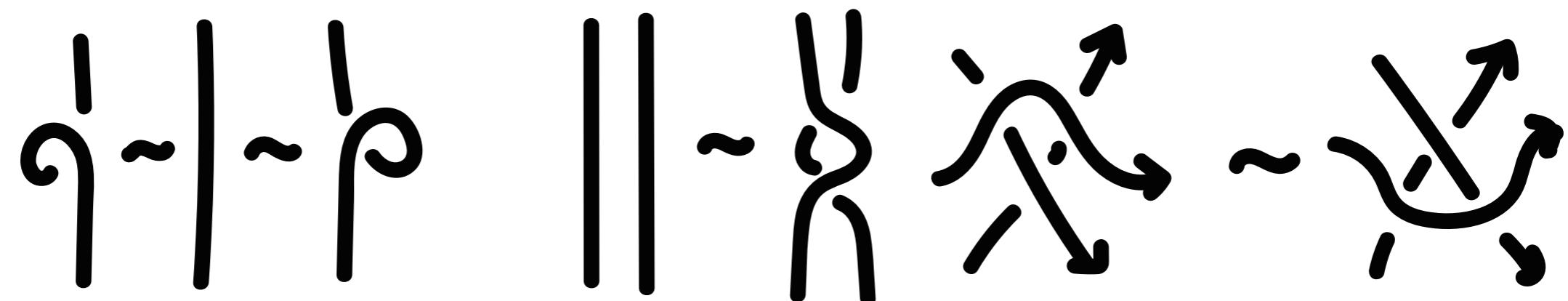
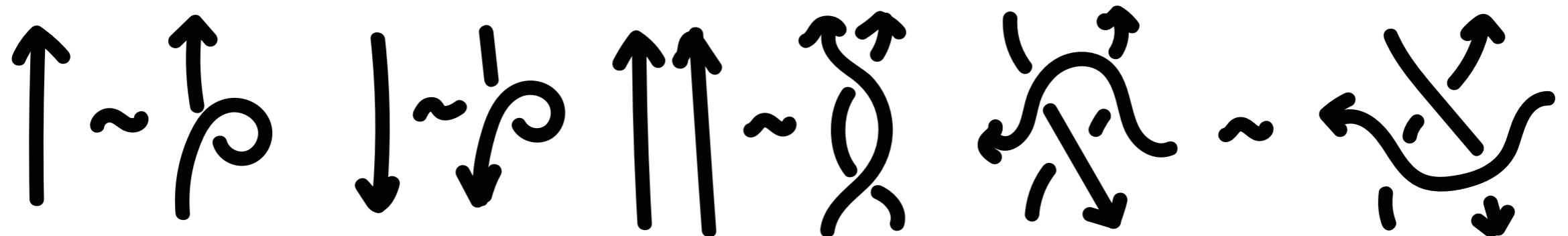


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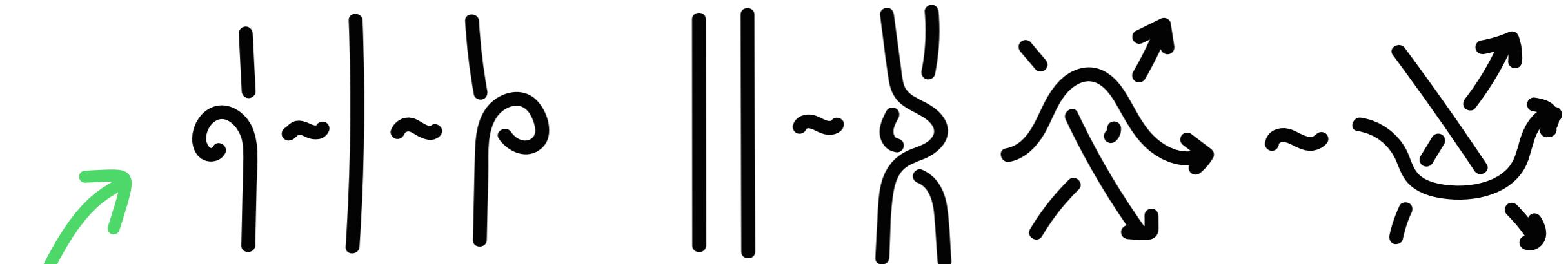
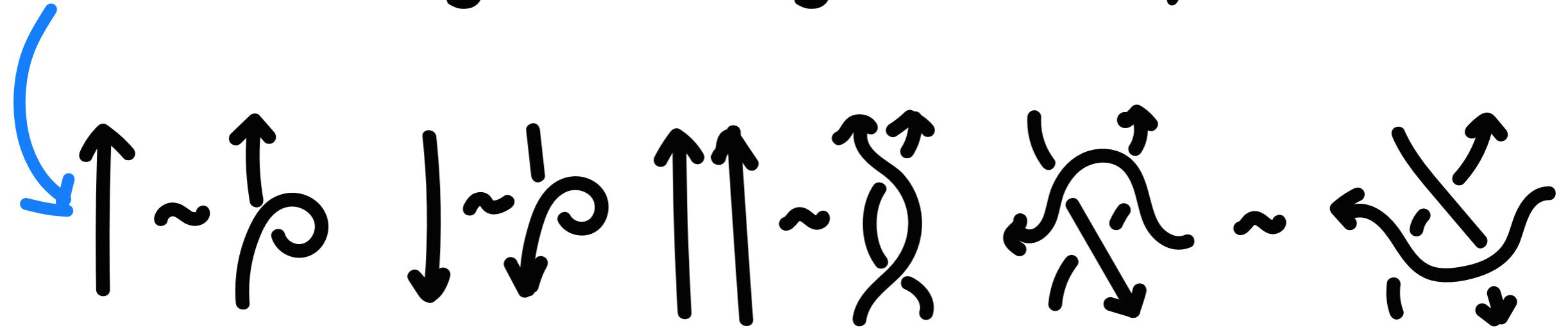
\mathcal{P}_3 (an example of P) as follows:

