On a question of Östlund — Diagrammatic Vassiliev invariant and RII number

Noboru Ito (NIT, Ibaraki College)
arXiv:2010.10793, 2011.14322
J.w.w. Yusuke Takimura

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A numerical knot diagram invariant extends to knots in the codimension one stratum by the formula:

$$v(K) = v(K_+) - v(K_-)$$

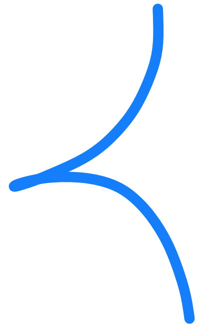
where K_+, K_- are created by resolving projection-degenerate knot (= non projection-generic knot) in positive respectively negative direction.

Revisiting Vassiliev invariant

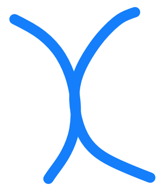
The invariant v is said to be of finite degree if there is a number n such that $v(K) = 0$ whenever K has more than n 1-degenerate germs. The smallest such n is the degree of v.

Def (projection-degenerate germs) [Östlund PhD thesis, 2001]

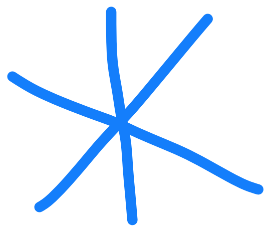
Ω_1: standard cusp; the first derivative of a projection of a knot vanishes, while 2^{nd} and 3^{rd} derivatives are linearly independent,



Ω_2: double point with first order self tangency



Ω_3: triple point with pairwise transversal crossings



Östlund wrote: "the concept of degree can be refined by considering the different types"

A knot diagram invariant of finite degree n in Ω_i $(i = 1,2,3)$ if it takes value zero on any projection-degenerate knot with more than n 1-degenerate germs of Ω_i.

Theorem (Östlund PhD thesis, 2001)

Let v be a knot diagram invariant that is unchanged under Ω_1- and Ω_3-moves, and of finite degree in Ω_2. Then v is a knot invariant.

Östlund's Theorem (Theorem 6 in thesis)

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In the view of finite degree invariants, Ω_2-move is superfluous.

In the view of finite degree invariants, Ω_2-move is superfluous. Östlund wrote:

If a non-trivial invariant that jumps only under Ω_2-moves is found, …
… it disproves the knot diagram counterpart of Vassiliev's conjecture for finite degree invariants of knots.

Östlund's Question

Östlund wrote:

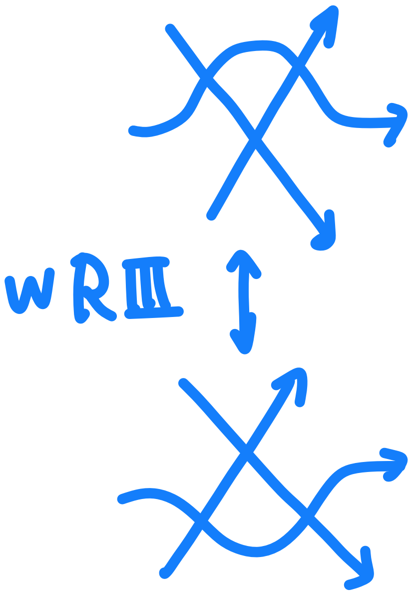
Whether all self-tangency moves of plane curves can be replaced by cusp- and triple point moves is a different question: The author knows of no potential counterexample to this statement.

Östlund further wrote:

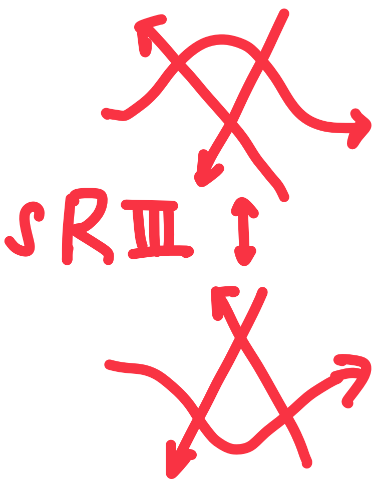
Can knot diagram invariants based on state sum models distinguish path-components of $\mathscr{K} \setminus \{$knots with projection self-tangency$\}$?

(So far, all such "quantum" invariants of knots and plane curves has been showed to be expressible in invariants of finite degree.)

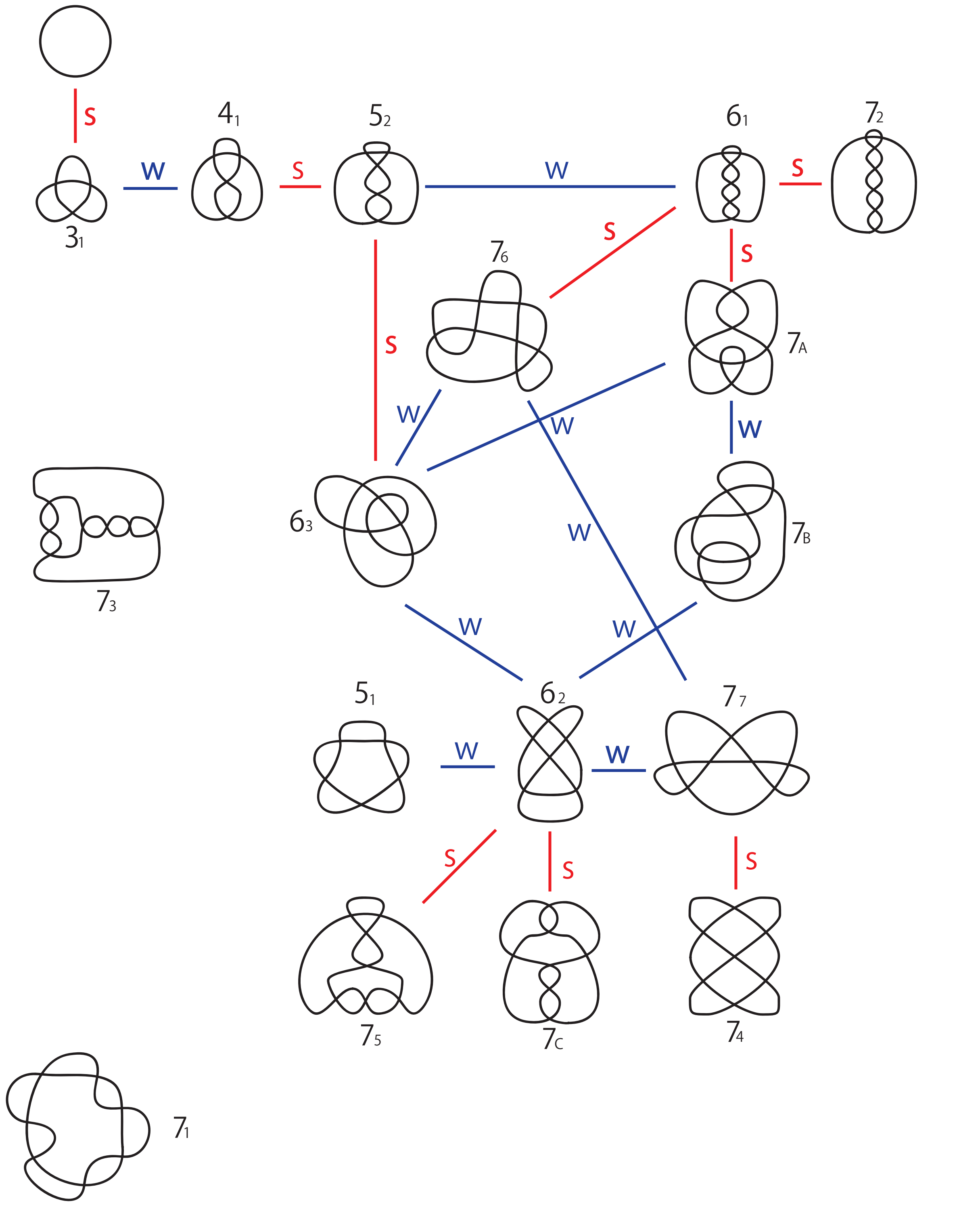
Let’s search for non-trivial knot projection under RI and RIII !

