A study of finite type invariants of higher-orders of classical and virtual knots

Noboru Ito (NIT, Ibaraki College)

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J.w.w. Migiwa Sakurai (Shibaura U.)

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3 filtrations of vector spaces

Vassiliev

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$$

GPV

$$GPV_1 \supset GPV_2 \supset \cdots \supset GPV_n \supset \cdots$$

Sakurai-I.

$$F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$$

Relationships ?

Vassiliev $$V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$$

GPV $$GPV_1 \supset GPV_2 \supset \cdots \supset GPV_n \supset \cdots$$

Sakurai-I. $$F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$$

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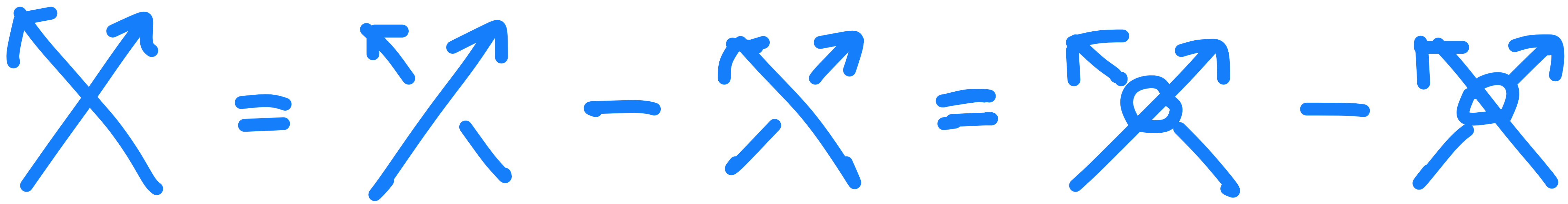

GPV $GPV_1 \supset GPV_2 \supset \cdots \supset GPV_n \supset \cdots$

GPV_n is filtered by double pts.

Sakurai-I. $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$

Relationships ?

Vassiliev $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$

canonical extension ?

GPV conj.

GPV $GPV_1 \supset GPV_2 \supset \cdots \supset GPV_n \supset \cdots$

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Virtual unknotting operations give us 2 kinds of filtrations of finite dim.-vector spaces:

GPV $GPV_1 \supset GPV_2 \supset \cdots \supset GPV_n \supset \cdots$

Sakurai-I. $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$

Theorem A (Sakurai-I.)

Any GPV_{2n+1} invariant is F_n invariant.

Theorem B (Sakurai-I.)

All invariants F_{n+1} are strictly stronger than those of F_n .

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Question. How to estimate the difference between GPV_i and F_j ?

Theorem C (Sakurai-I.)

All invariants F_{n+1} are strictly stronger than those of GPV_{2n+1}.

Theorem A (Sakurai-I.)