$$\begin{aligned} & \left\langle \text{[Diagram]} \right\rangle + \text{[Diagram]} + \text{[Diagram]} + \text{[Diagram]} \\ & + \text{[Diagram]} + \text{[Diagram]} + \text{[Diagram]} + \text{[Diagram]} \\ & + \text{[Diagram]} + \text{[Diagram]} + \text{[Diagram]} - \text{[Diagram]} \\ & + \text{[Diagram]} - \text{[Diagram]}, \cdot \rightarrow \text{[Diagram]} \leftarrow \text{typo.} \end{aligned}$$

Computation for more reducing risks, we switch relations of P_n to the minimum generating set.



Minimum generating Set [Polyak 2010]



Relations in Goussarov-Polyak-Viro[2000]

Set $P = M$ induced by the minimum generating set of Reidemeister moves, and add relation corresponding to:

$$\langle \text{○} \rightarrow \text{○}, \cdot \rangle = \langle \text{○} \leftarrow \text{○}, \cdot \rangle$$

(linking number relation)

Theorem (Kotorii-I.)

Let M be the module of minimum relations. M gives virtual knot invariants of degree n .

In $n = 3$ case, M divided by LK rel. implies Polyak-Viro formula that is a classical knot invariant and not a virtual knot invariant.

Theorem (Kotorii-I.)

A Let M be the module of minimum relations. M gives virtual knot invariants of degree n .

B In $n = 3$ case, M divided by LK rel. implies Polyak-Viro formula that is a classical knot invariant and not a virtual knot invariant.

By applying Part A (Kotorii-I.) to
 $n = 3$ case, we have 7 independent
formulas via computer aid by
Takamura.

By hand, only using “connected”
Gauss diagram, we have 5 formulas:

$$\tilde{v}_{3,1}(\cdot) := \langle$$
 $, \cdot \rangle,$
 $(= 2v_3 + v_2)$

$$\tilde{v}_{3,2}(\cdot) := \langle$$
 $, \cdot \rangle,$
 $(= 2v_3)$

$$\begin{aligned} \tilde{v}_{3,3}(\cdot) &:= \langle - \text{ (circle with two crossed arrows)} + \text{ (circle with two crossed arrows)} \\ &\quad + \text{ (circle with two crossed arrows)} + \text{ (circle with three arrows)} \\ &\quad - \text{ (circle with three arrows)} - \text{ (circle with two arrows)} + \text{ (circle with two arrows)} \\ &\quad + \text{ (circle with two arrows)} + \text{ (circle with two arrows)} - \text{ (circle with two arrows)}, \cdot \rangle, \\ & (= 0) \end{aligned}$$

$$\begin{aligned} \tilde{v}_{3,4}(\cdot) &:= \langle - \text{ (circle with two crossed arrows)} + \text{ (circle with two crossed arrows)} \\ &\quad - \text{ (circle with three arrows)} + \text{ (circle with two arrows)} + 3 \text{ (circle with two arrows)} \\ &\quad + 2 \text{ (circle with two arrows)} - \text{ (circle with three arrows)} + \text{ (circle with two arrows)} \\ &\quad + \text{ (circle with two arrows)} \\ &\quad - \text{ (circle with two arrows)} - \text{ (circle with two arrows)}, \cdot \rangle, \\ & (= 2v_3) \end{aligned}$$