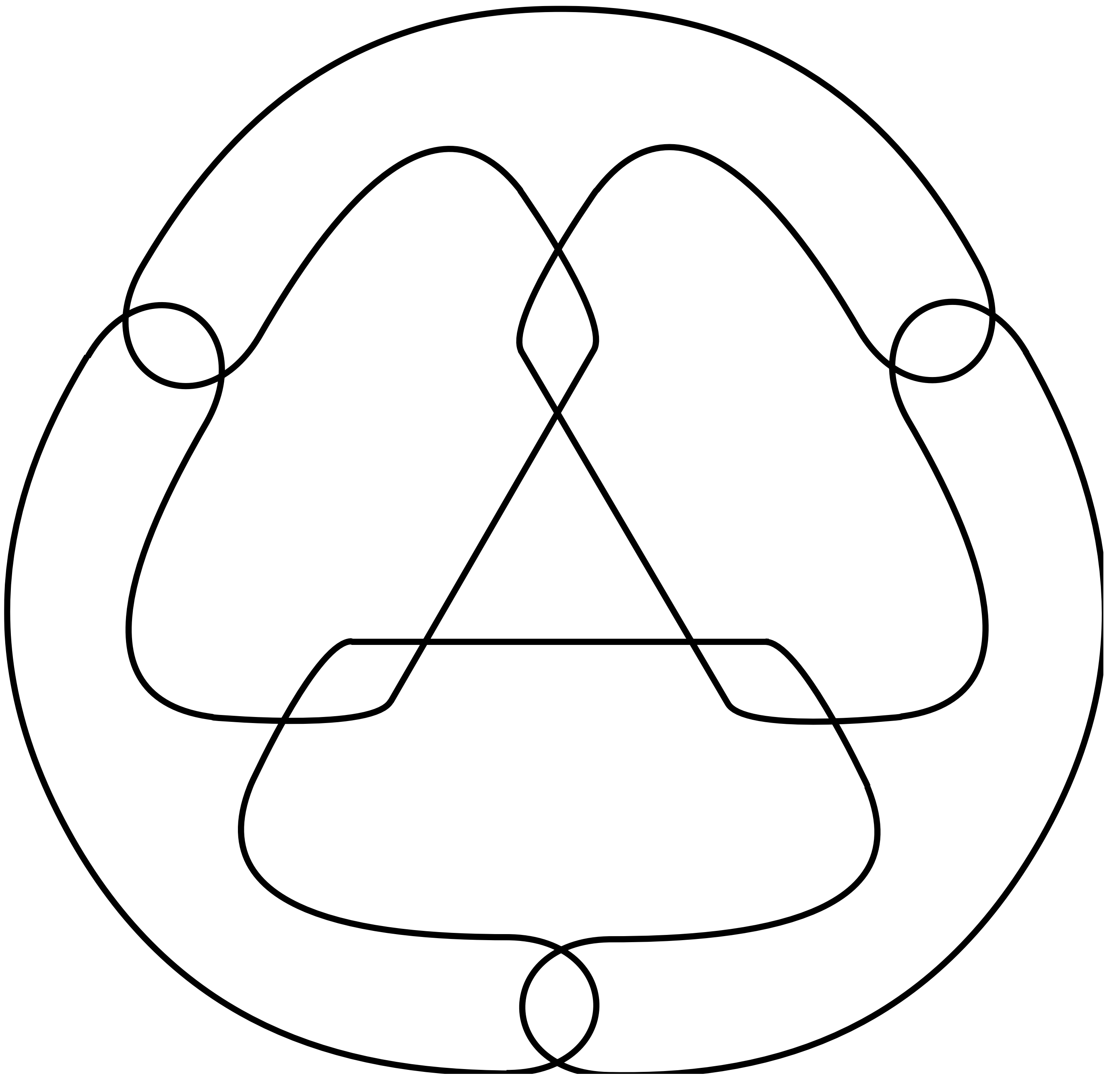


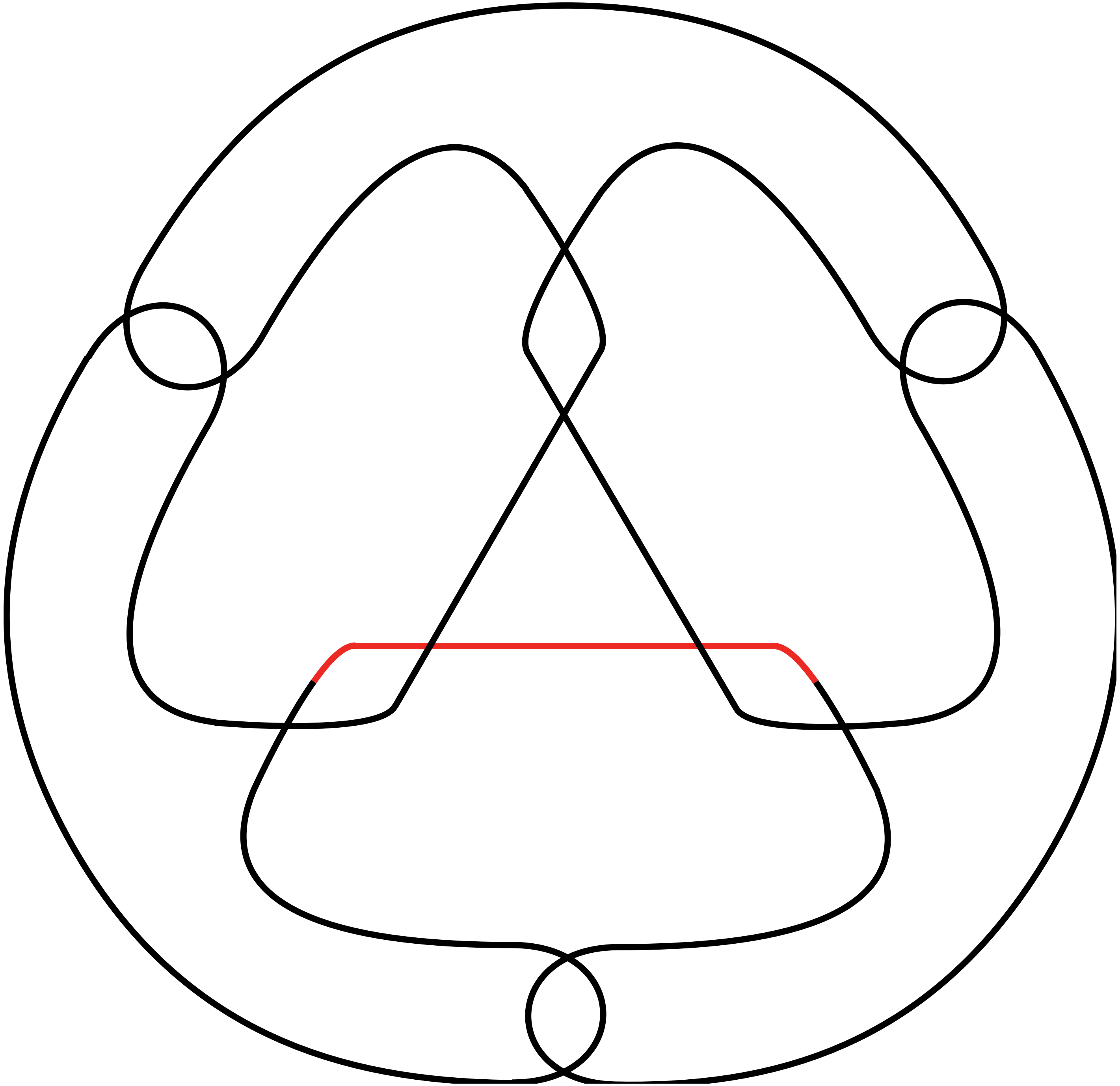
w : RIs & wRIII

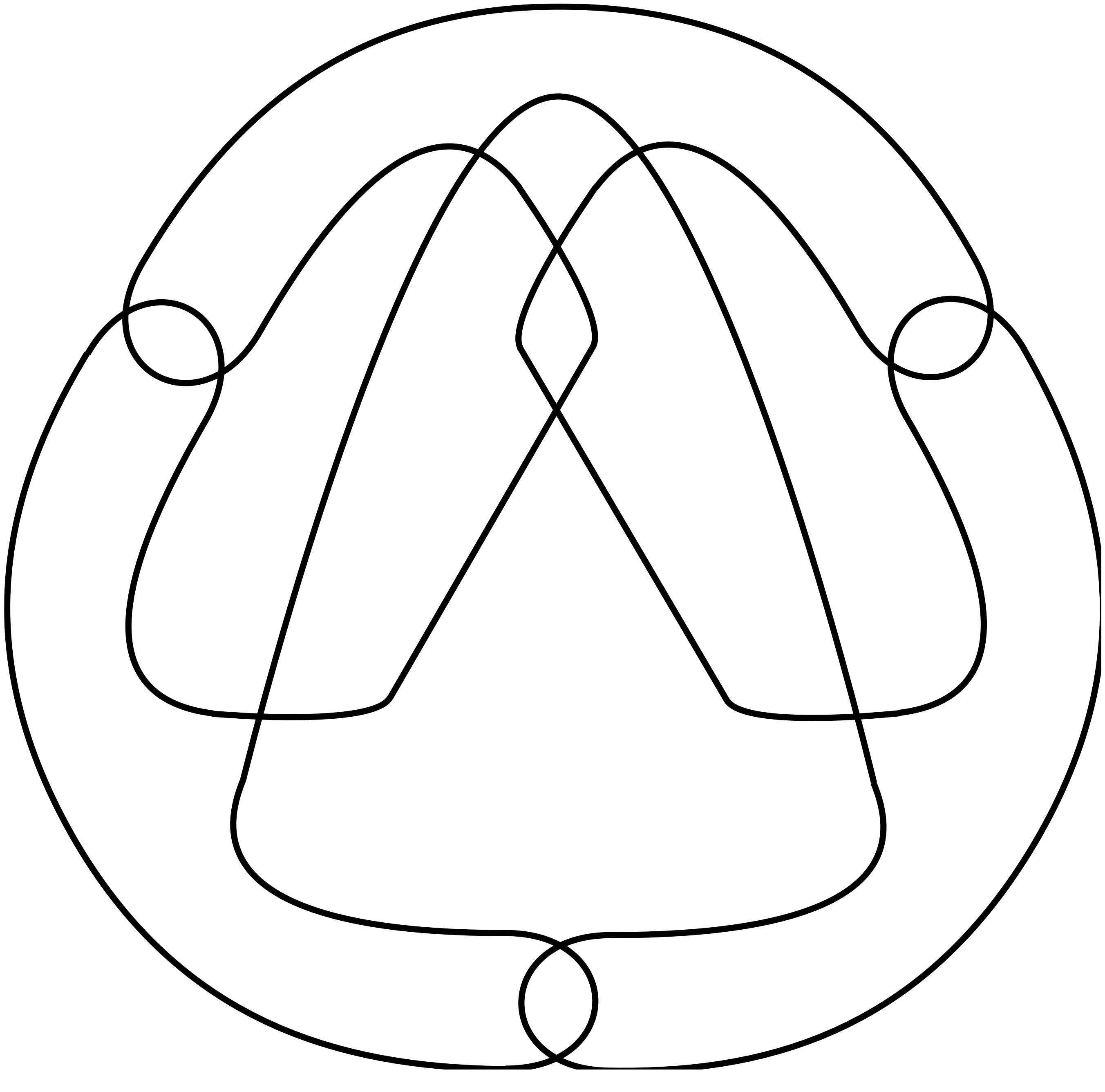


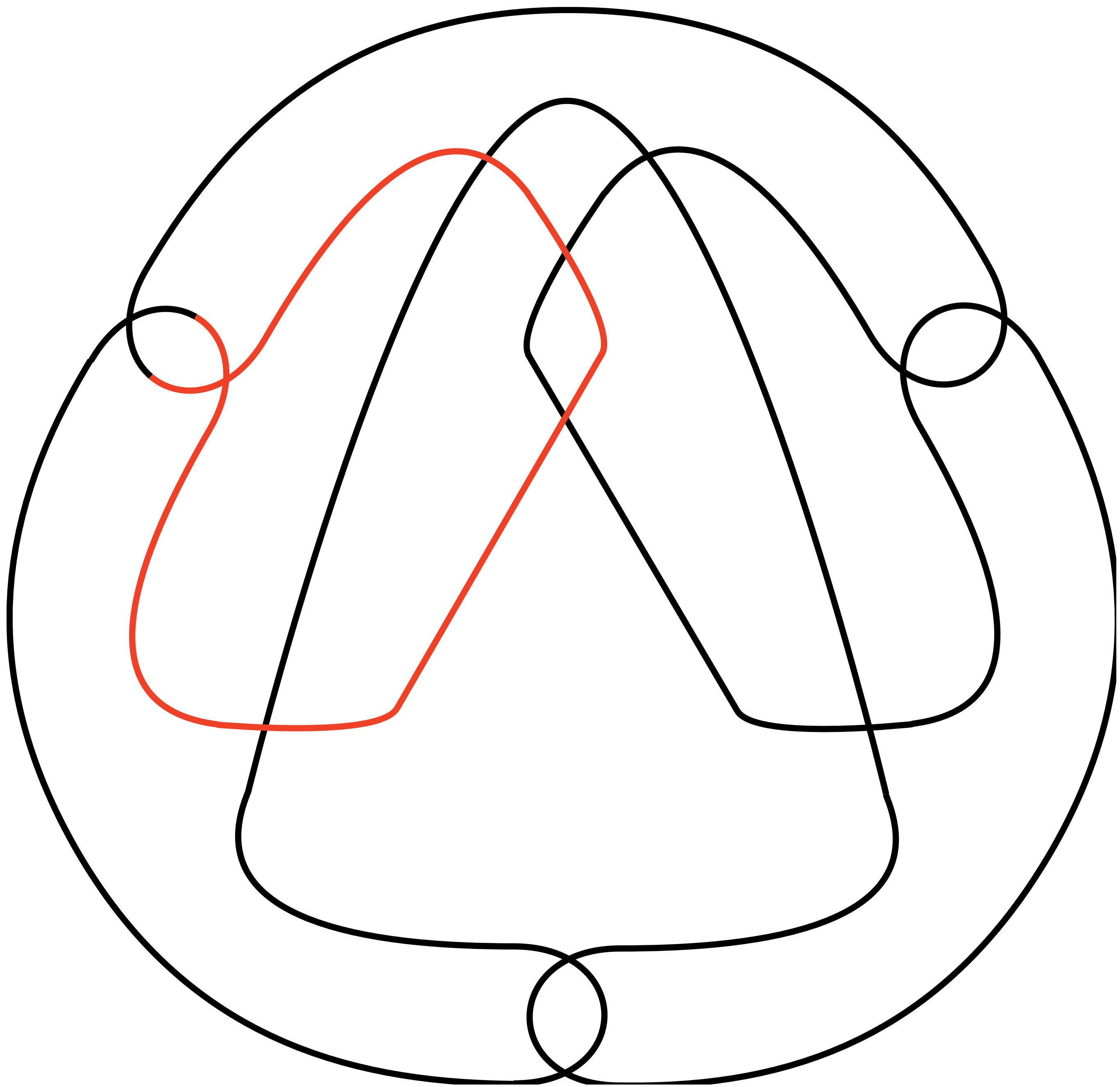
s : RIs & sRIII