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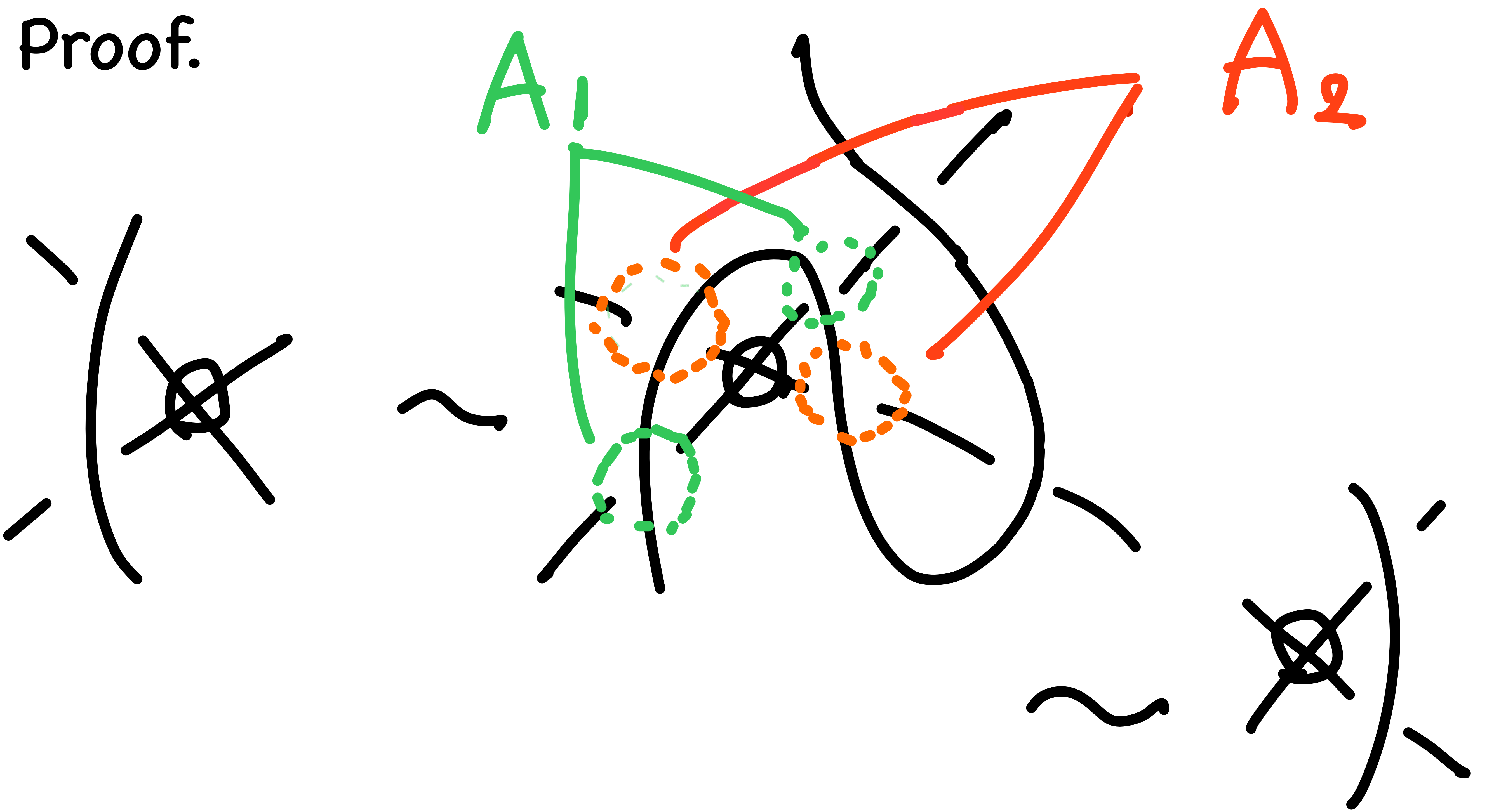
Proof.