$$\begin{aligned}
\tilde{v}_{3,5}(\cdot) &:= \langle - \begin{array}{c} + \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} + \\ \diagup \quad \diagdown \\ \bullet \end{array} - \begin{array}{c} + \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} + \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} + \\ \diagup \quad \diagdown \\ \bullet \end{array} - \\
&\quad - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array} - \\
&\quad - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ - \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ - \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ - \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ - \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ - \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ - \end{array} - \\
&\quad (= -2v_3 - v_2), \cdot, \rangle.
\end{aligned}$$

Part B (Kotorii-I.)

Using linking number relation,
relations to be checked decrease
to almost the half of them, and
we have 19 Gauss diagram
formulas including Polyak-Viro
formula (1994), i.e.

$$\nu_{PV} = \nu_{3,16} + \nu_{3,18}.$$

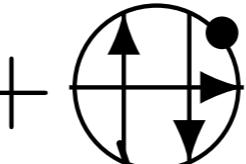
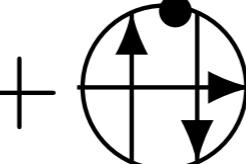
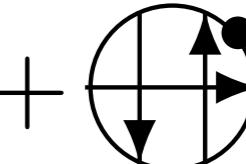
$$v_{3,1} := \left\langle - \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 6} \end{array} \right\rangle,$$

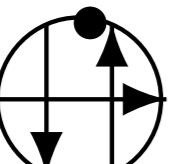
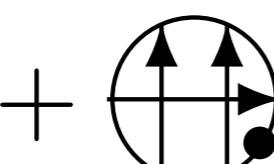
$$v_{3,3} := \left\langle \begin{array}{c} + + \\ \diagup \diagdown \end{array} \right. + \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} - + \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} + \begin{array}{c} - - \\ \diagup \diagdown \end{array} -$$

- - - , .

$$\left. \begin{array}{c} - \\ \diagup \diagdown \end{array} \right\rangle,$$

$$v_{3,4} := \left\langle \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} - + \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} - \begin{array}{c} + - \\ \diagup \diagdown \end{array} + 2 \begin{array}{c} \cdot \\ \diagup \diagdown \end{array} + \right.$$

+
 -  +  +  +  +  +

 +  + , .

$$\left. \right\rangle,$$

$$v_{3,5} := \left\langle \begin{array}{ccccccc} + & + & + & + & - & - & - \\ 2 \text{ } \circlearrowleft & - 2 \text{ } \circlearrowleft & + 2 \text{ } \circlearrowleft & + 2 \text{ } \circlearrowleft & - \text{ } \circlearrowleft & - \text{ } \circlearrowleft & - \\ 3 \text{ } \circlearrowleft & - 2 \text{ } \circlearrowleft & + 2 \text{ } \circlearrowleft & - \text{ } \circlearrowleft & - 2 \text{ } \circlearrowleft & - \text{ } \circlearrowleft & - \\ \circlearrowleft & - \text{ } \circlearrowleft & - 2 \text{ } \circlearrowleft & + \text{ } \circlearrowleft & \circlearrowleft & \cdot & \end{array} \right\rangle,$$

$$v_{3,6} := \left\langle \begin{array}{ccccccc} + & + & - & - & + & + & + \\ \circlearrowleft & - \text{ } \circlearrowleft & + \text{ } \circlearrowleft & - \text{ } \circlearrowleft & + 2 \text{ } \circlearrowleft & + \text{ } \circlearrowleft & + \\ \circlearrowleft & + 2 \text{ } \circlearrowleft & + \text{ } \circlearrowleft & + \text{ } \circlearrowleft & + 2 \text{ } \circlearrowleft & + \text{ } \circlearrowleft & , \cdot \end{array} \right\rangle,$$

$$v_{3,7} := \left\langle -\begin{array}{c} + + \\ \diagup \diagdown \end{array} + 2 \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \right.$$

$$\begin{array}{c} + - \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} - 2 \begin{array}{c} - - \\ \diagup \diagdown \end{array} + \begin{array}{c} - - \\ \diagup \diagdown \end{array} + 3 \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + 2 \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} -$$

$$2 \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} - \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} +$$

$$\begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array}, \cdot, \right.$$