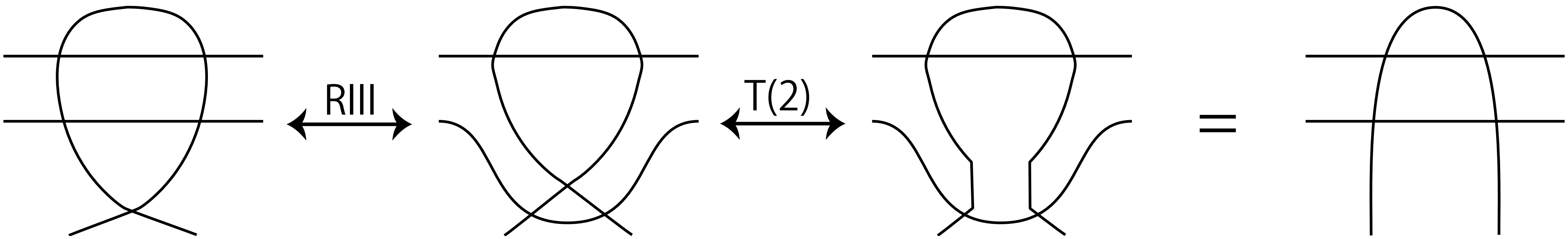


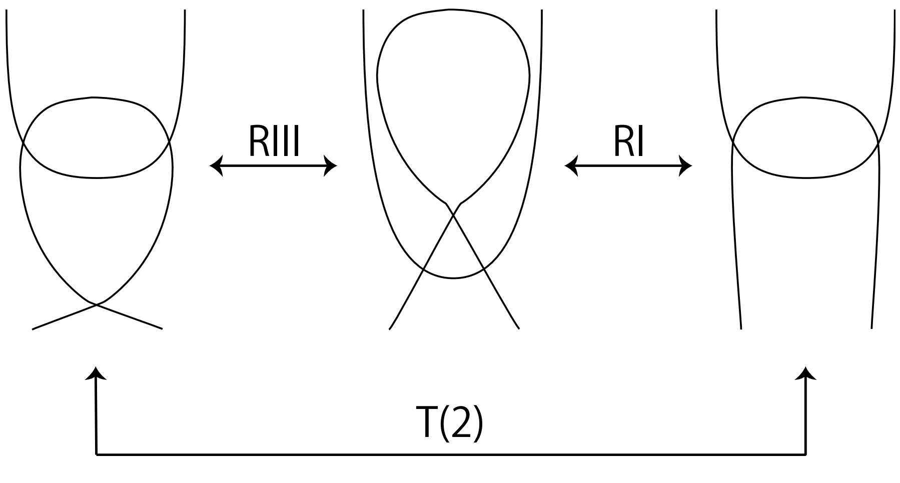


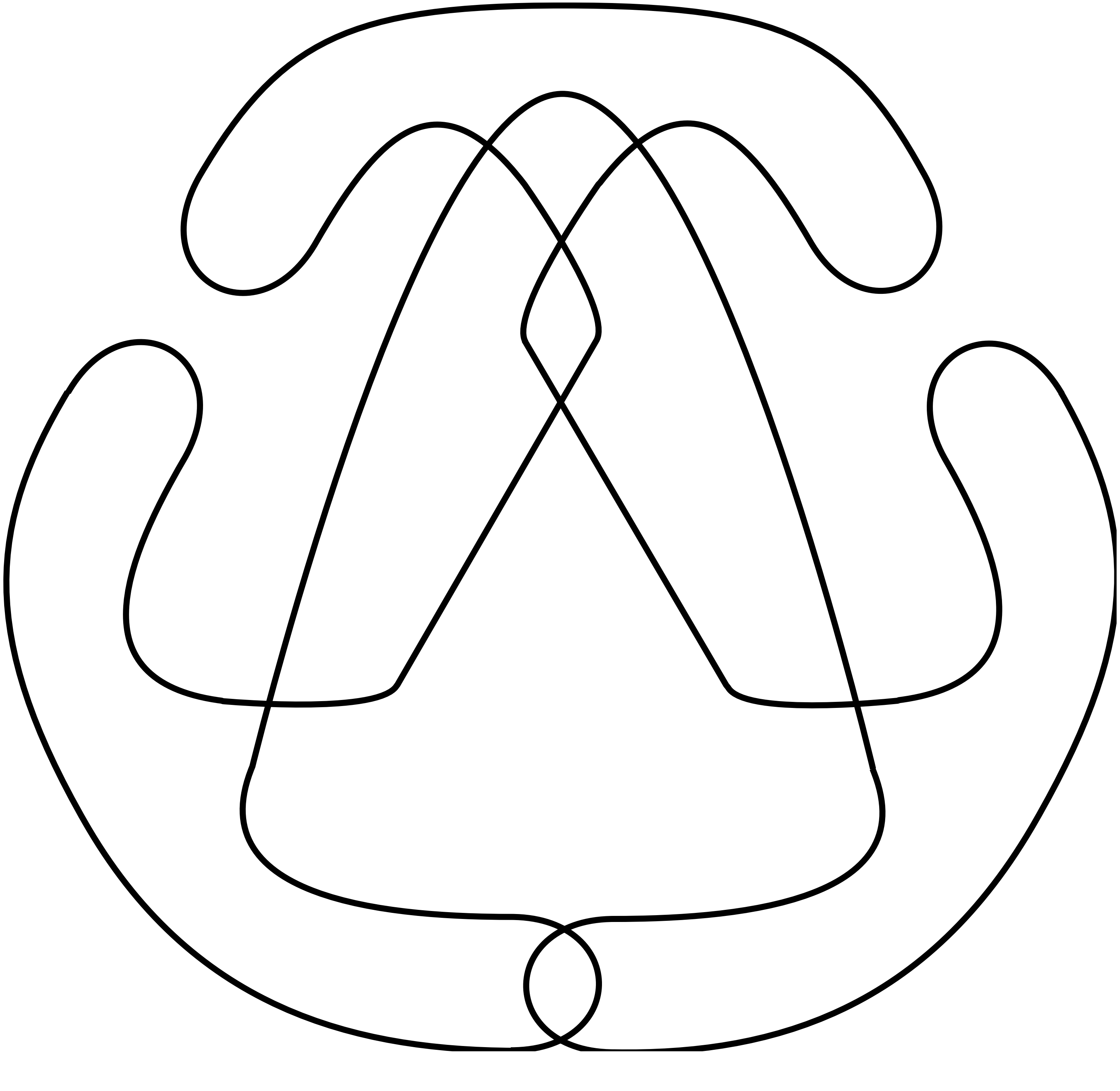


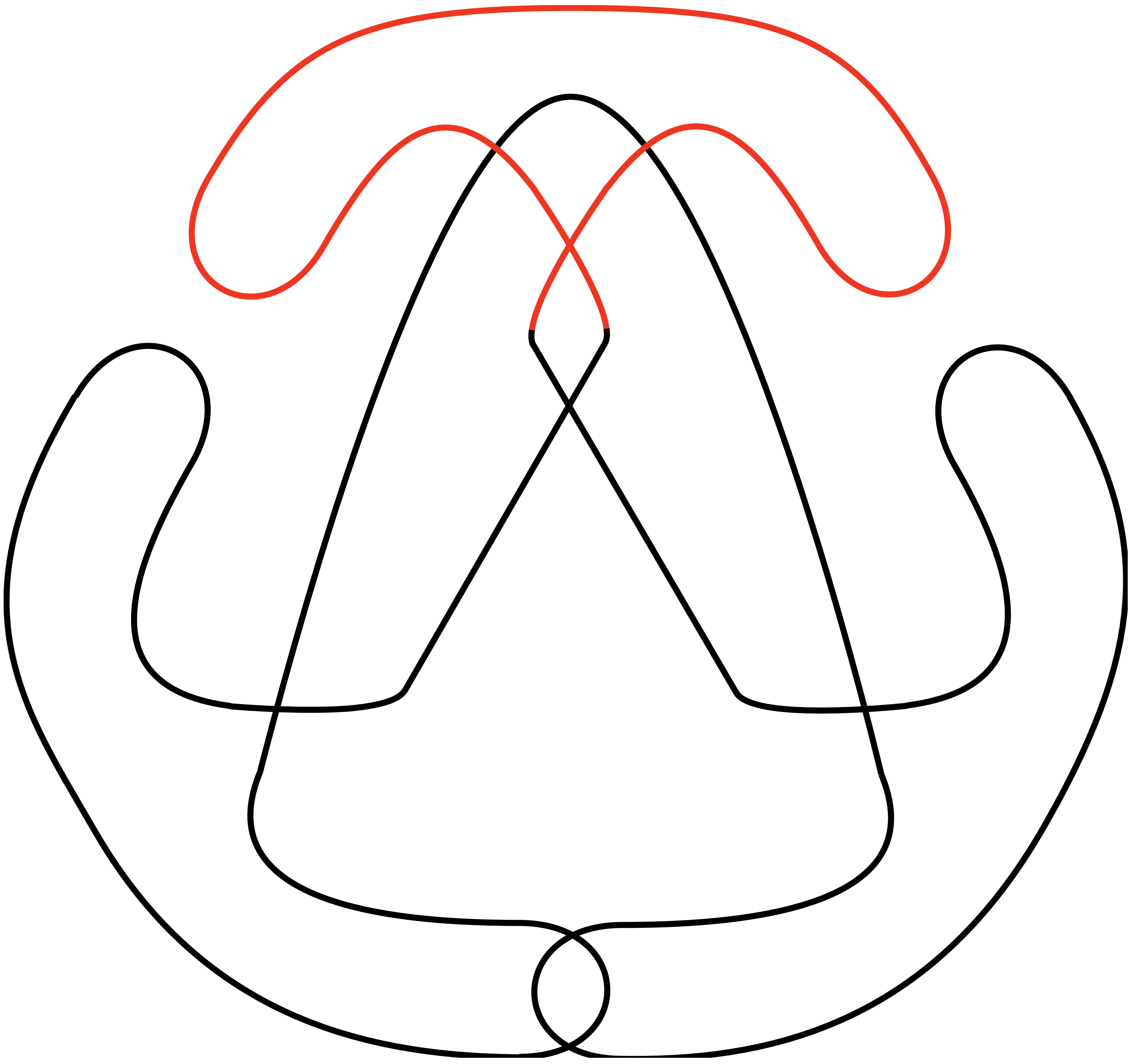


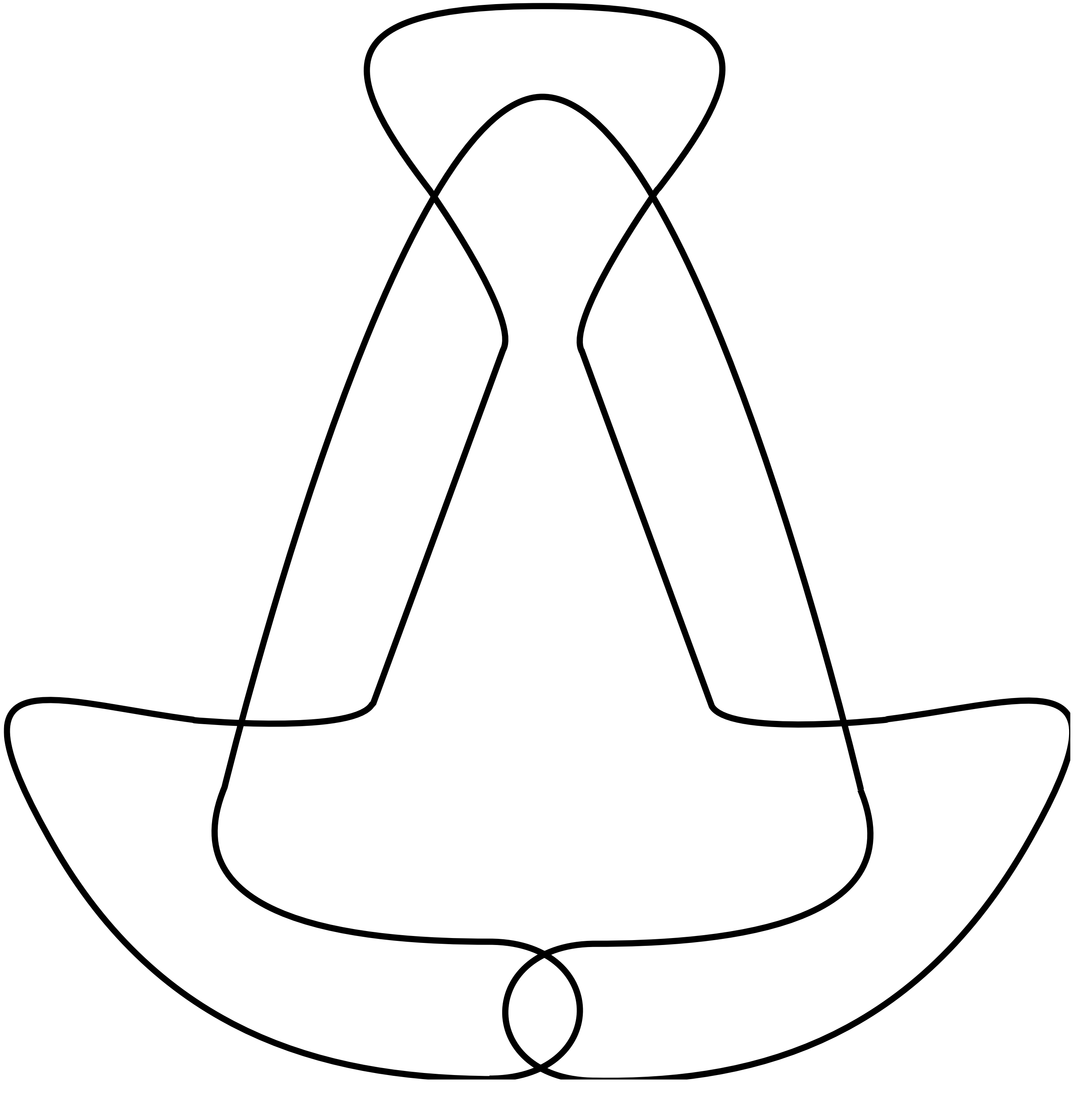
By applying T(2), we have (∗):