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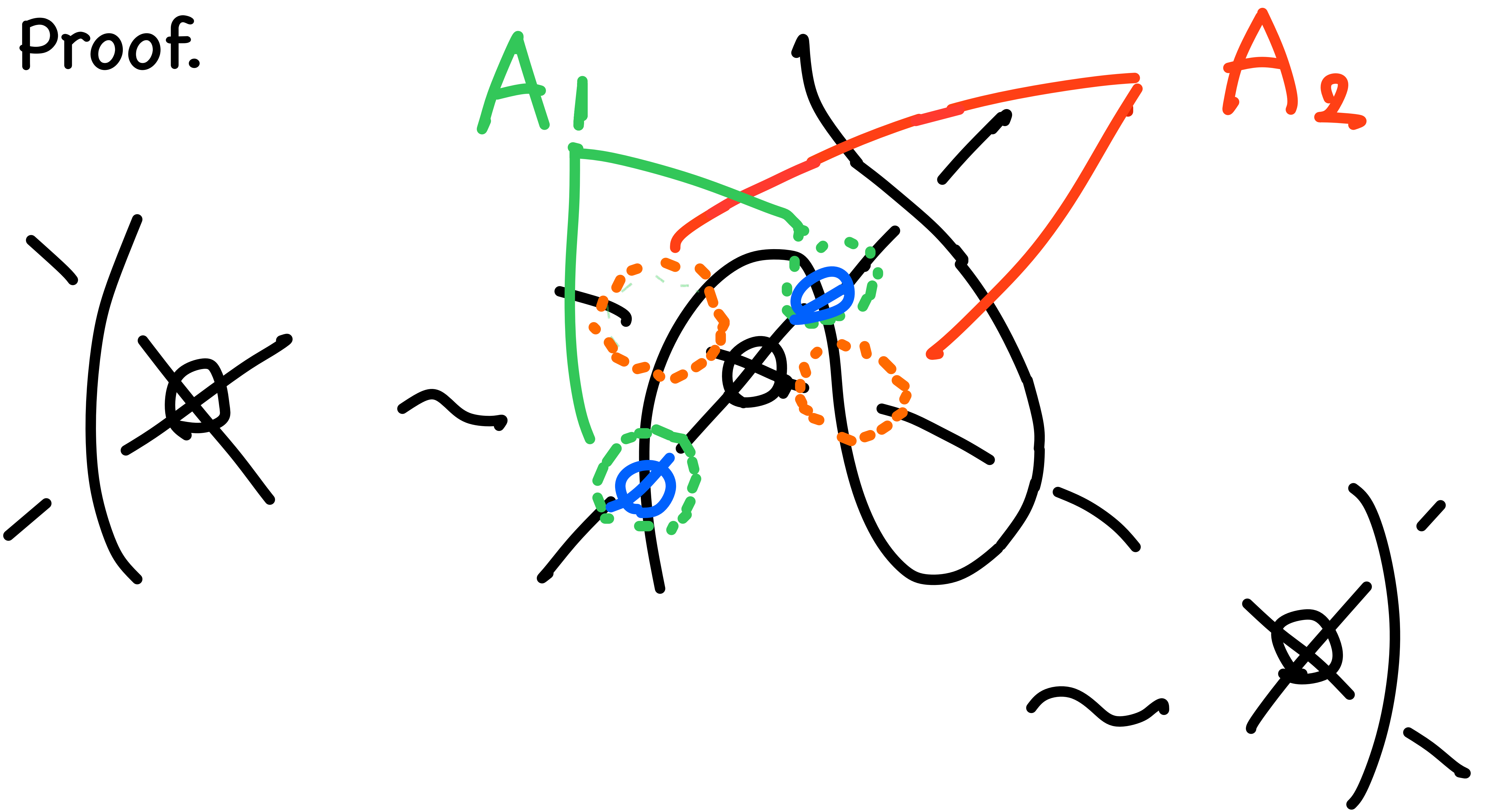