$$v_{3,8} := \left\langle \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} - \begin{array}{c} + - \\ \diagup \diagdown \end{array} - \right.$$

$$\begin{array}{c} \bullet \\ \diagup \diagdown \end{array} - \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} - \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} -$$

$$\begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array}, \cdot, \right.$$

$$v_{3,9} := \left\langle - \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} + - \\ \diagup \diagdown \end{array} - \right.$$

$$\begin{array}{c} + - \\ \diagup \diagdown \end{array} - \begin{array}{c} - - \\ \diagup \diagdown \end{array} - \begin{array}{c} - - \\ \diagup \diagdown \end{array} + \begin{array}{c} + \\ \bullet \end{array} -$$

$$\begin{array}{c} + \\ \bullet \end{array} + \begin{array}{c} + \\ \bullet \end{array} - \begin{array}{c} + \\ \bullet \end{array} - \begin{array}{c} + \\ \bullet \end{array} - \begin{array}{c} + \\ \bullet \end{array} + \begin{array}{c} + \\ \bullet \end{array} + \begin{array}{c} + \\ \bullet \end{array} + \begin{array}{c} + \\ \bullet \end{array} +$$

$$\begin{array}{c} + \\ \bullet \end{array} + \begin{array}{c} + \\ \bullet \end{array}, \cdot \right\rangle,$$

$$v_{3,10} := \left\langle 2 \begin{array}{c} + + \\ \diagup \diagdown \end{array} - 2 \begin{array}{c} + + \\ \diagup \diagdown \end{array} + 2 \begin{array}{c} + - \\ \diagup \diagdown \end{array} + 2 \begin{array}{c} - - \\ \diagup \diagdown \end{array} - \right.$$

$$\begin{array}{c} + \\ \bullet \end{array} - 3 \begin{array}{c} + \\ \bullet \end{array} - 2 \begin{array}{c} + \\ \bullet \end{array} + 2 \begin{array}{c} + \\ \bullet \end{array} - \begin{array}{c} + \\ \bullet \end{array} -$$

$$2 \begin{array}{c} + \\ \bullet \end{array} - \begin{array}{c} + \\ \bullet \end{array} - \begin{array}{c} + \\ \bullet \end{array} - 2 \begin{array}{c} + \\ \bullet \end{array} + \begin{array}{c} + \\ \bullet \end{array}, \cdot \right\rangle,$$

$$v_{3,11} := \left\langle \begin{matrix} + & + & - & - \\ \text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} & \text{Diagram 4} \end{matrix} \right. , \cdot \left. \begin{matrix} + & + & + & + \\ \text{Diagram 5} & \text{Diagram 6} & \text{Diagram 7} & \text{Diagram 8} \end{matrix} \right\rangle,$$

$$v_{3,12} := \left\langle \begin{matrix} + & + & + & + \\ \text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} & \text{Diagram 4} \end{matrix} \right. , \cdot \left. \begin{matrix} + & + & + & + \\ \text{Diagram 5} & \text{Diagram 6} & \text{Diagram 7} & \text{Diagram 8} \end{matrix} \right\rangle,$$

$$v_{3,13} := \left\langle -\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array}, \cdot \right\rangle,$$

$$\begin{aligned} v_{3,14} := & \left\langle \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} - \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} - \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + - \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \right. \\ & - \begin{array}{c} - - \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} - + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + - \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} - \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + - \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \\ & \left. \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} - \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} + + \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bullet \end{array}, \cdot \right\rangle, \end{aligned}$$

$$v_{3,15} := \left\langle \begin{array}{ccccccc} + & + & + & - & + & + & - \\ \text{Diagram: } & \text{Diagram: } \\ + & - & - & - & - & - & - \\ \text{Diagram: } & \text{Diagram: } \\ - & + & - & + & - & - & - \\ \text{Diagram: } & \text{Diagram: } \\ - & - & + & - & - & - & - \\ \text{Diagram: } & \text{Diagram: } \\ - & + & - & + & - & - & - \\ \text{Diagram: } & \text{Diagram: } \\ + & - & + & + & - & - & - \\ \text{Diagram: } & \text{Diagram: } \\ + & - & + & + & - & - & - \\ \text{Diagram: } & \text{Diagram: } \\ , & \cdot & , & \cdot & , & \cdot & , \end{array} \right\rangle,$$

$$v_{3,16} := \left\langle \begin{array}{ccccc} + & + & + & + & + \\ \text{Diagram: } & \text{Diagram: } & \text{Diagram: } & \text{Diagram: } & \text{Diagram: } \\ + & + & + & + & + \\ \text{Diagram: } & \text{Diagram: } & \text{Diagram: } & \text{Diagram: } & \text{Diagram: } \\ , & \cdot & , & \cdot & , \end{array} \right\rangle,$$

$$v_{3,17} := \left\langle -\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array}, \cdot \right\rangle,$$

$$v_{3,18} := \left\langle \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array}, \cdot \right\rangle,$$