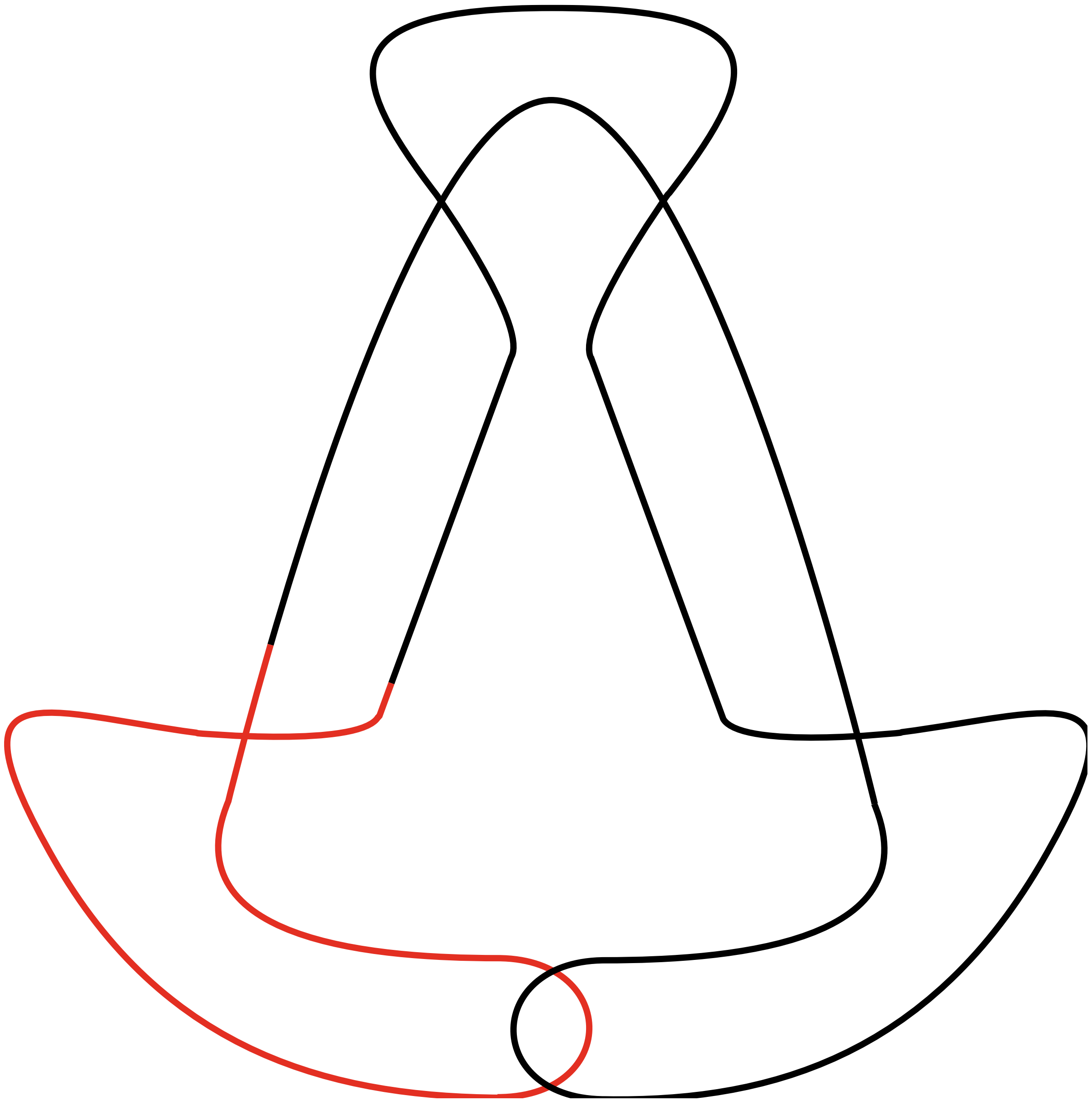


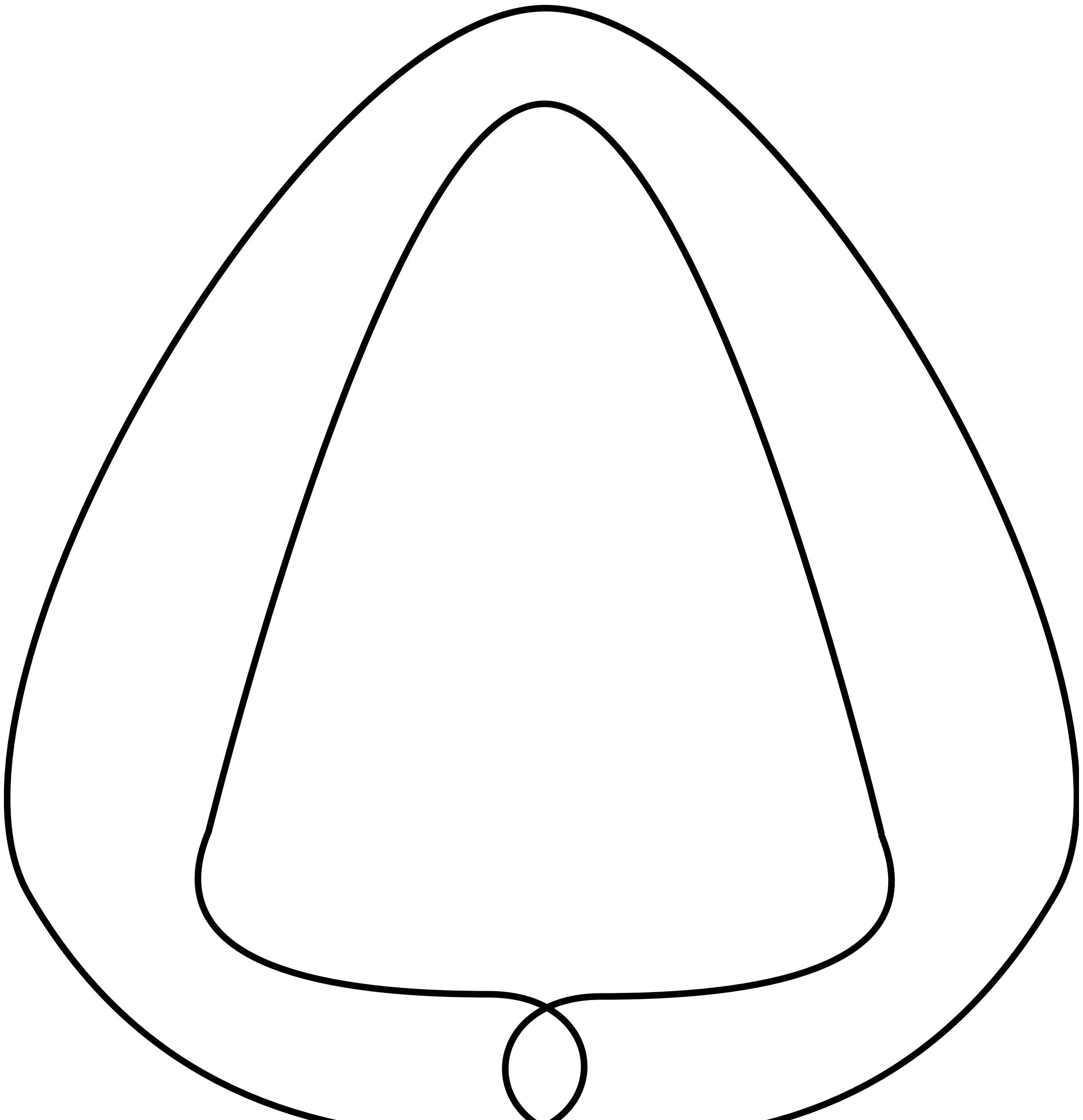


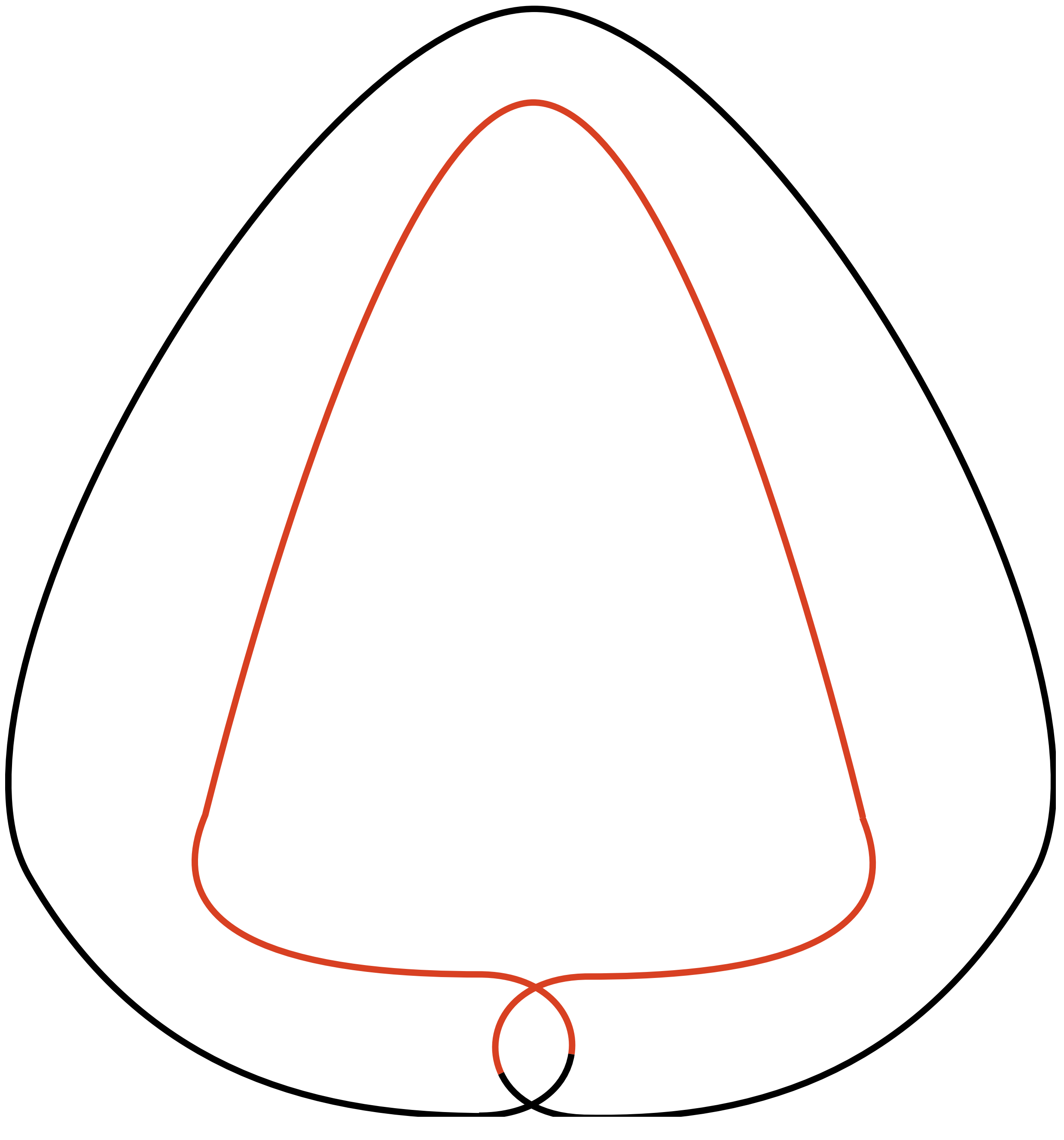


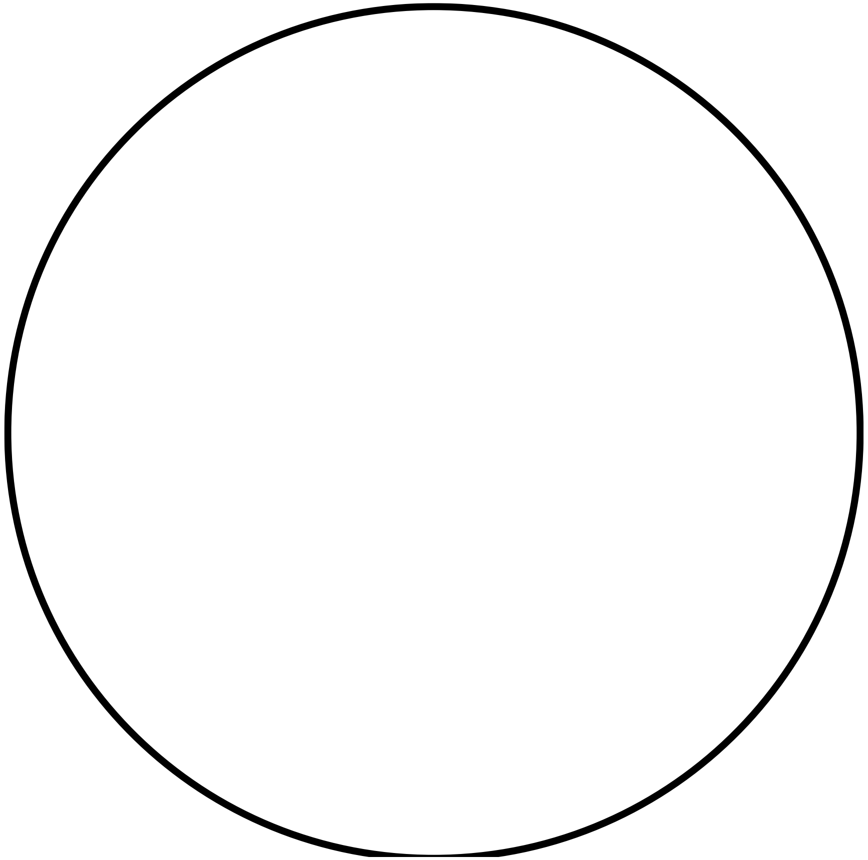


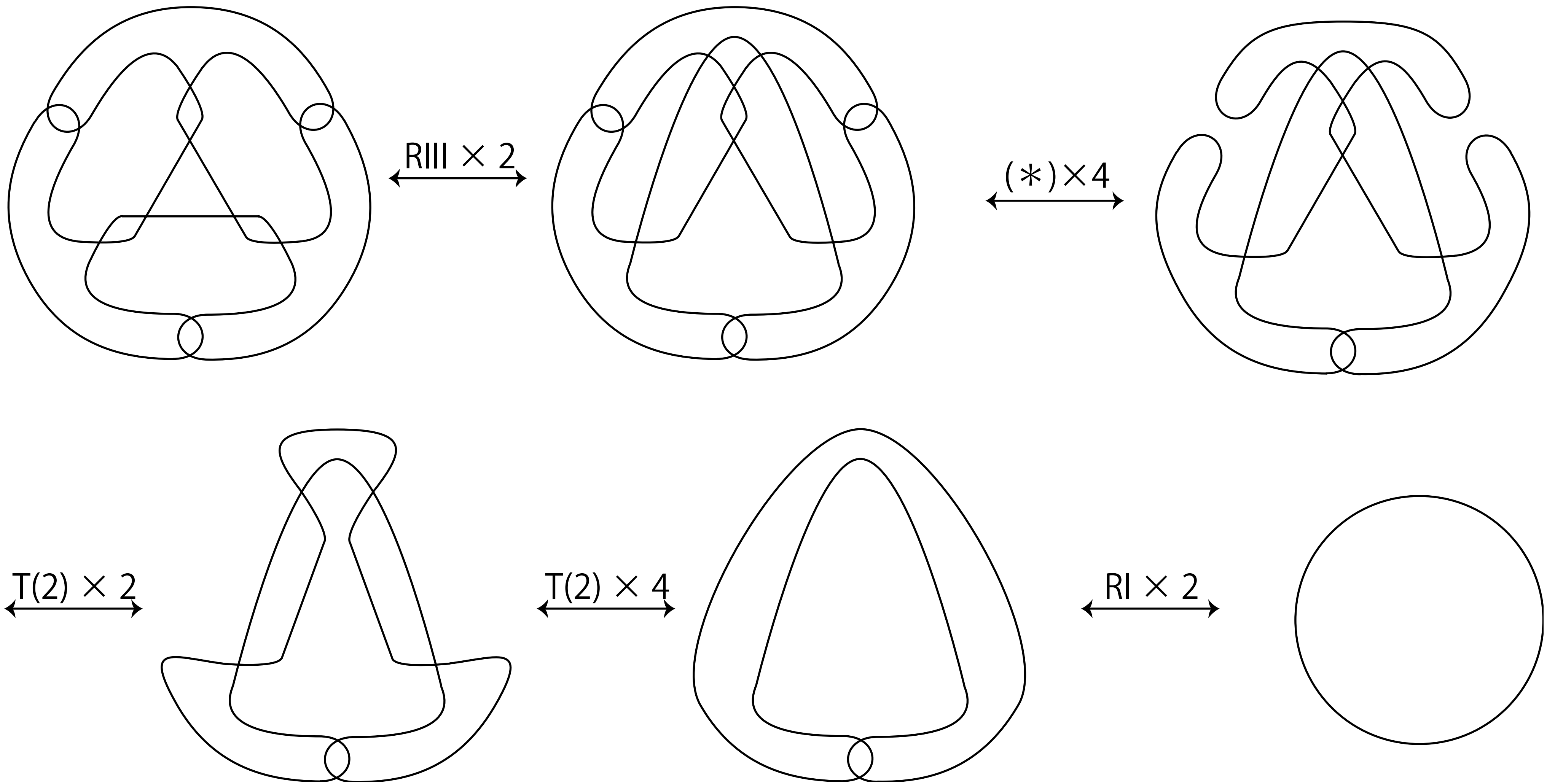




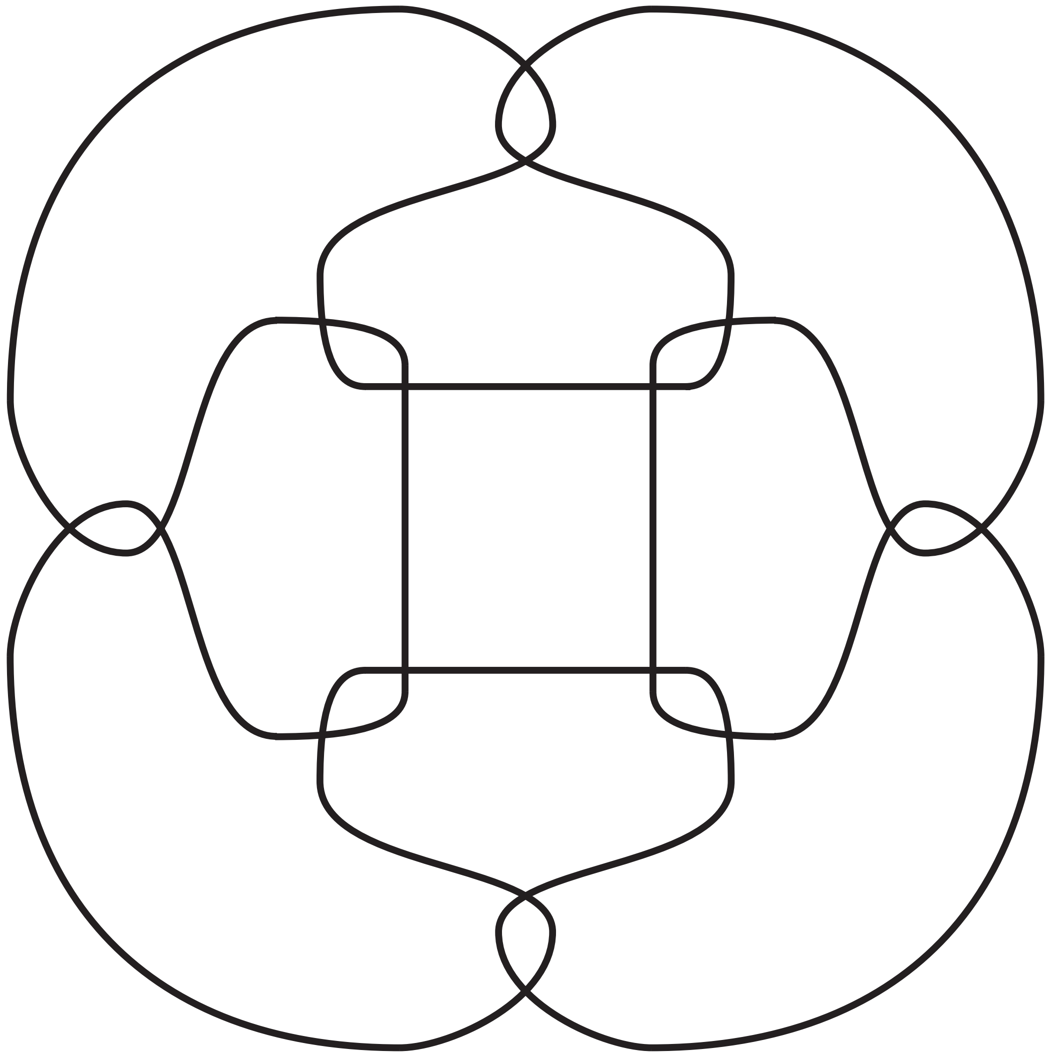




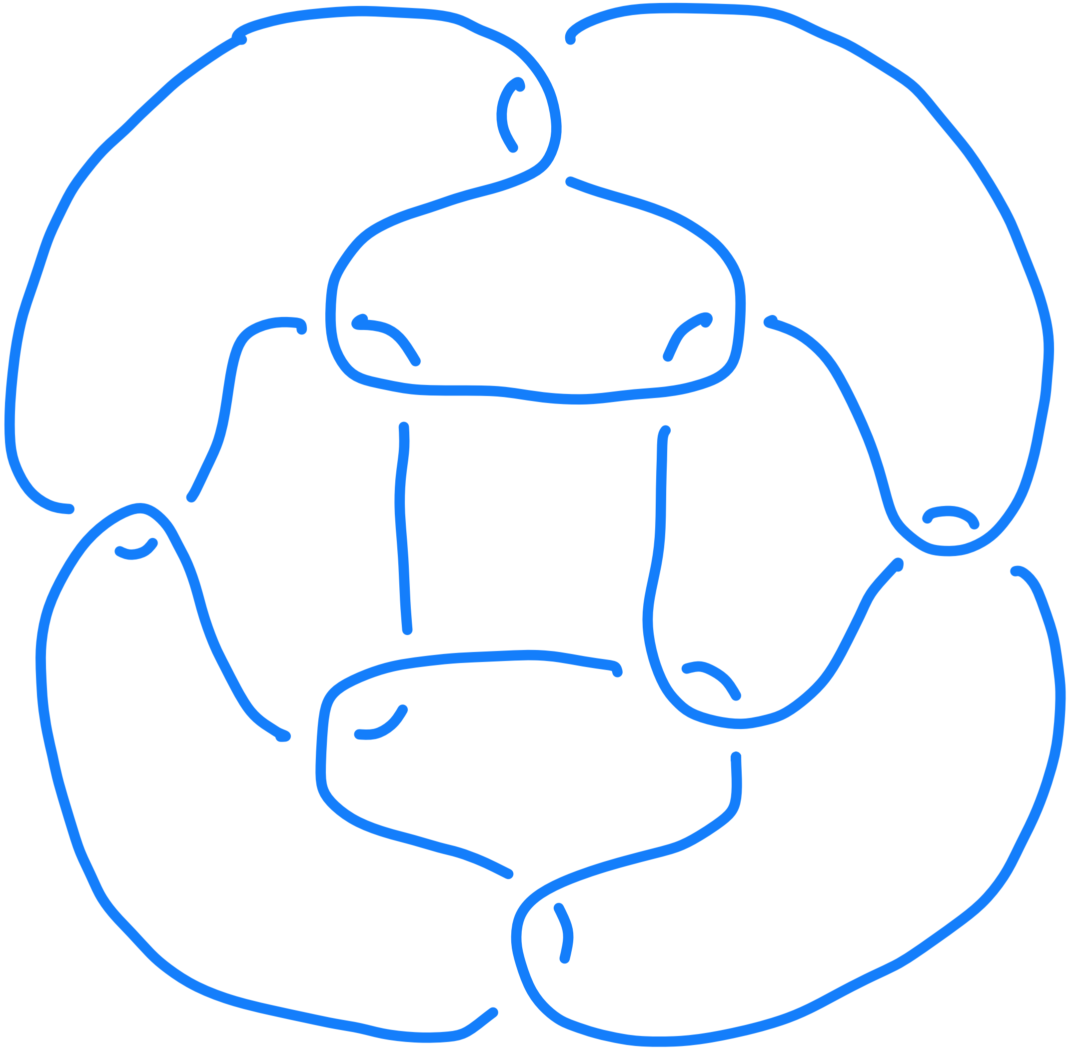



RIII × 2
(∗)×4
T(2) × 2
T(2) × 4
RI × 2

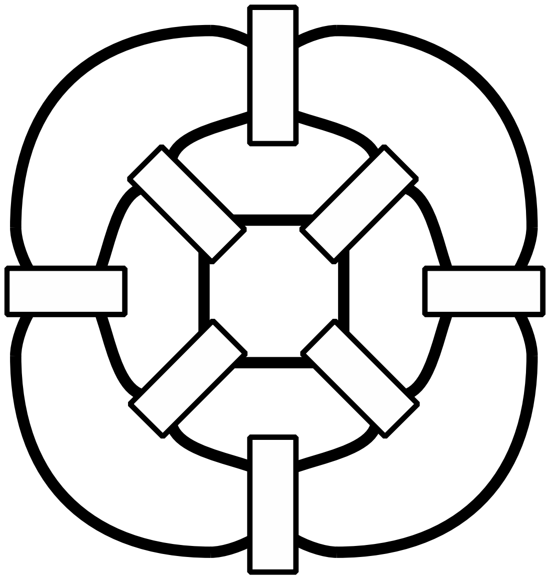
Non-trivial curve given by
Hagge-Yazinski (arXiv: 0812.1241)