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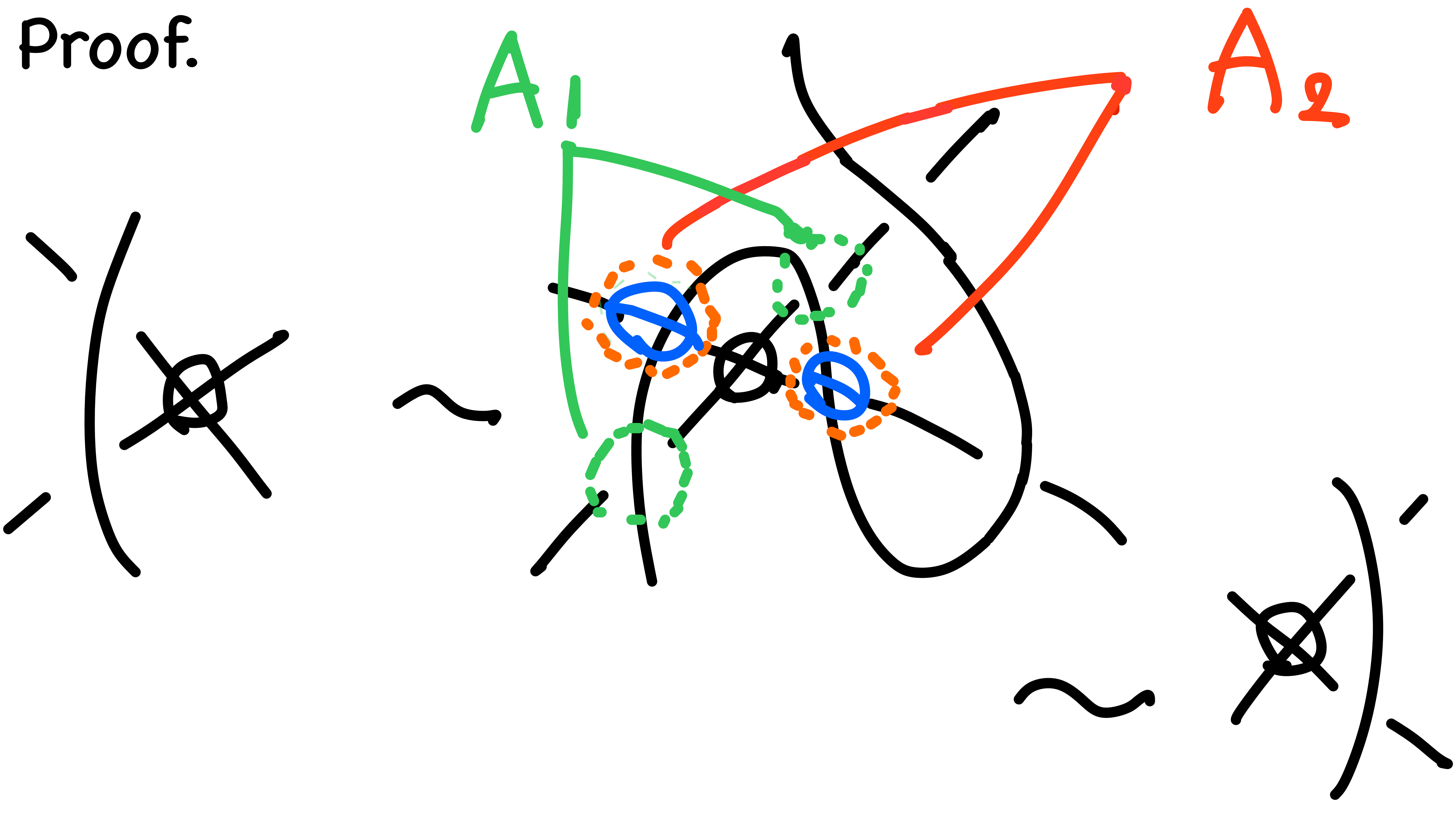