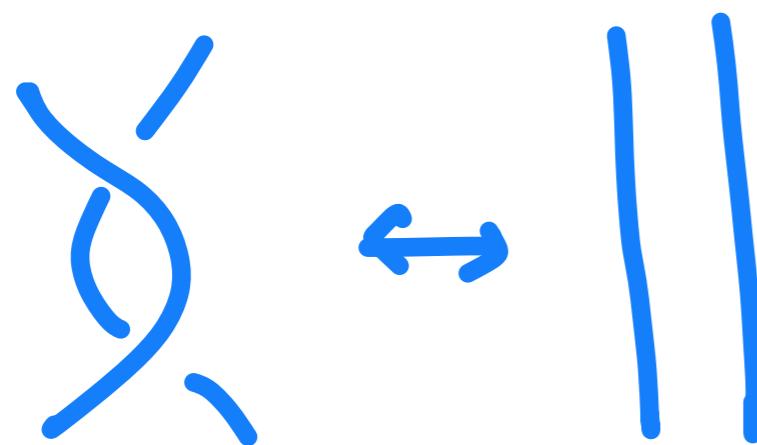
$$v_{3,19} := \left\langle -\begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + + \\ \diagup \diagdown \end{array} - \begin{array}{c} - \bullet + \\ \diagup \diagdown \end{array} - \begin{array}{c} - + + \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} - \right.$$

$$\begin{array}{c} + - \\ \diagup \diagdown \end{array} + \begin{array}{c} - \\ \bullet \end{array} - \begin{array}{c} - \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} - \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} - \begin{array}{c} \bullet \\ \diagup \diagdown \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \end{array}, \dots \right\rangle,$$

$$v_{2,1} := \left\langle \begin{array}{c} + + \\ \diagup \diagdown \end{array} + \begin{array}{c} - + \\ \diagup \diagdown \end{array} + \begin{array}{c} + - \\ \diagup \diagdown \end{array} + \begin{array}{c} - - \\ \diagup \diagdown \end{array}, \dots \right\rangle$$

$$v_{2,2} := \left\langle \begin{array}{c} + \bullet + \\ \diagup \diagdown \end{array} + \begin{array}{c} - \bullet + \\ \diagup \diagdown \end{array} + \begin{array}{c} + \bullet - \\ \diagup \diagdown \end{array} + \begin{array}{c} - \bullet - \\ \diagup \diagdown \end{array}, \dots \right\rangle.$$

Thank you for your attention!



$$\left| \begin{array}{c} \varepsilon \\ \rightarrow \\ -\varepsilon \end{array} \right| + \left| \begin{array}{c} \varepsilon \\ \rightarrow \\ -\varepsilon \end{array} \right| + \left| \begin{array}{c} -\varepsilon \\ \rightarrow \\ \varepsilon \end{array} \right| = 0$$

Theorem (Kotorii-I.)

Thank you for your attention!

A Let M be the module of minimum relations. M gives virtual knot invariants of degree n .

B In $n = 3$ case, M divided by LK rel. implies Polyak-Viro formula that is a classical knot invariant and not a virtual knot invariant.

**Does GPV-construction imply complete
invariant of virtual knots ? (in
discussion with Prof. Bar-Nathan)**

$\mathcal{A}, \mathcal{K}, \mathcal{D}$: sets arrow diagrams, virtual knots,
Gauss diagrams.

$(,)$: scalar product, $\langle A, D \rangle := (A, I(D))$.

$(\mathcal{K}, \mathcal{K})$: canonically complete invariant.
 s_{\parallel}

$(\mathcal{K}, I(\mathcal{K}))$ GPV shows I is an isomorphism.
 $\text{Hom}(\mathcal{K}, \mathbb{Z}) \hookrightarrow \text{Hom}(A, \mathbb{Z})$

$$(\mathcal{A}, I(\mathcal{K})) =: \langle \mathcal{A}, \mathcal{K} \rangle$$

(truncation) \rightarrow GPV-invariants presented by

$$(\mathcal{A}, I(\mathcal{D})) =: \langle \mathcal{A}, \mathcal{D} \rangle.$$