There exists a knot diagram of the unknot that needs RII to be transformed into the trivial as follows:

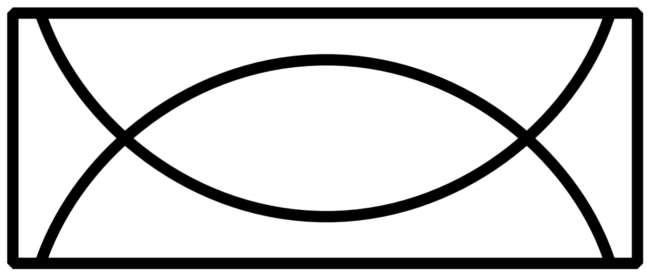


P_{HY} can be generalized:



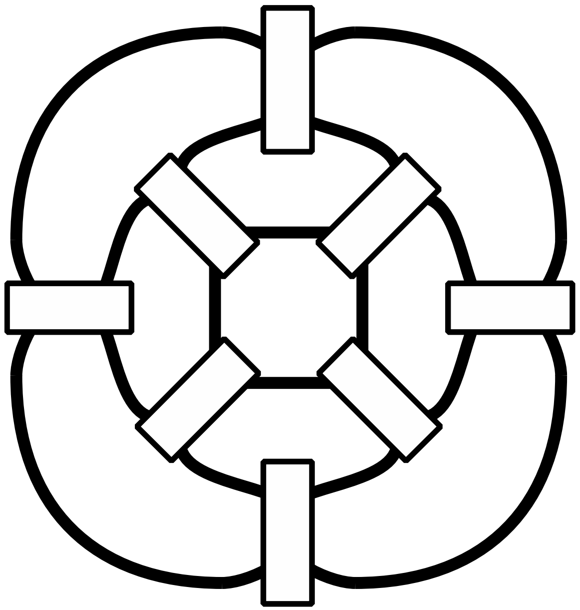
$P(1, m, 4)$

$\mathbf{b}_m$



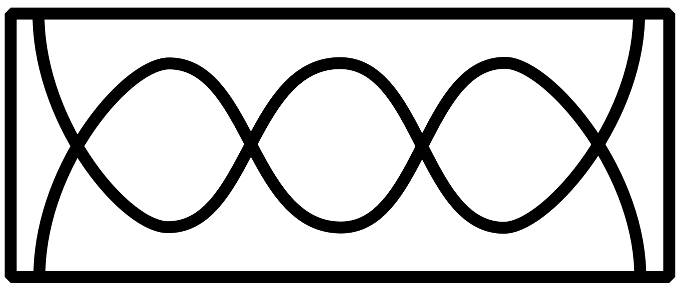
$m = 1$

P_{HY} can be generalized:



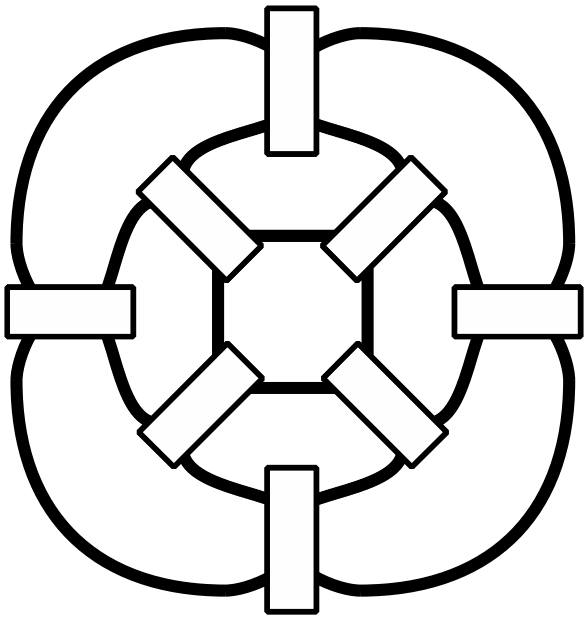
$P\,(1,\ m,\ 4)$

$$b_m$$



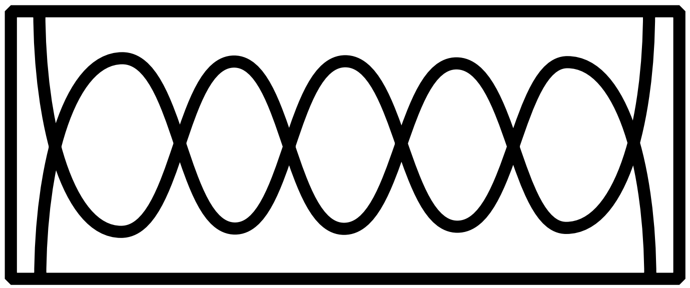
$m = 2$

P_{HY} can be generalized:



P (1, m, 4)

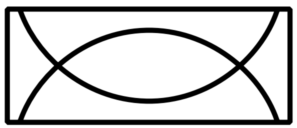
$\mathbf{b}_{\mathrm{m}}$



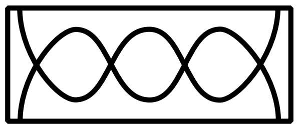
m = 3

P_{HY} can be generalized:

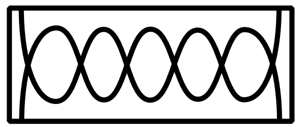
b_m



m = 1

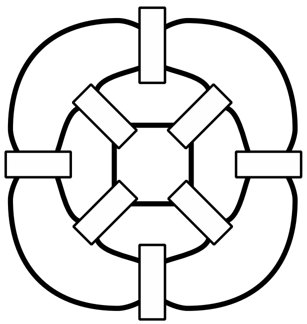


m = 2

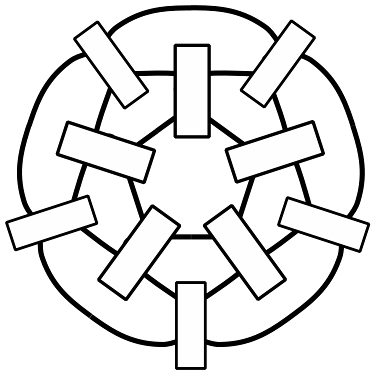


m = 3

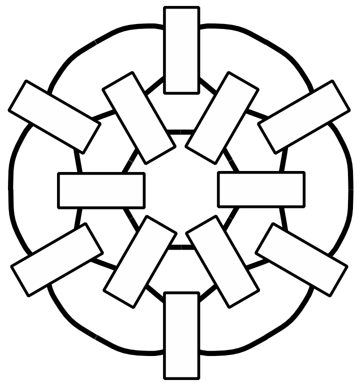
…



P (1, m, 4)



P (1, m, 5)

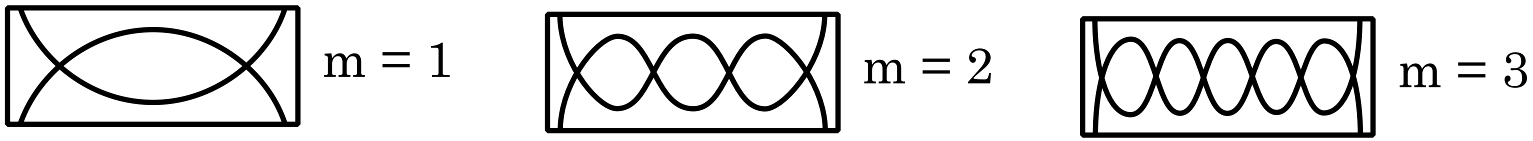


P (1, m, 6)

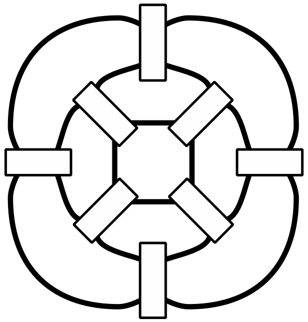
…

P_{HY} can be generalized:

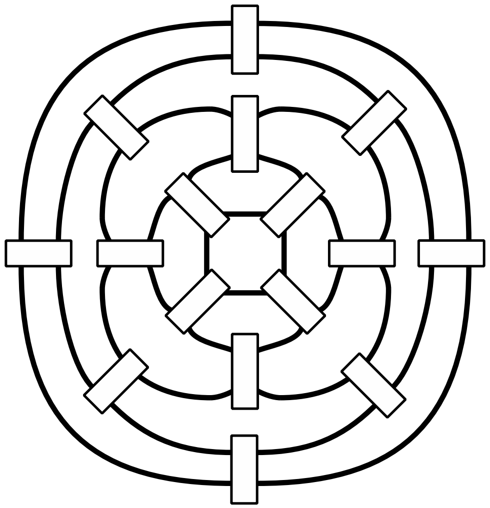
b_m



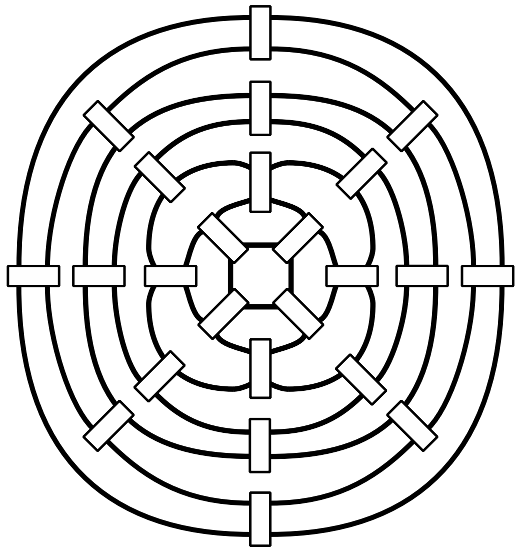
$\cdots$



P (1, m, 4)



P (2, m, 4)

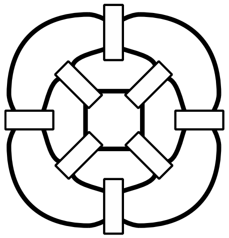


P (3, m, 4)

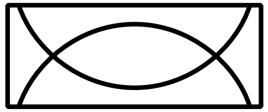
$\cdots$

P_{HY} can be generalized:

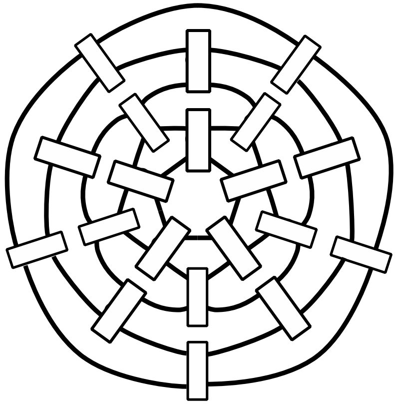
b_m



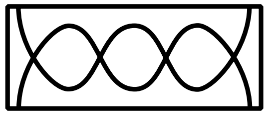
P (1, m, 4)



m = 1

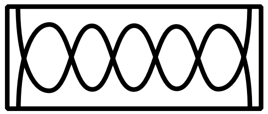


P (2, m, 5)



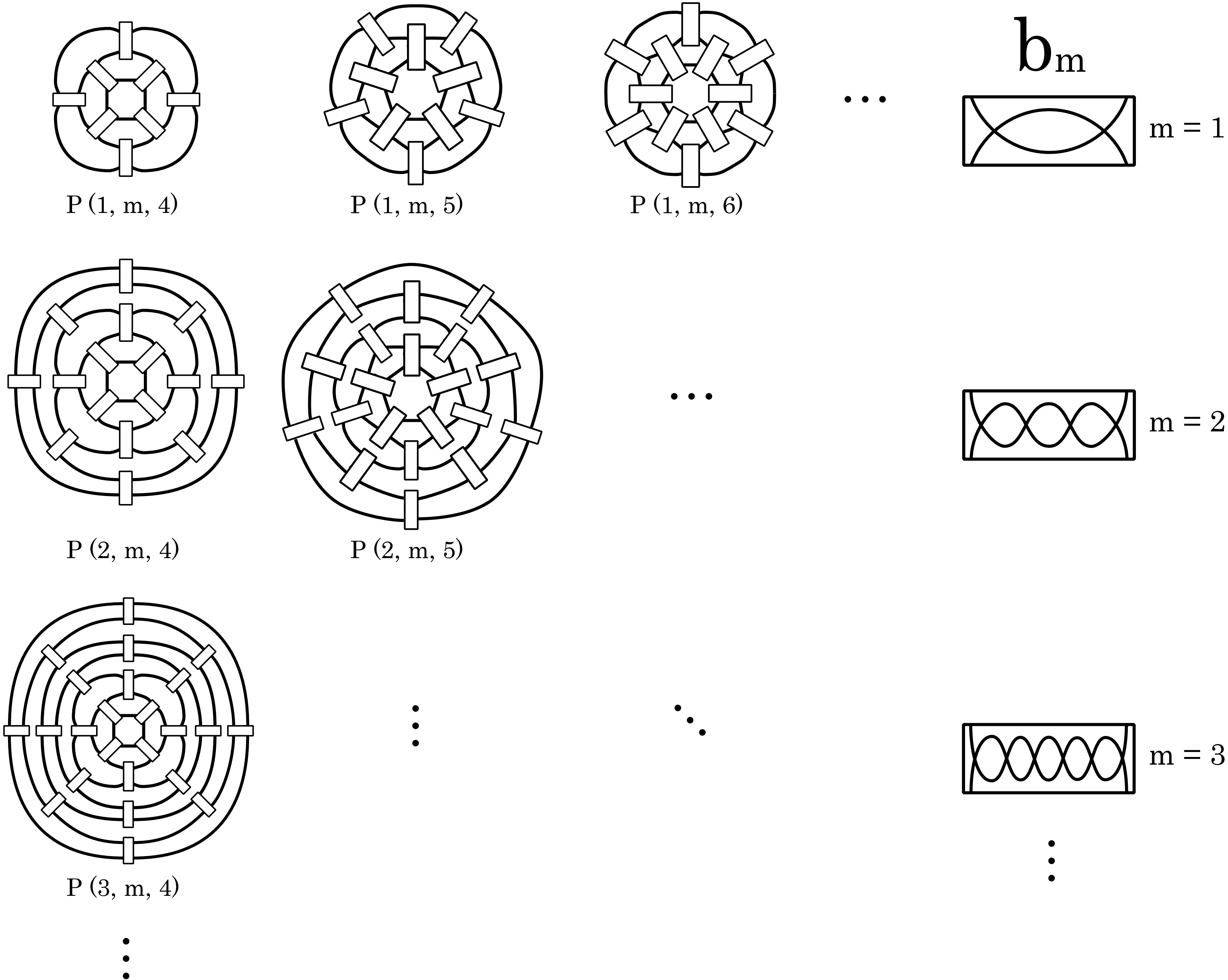
m = 2

$\ddots$



m = 3

$\vdots$



P (1, m, 4)
P (1, m, 5)
P (1, m, 6)
b_m
m = 1
P (2, m, 4)
P (2, m, 5)
m = 2
P (3, m, 4)
m = 3