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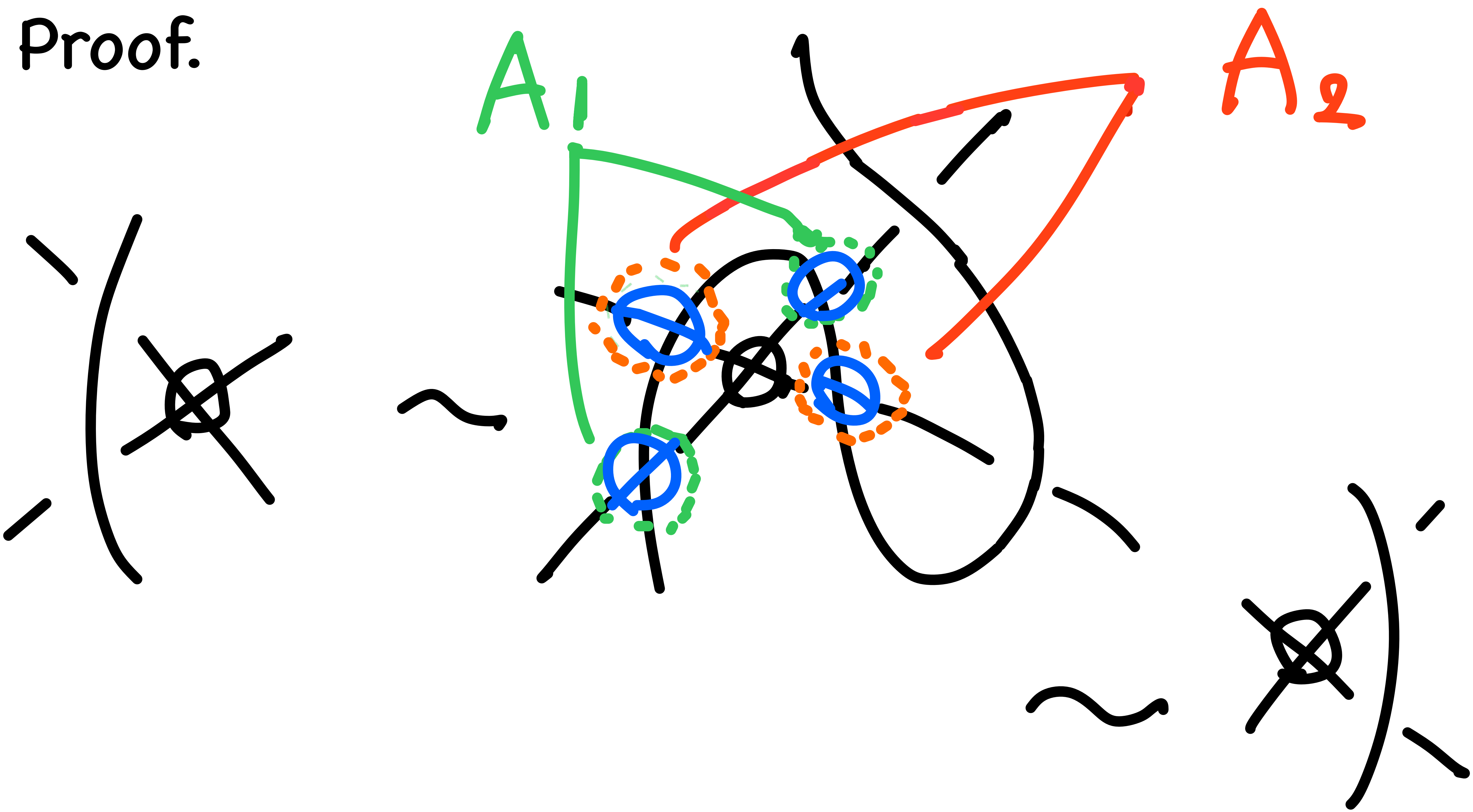

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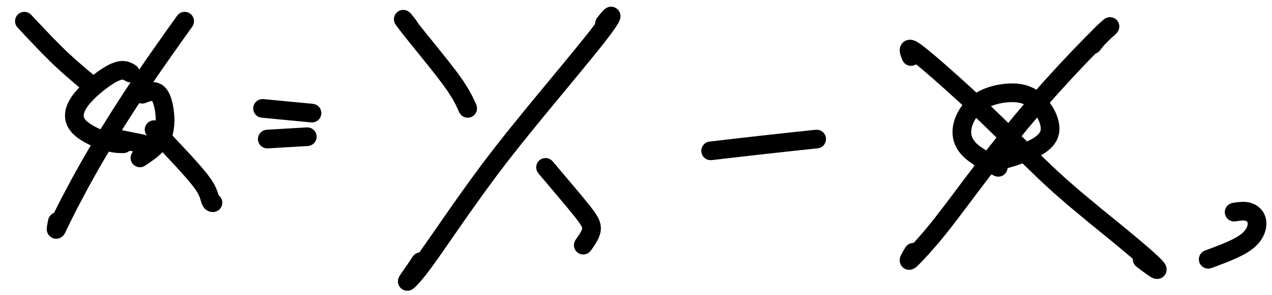
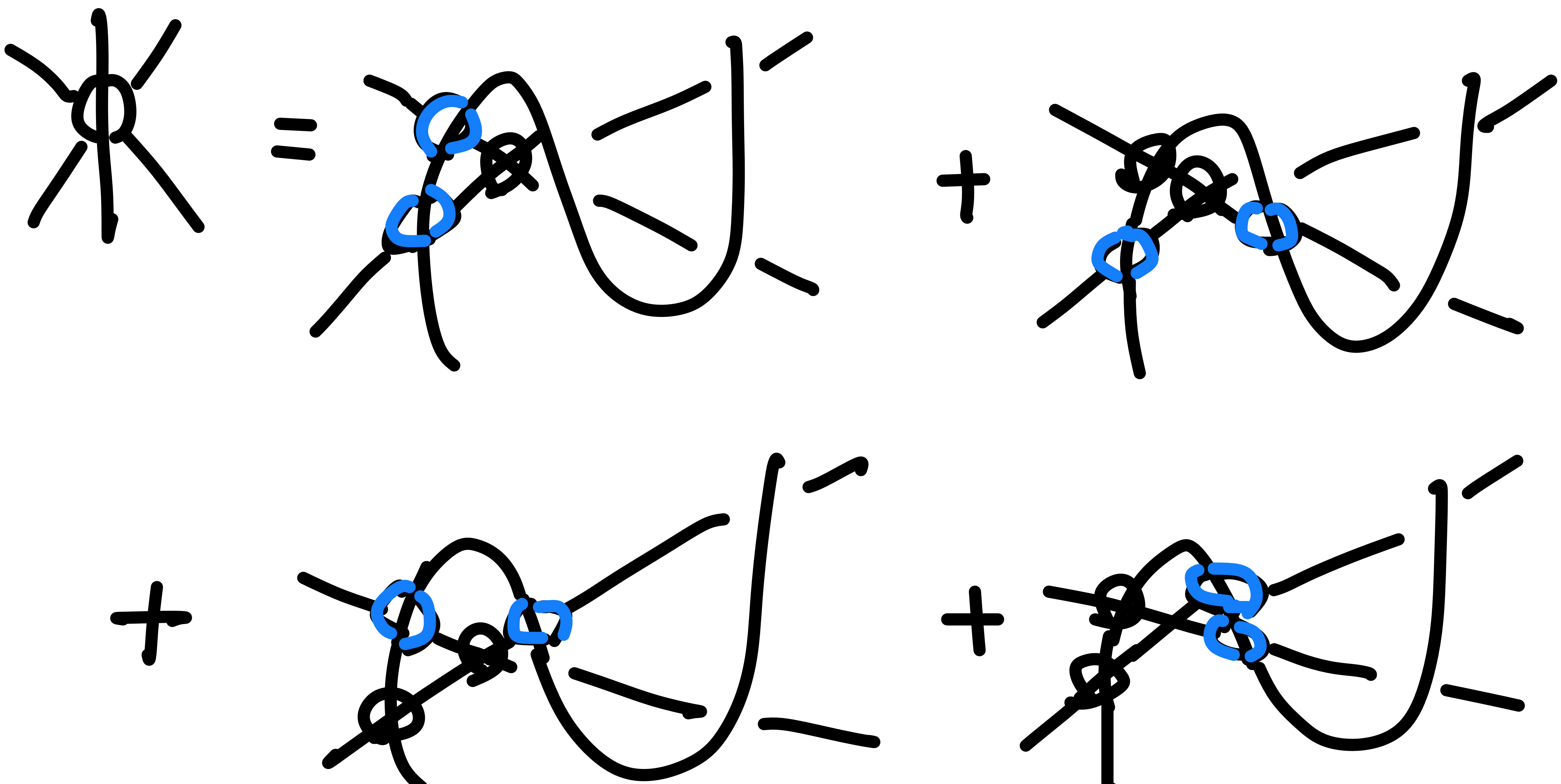
Proof. Under 