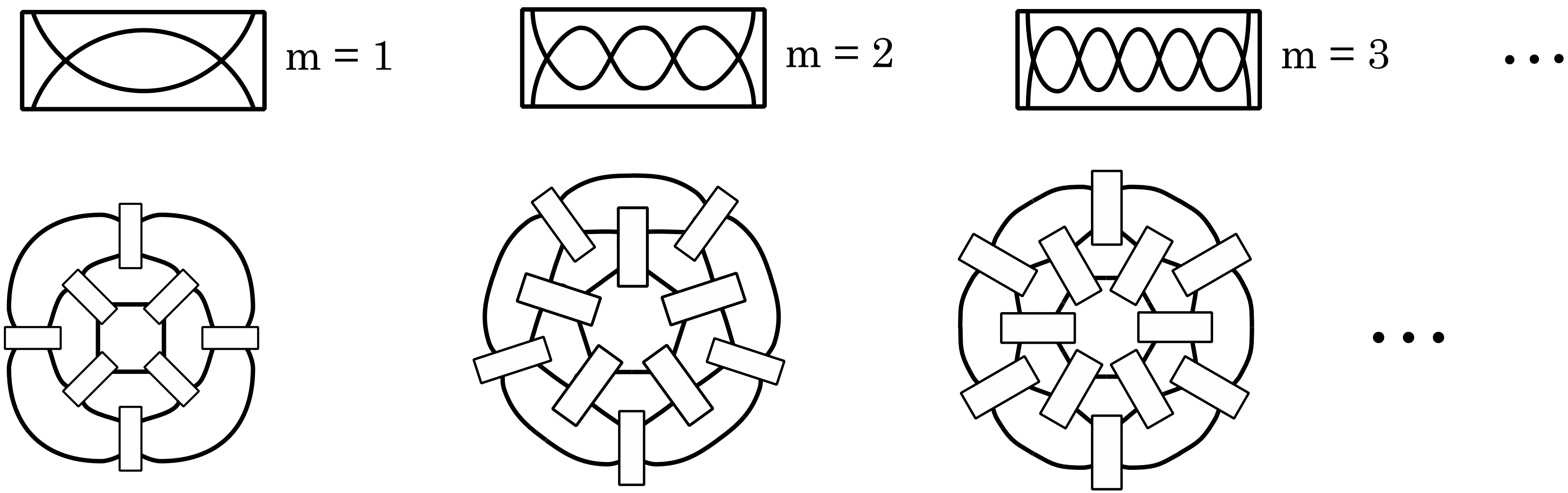
Invariant of knot projections under RI and RIII.

Def (Takimura-I. arXiv:2010.10793). The RII number is the minimum number of deformations of negative RII in sequences to obtain the standard embedding of the circle from a knot projection.

Theorem (Takimura-I. Kobe J. M. in Press now, arXiv:2010.10793)

$\mathrm{RII}(P(m,n)) = m$ **for the following:**

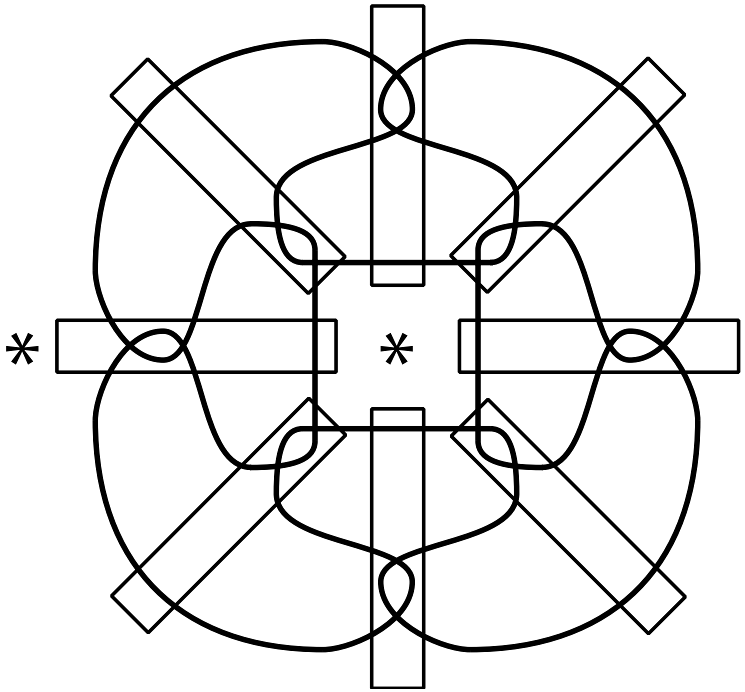
$\mathrm{b_m}$

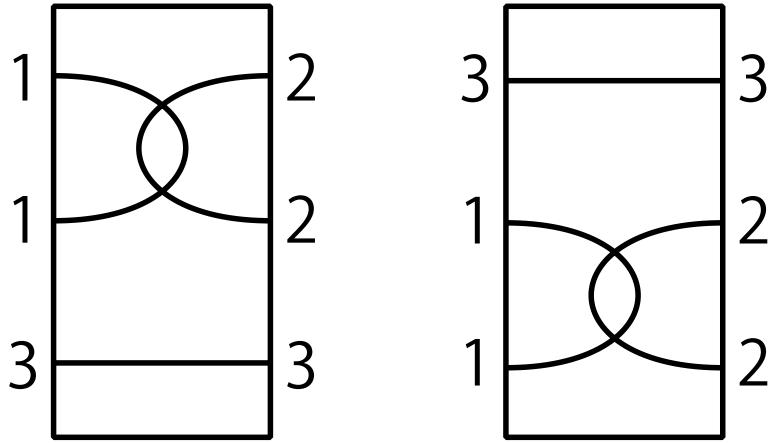


$P(m,4)$ $P(m,5)$ $P(m,6)$

$$\mathrm{RII}(P(m,n)) \geq 1.$$

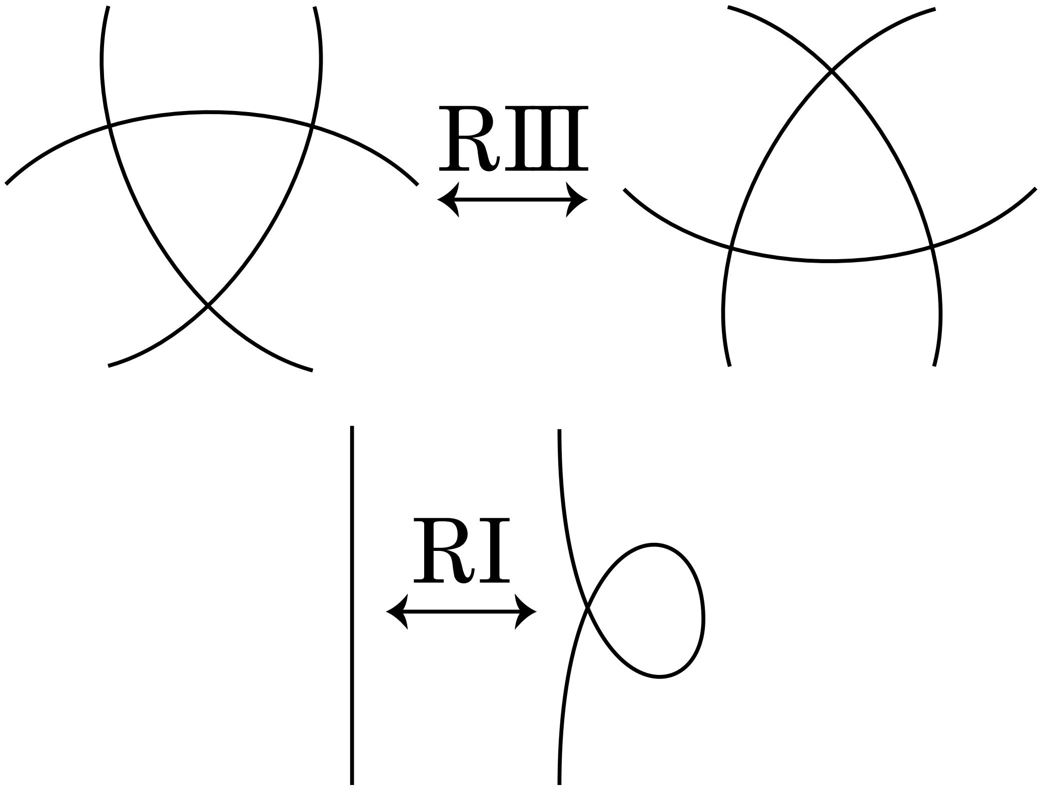
The number of intersections between "string 1" and "string 2" are unchanged under RI and RIII (completely) including Box.

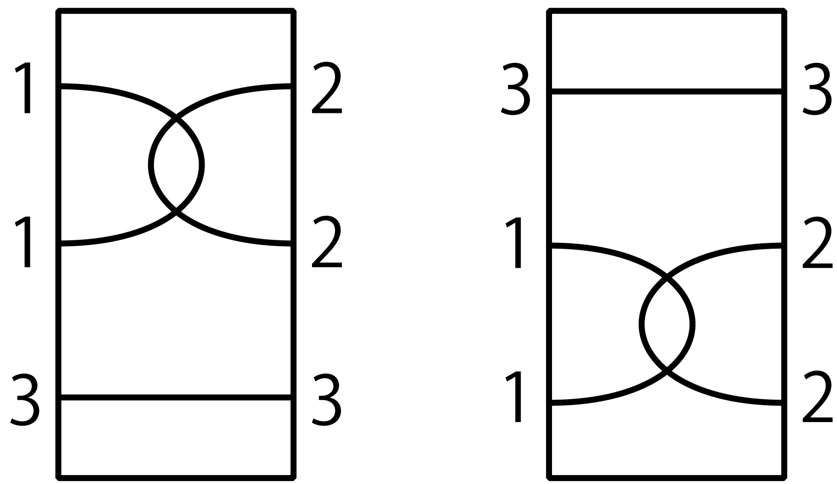




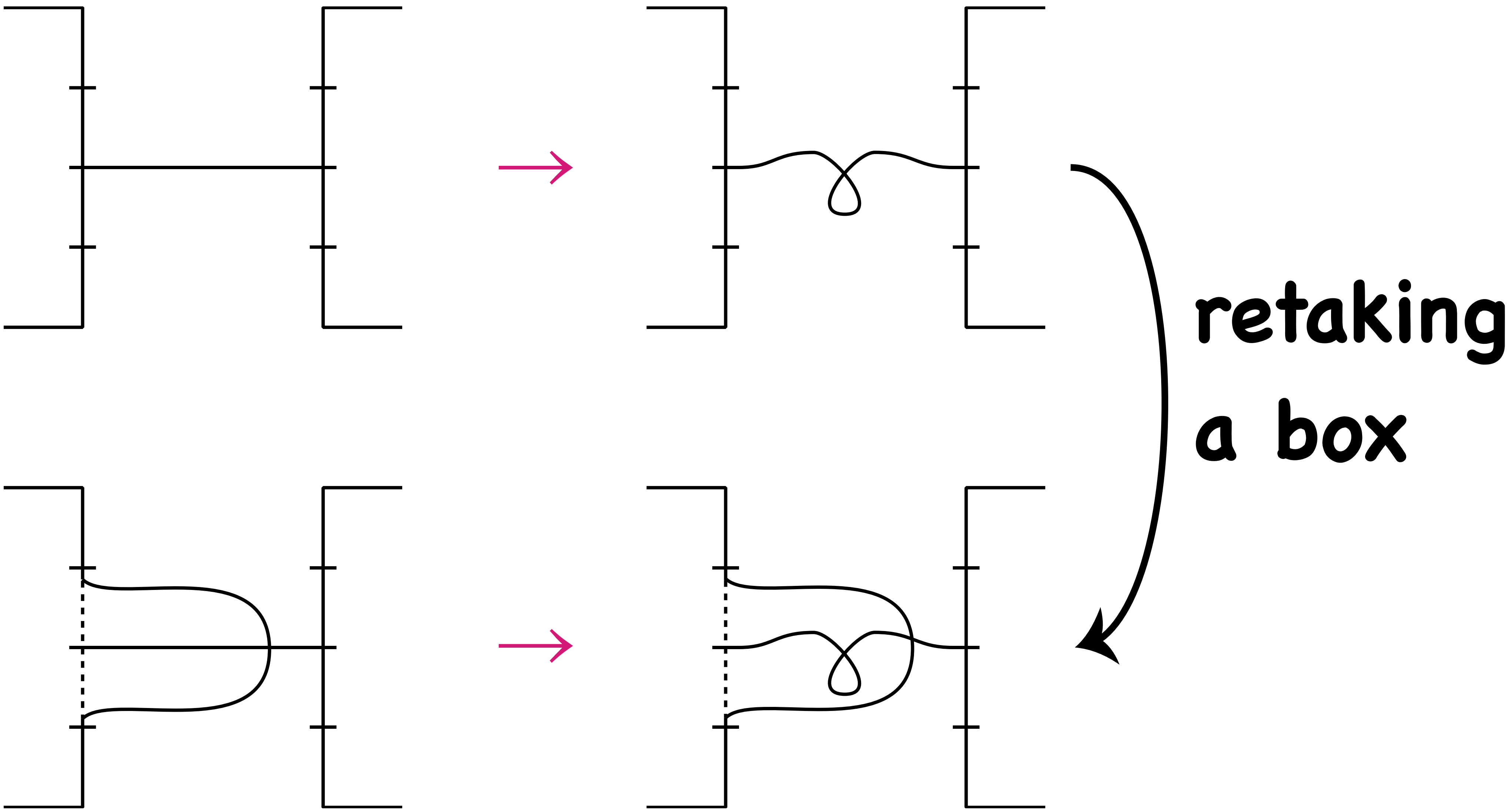
Polygon "*" has at least four sides.

The number of intersections between “string 1” and “string 2” are unchanged under RI and RIII (completely) including Box.

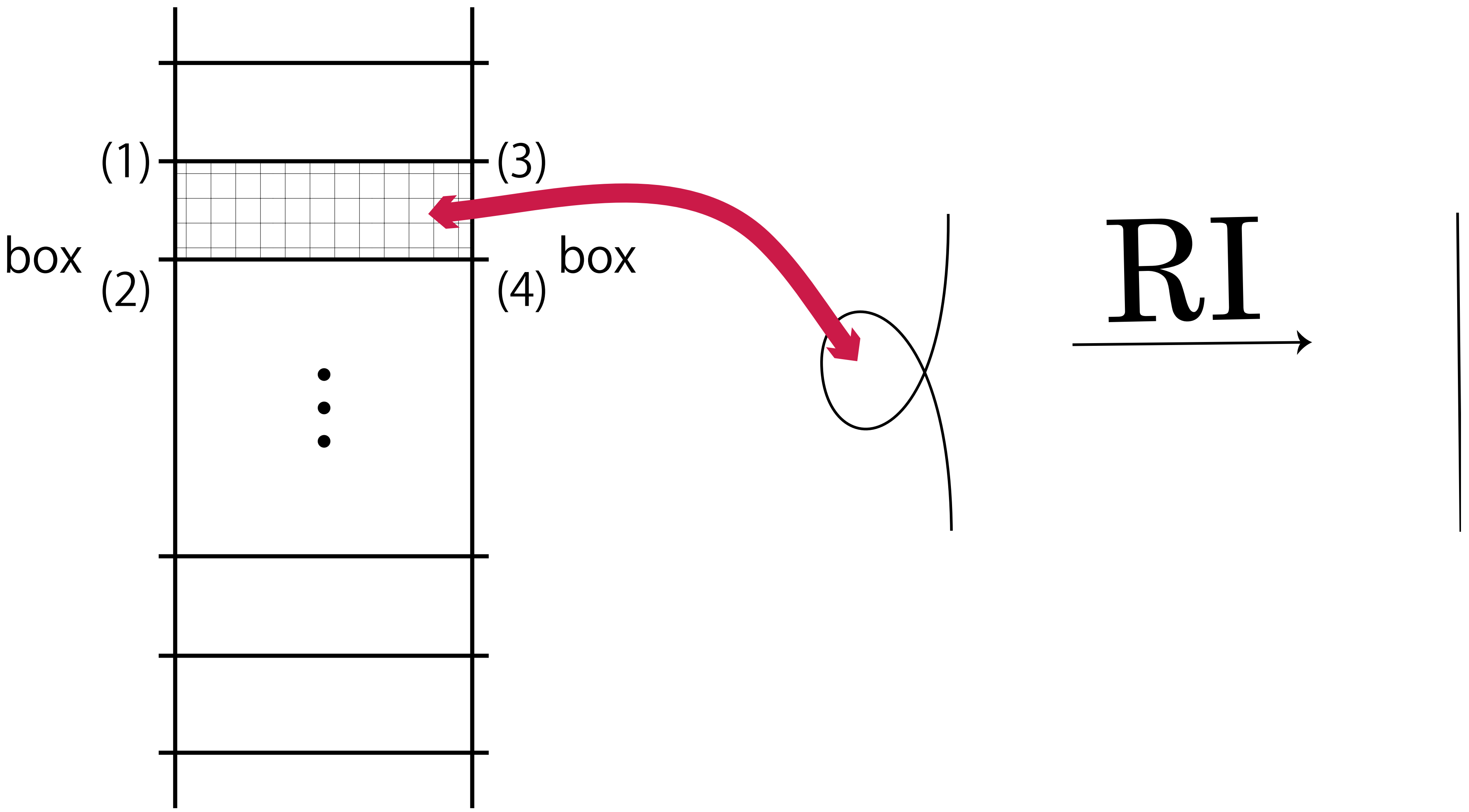




For positive RI not within Box,

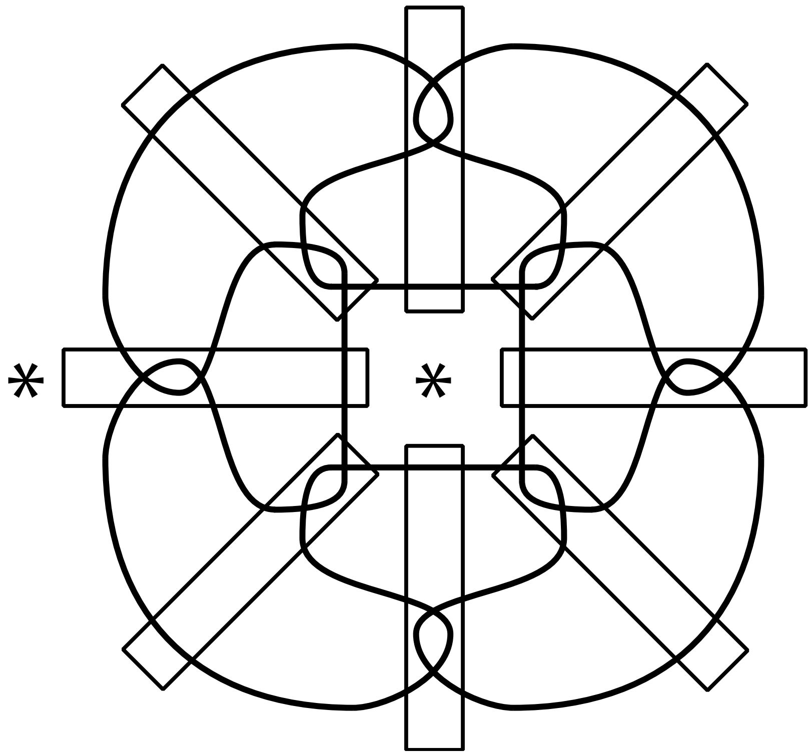


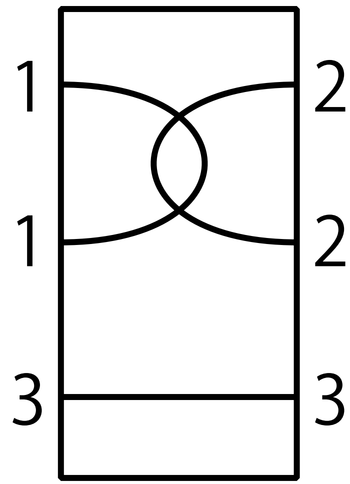
For negative RI not within Box,

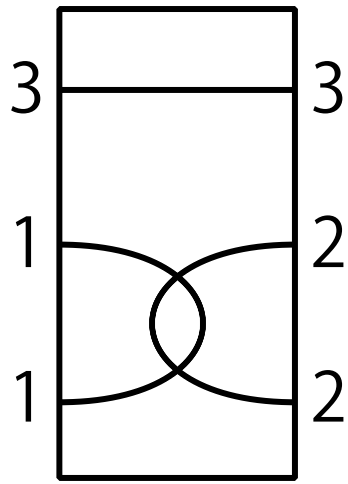


String 1 or 2 has at least two crossings.

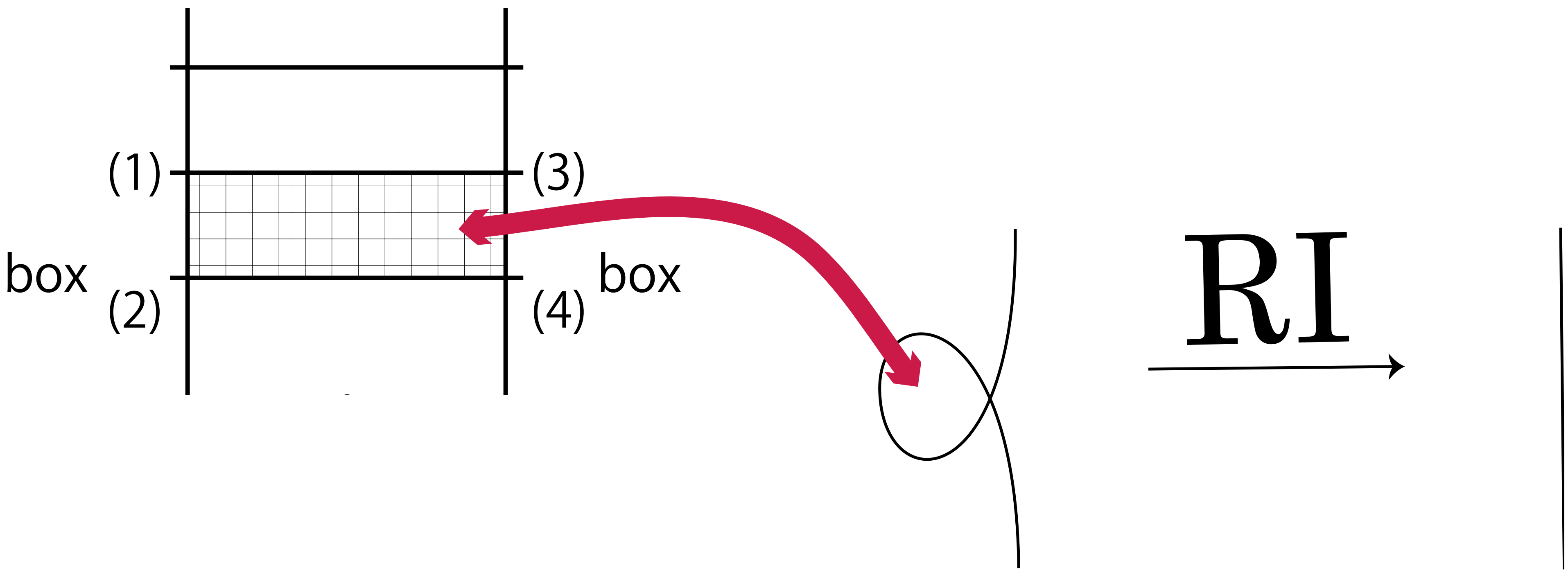
String 3 connects string 1 or 2.





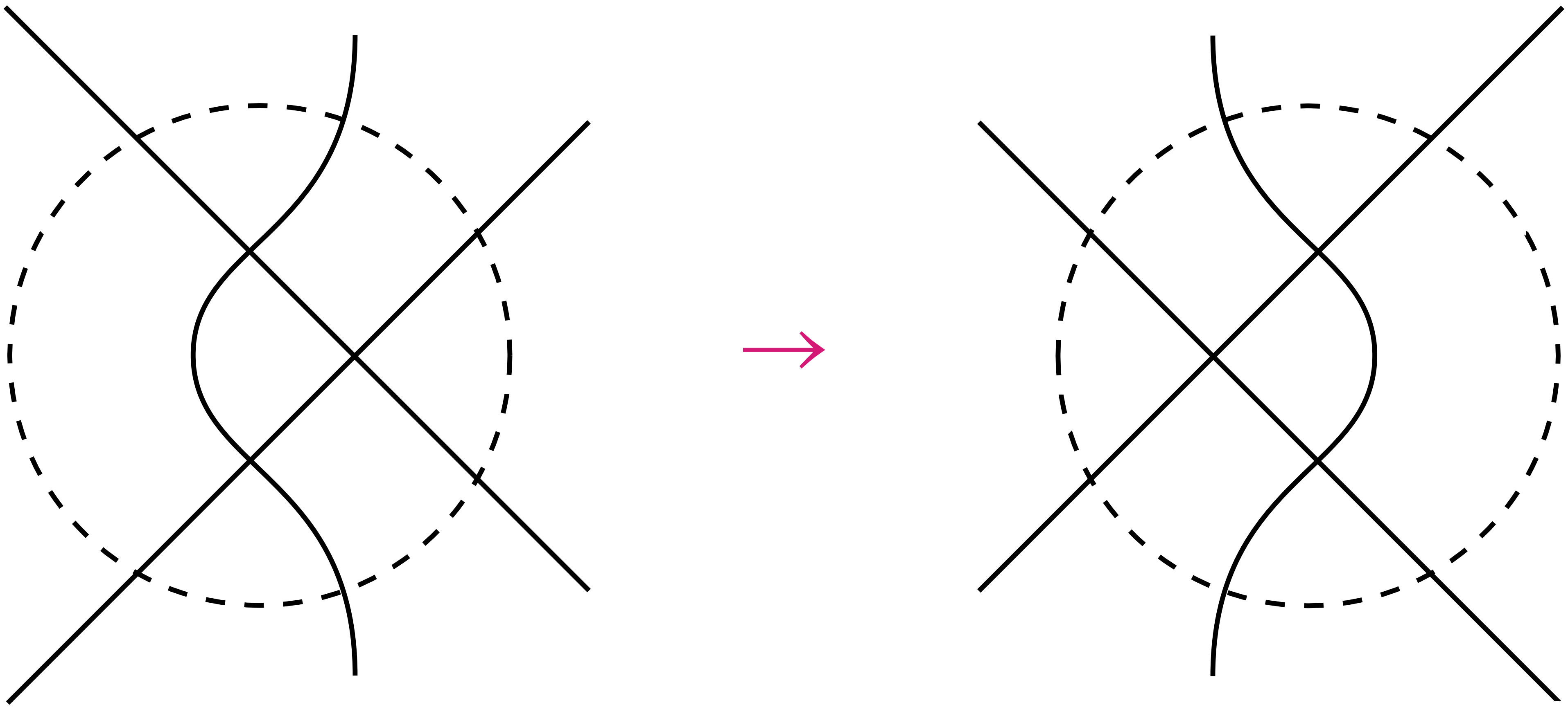


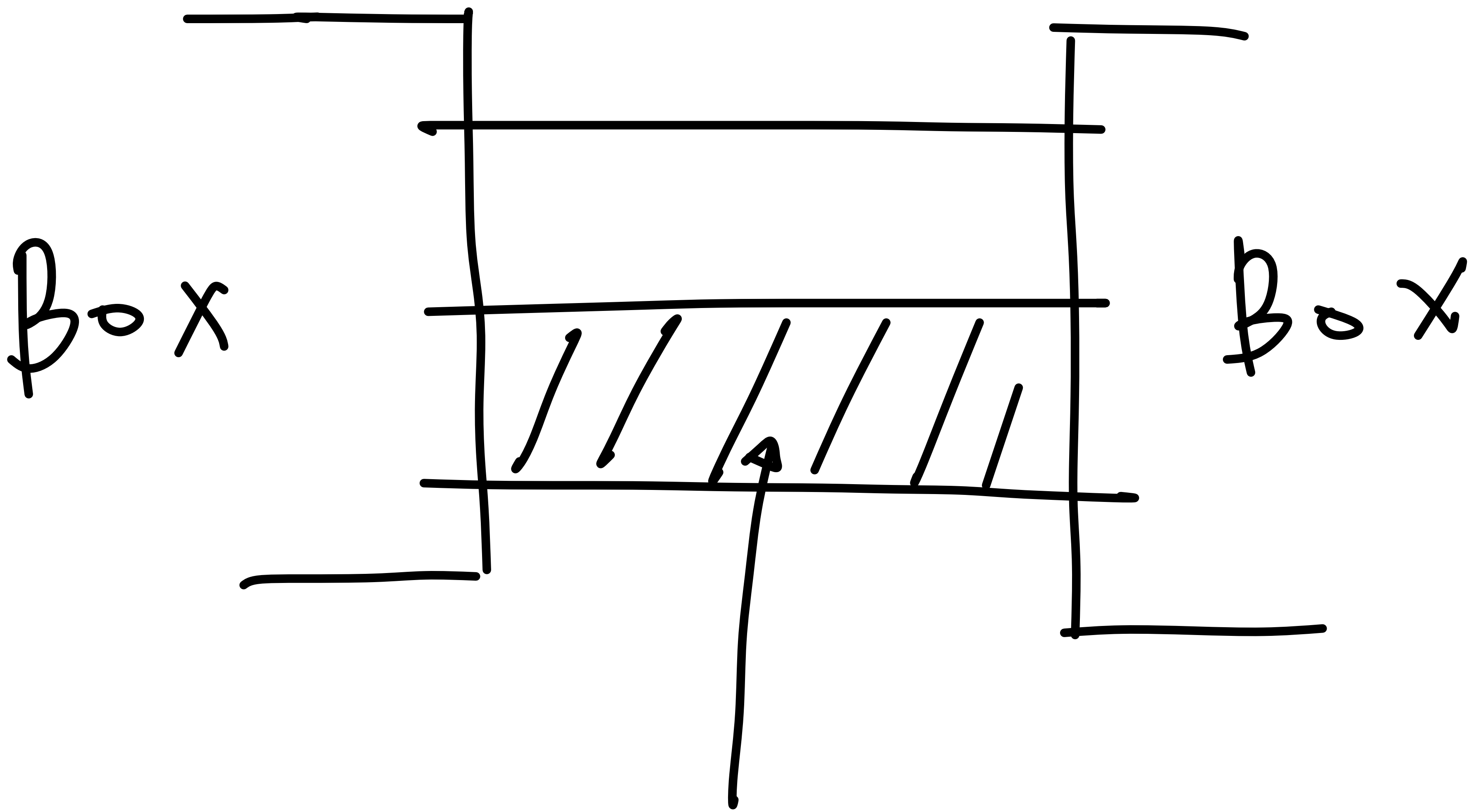
If there is a 1-gon not within Box,



which implies contradiction.