Theorem A (Sakurai-I.)

Any GPV_{2n+1} invariant is F_n invariant.

Proof.

$$v_{2n+1}^{GPV}(D_{n+1}^{\text{triple}}) = \sum v_{2n+1}^{GPV}(D_{2(n+1)}^{\text{semi-virtual}}) = 0. \square$$

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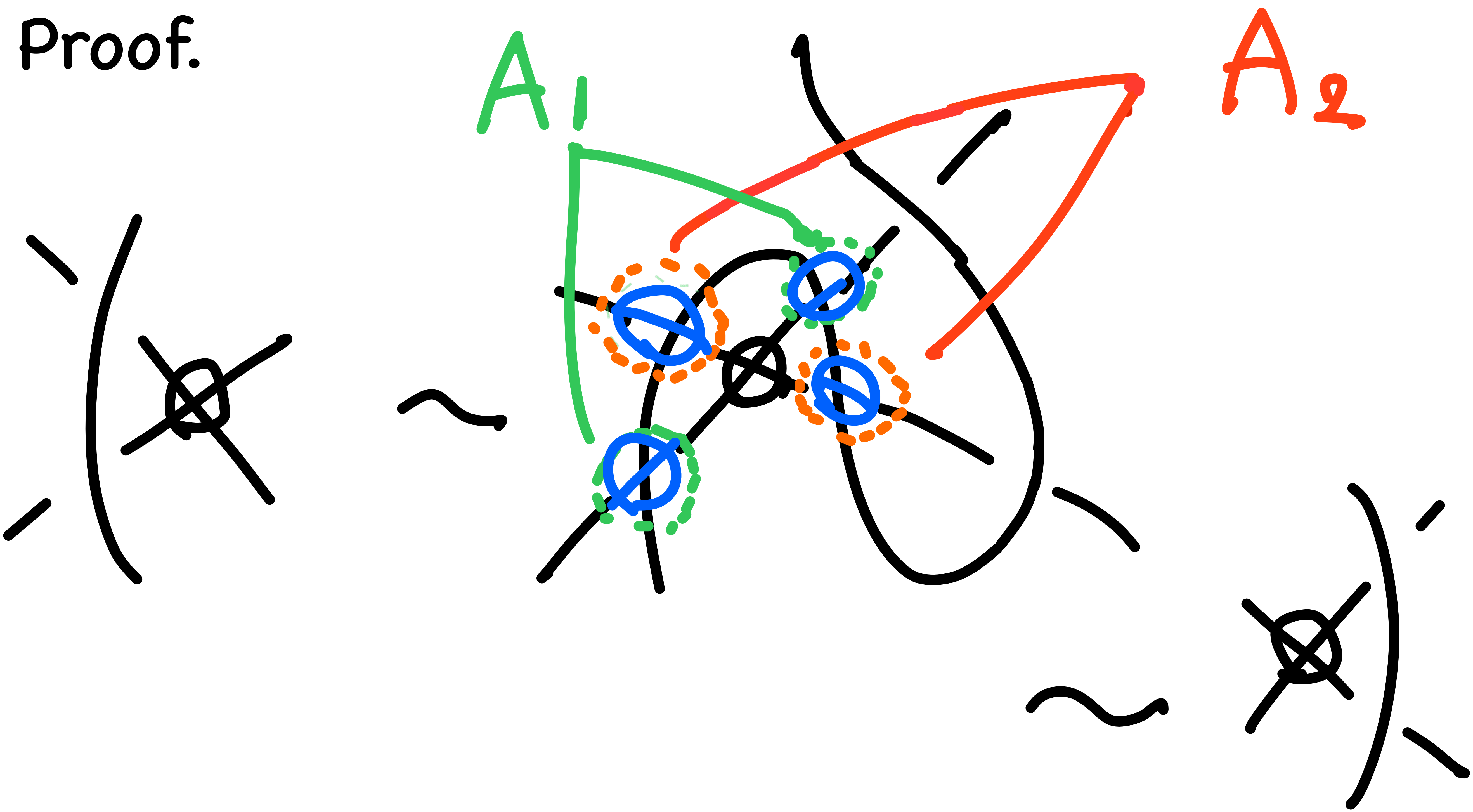


Theorem B (Sakurai-I.)

All invariants F_{n+1} are strictly stronger than those of F_n .

Proof.

Let c_{2n} be the coefficient of degree $2n$ of Conway polynomial. Then, $c_{2n} \in GPV_{2n}$.

We will prove $c_{2n} \in GPV_{2n} \backslash GPV_{2n-1}$.

Let $\lambda_i^{(2m)}$ be a coefficient. By $GPV_{2n} \subset F_n$, we have a presentation:

$$c_{2m} = \lambda_m^{(2m)} v_m^F + \sum_{i \le m-1} \lambda_i^{(2m)} v_i^F .$$

In particular,

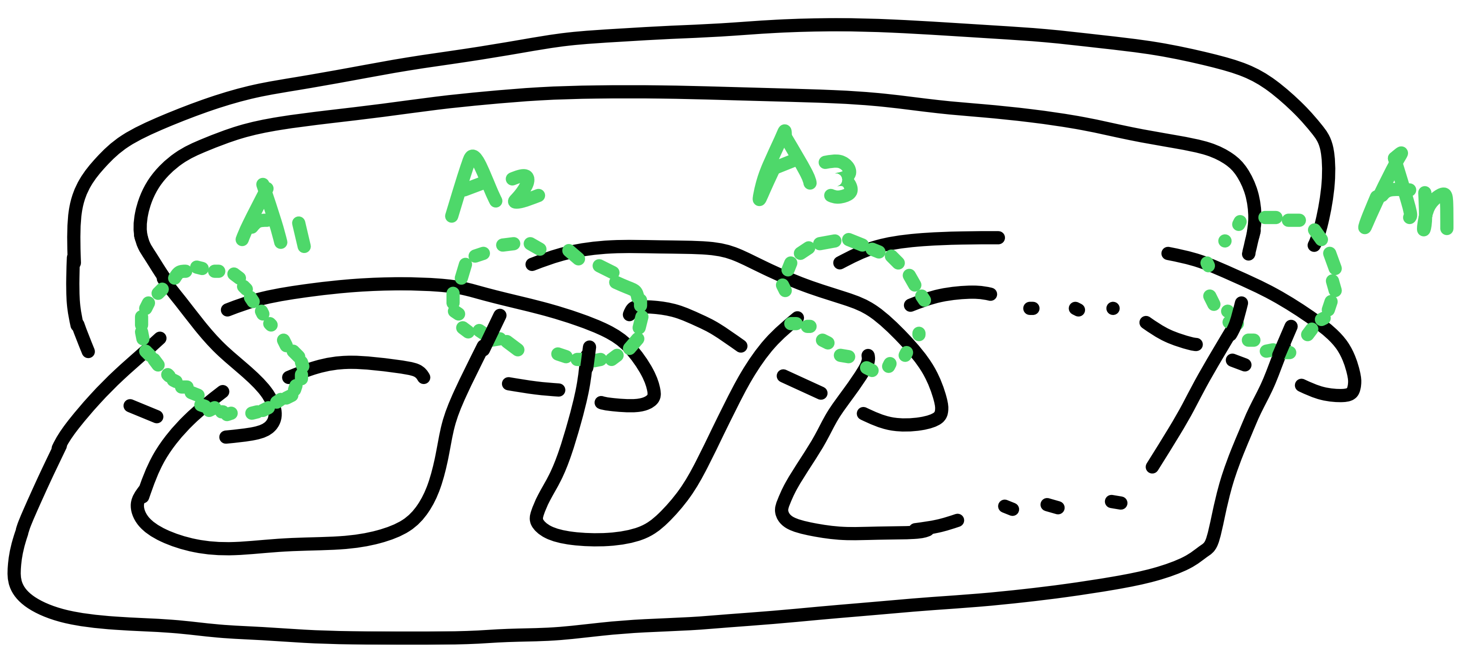
$$c_{2n+2} = \lambda_{n+1}^{(2n+2)} v_{n+1}^F + \sum_{i \le n} \lambda_i^{(2n+2)} v_i^F .$$

Suppose that $\lambda^{(2i)} \neq 0 \ (i \le n)$.

Then,

$$c_{2n+2} = \lambda_{n+1}^{(2n+2)} v_{n+1}^{F} + \sum_{i \leq n} \mu_i c_{2i} \ (\mu_i \neq 0) .$$

If $\lambda_{n+1}^{(2n+2)} = 0$, for K_{2n+2}, the above is zero, whereas $c_{2n+2}(K_{2n+2}) = -2$, which implies the contradiction.



Then, for $c_{2m} = \lambda_m^{(2m)} v_m^F + \sum_{i \leq m-1} \lambda_i^{(2m)} v_i^F$,

If $\lambda_i^{2i} \neq 0 \ (i \leq n)$, then $\lambda_{n+1}^{2(n+1)} \neq 0$.
(Thus, induction works.)

It implies $c_{2n} \in F_n \backslash F_{n-1}$. $\square$

Corollary (Sakurai-I.) from the proof and Theorem A.

All invariants GPV_{2n+2} are strictly stronger than those of GPV_{2n} .

For example, each coefficient of the Conway polynomials of knots is in GPV_{2n} .

Results for higher-order

- Any GPV_{2n+1} invariant is F_n invariant.
- All invariants F_{n+1} are strictly stronger

than those of F_n .

- All invariants GPV_{2n+2} are strictly stronger

than those of GPV_{2n} .

GPV $$GPV_1 \supset GPV_2 \supset \cdots \supset GPV_n \supset \cdots$$

Sakurai-I. $$F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$$

Thank you for your attention!

- Any GPV_{2n+1} invariant is F_n invariant.
- All invariants F_{n+1} are strictly stronger than those of F_n .
- All invariants GPV_{2n+2} are strictly stronger than those of GPV_{2n} .