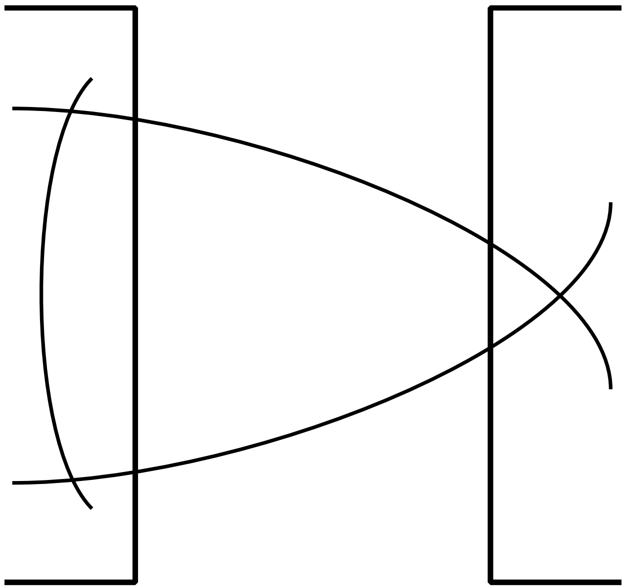
RIII not within a box



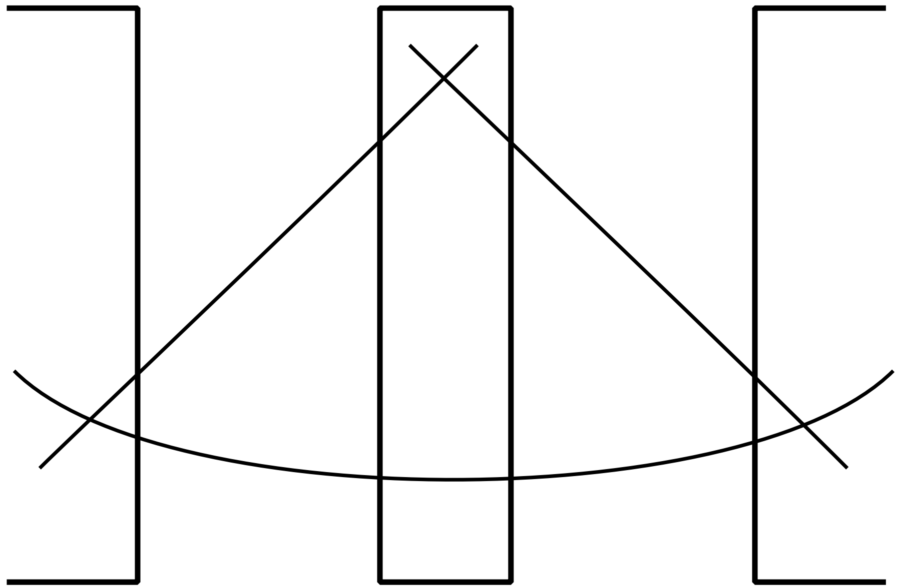


Part of 3-gon

Possible two cases:



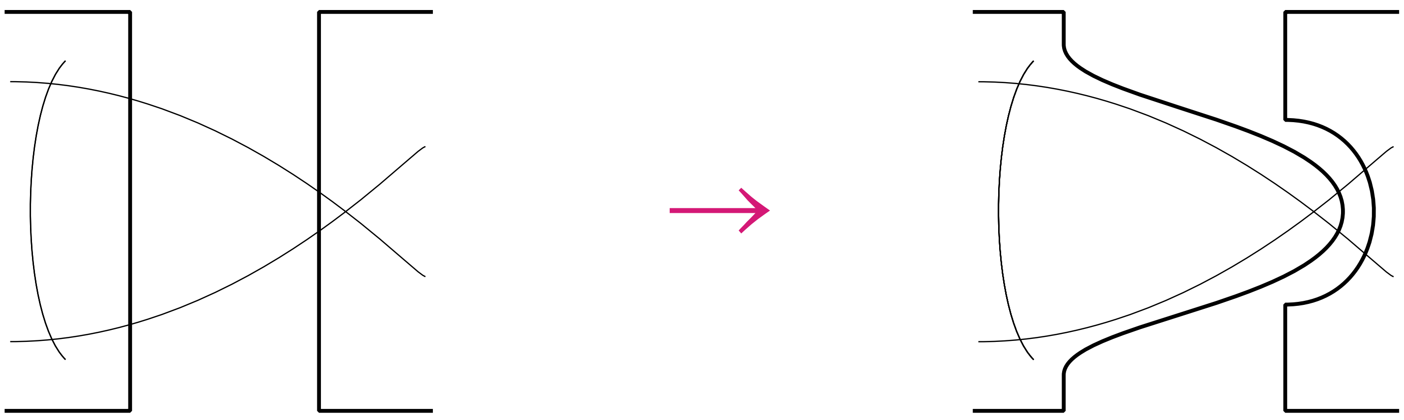
case1



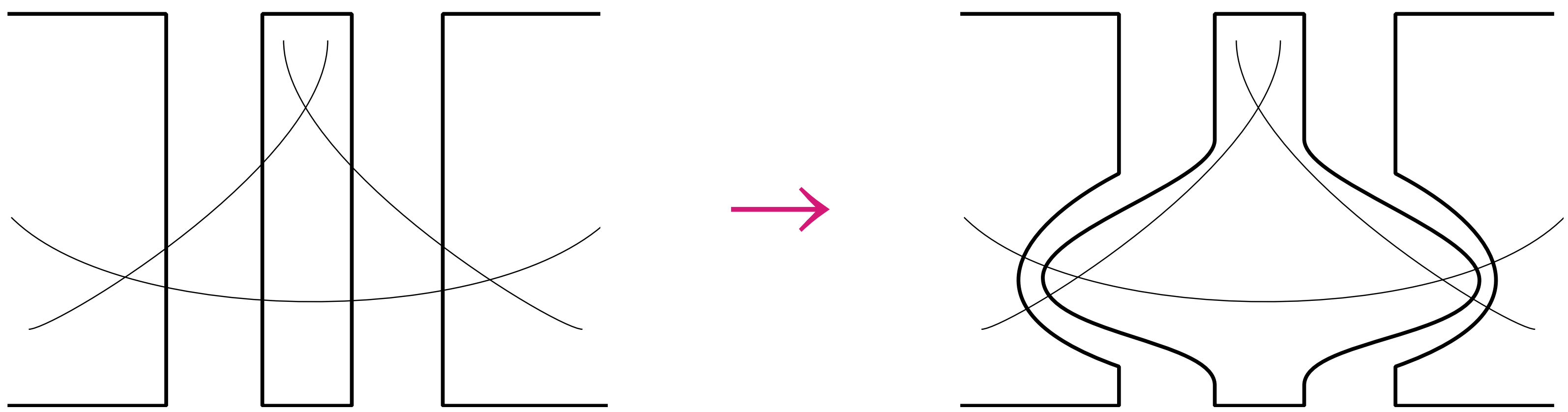
case2

Retaking Box

case1



case2



Sketch of Proof: $\mathrm{RII}(P(m,n)) \geq 1.$

Induction of the number of RI and RIII. Assumption of induction implies:

1. RI and RIII within Box $\rightarrow$ Least Intersections hold.
2. Positive RI not within Box $\rightarrow$ Retaking Box.

   Negative RI not within Box $\rightarrow$ Non Existence.

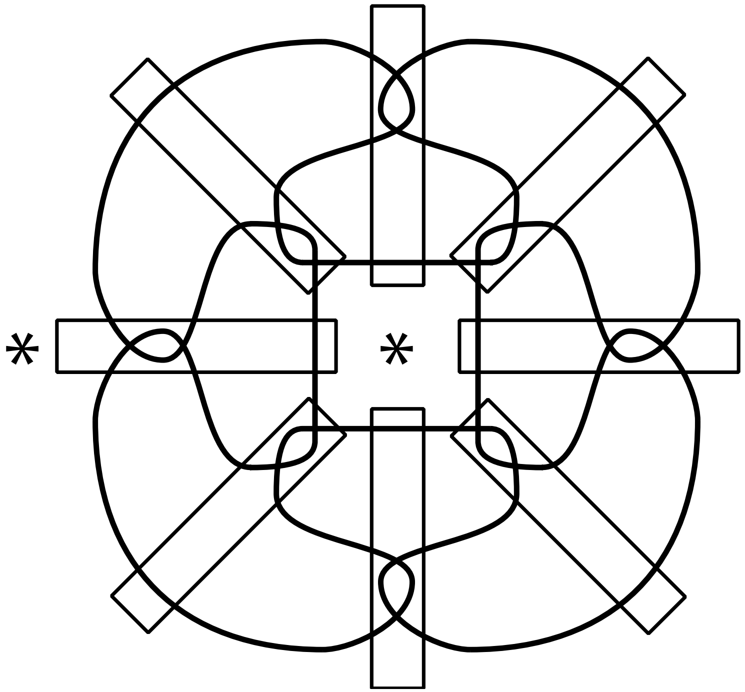
   RIII not within Box $\rightarrow$ Retaking Box.

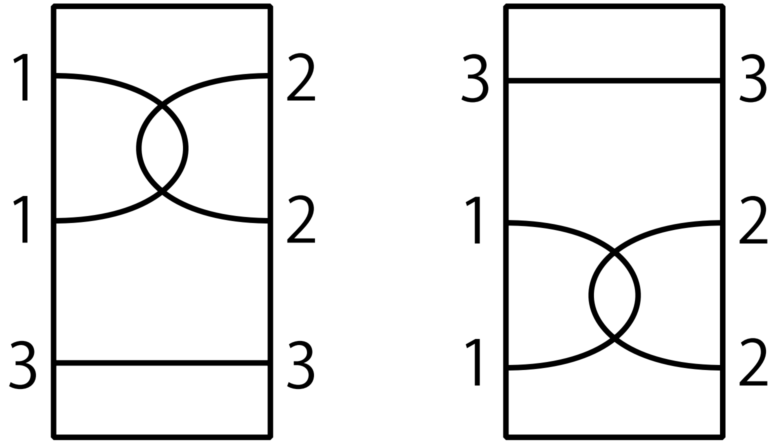
Generalize $\mathrm{RII}(P(m,n)) \geq 1$ to $\mathrm{RII}(P(m,n)) \geq m$.

Lemma. $P(m,n) = Q_0 \to Q_1 \to \cdots \to Q_r$ consists of a single negative RII, some RIs, RIIIs. If Q_k is negative RII, Q_i $(0 \leq i \leq k)$ preserves (m,n) box property and Q_i $(k+1 \leq i \leq r)$ preserves $(m-1,\, n)$ box property.

Box Property

The number of intersections between "string 1" and "string 2" are unchanged under RI and RIII (completely) including Box.



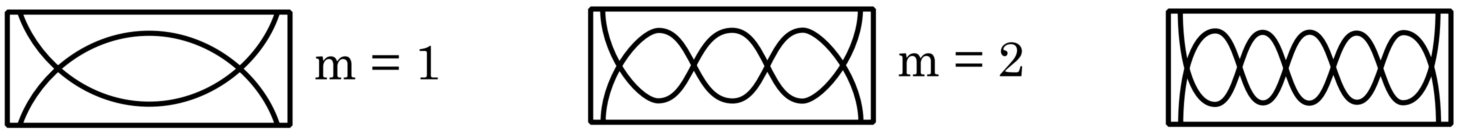


Polygon "*" has at least four sides.

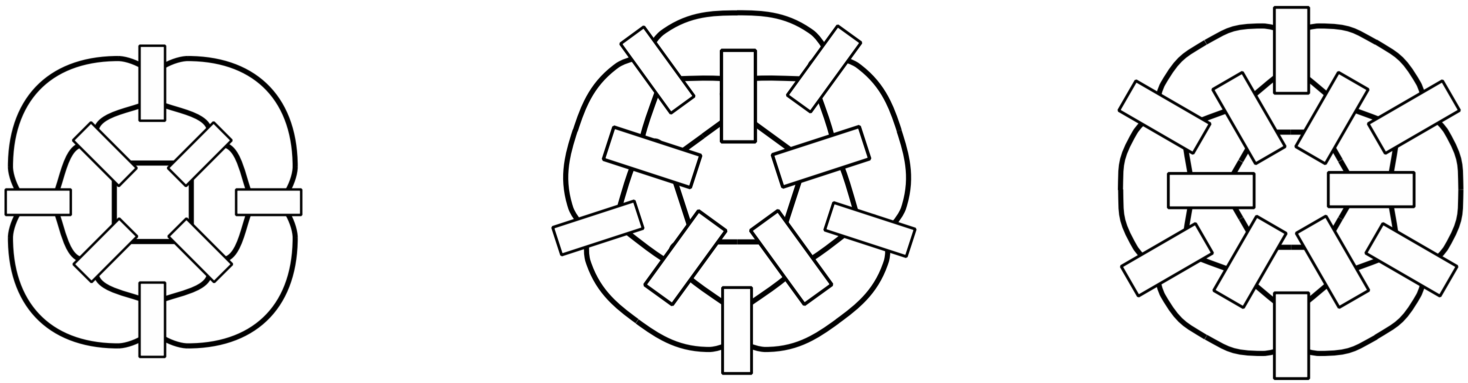
Theorem (Takimura-I. Kobe J. M. in Press now, arXiv:2010.10793)

$\mathrm{RII}(P(m,n)) = m$ **for the following:**

$\mathrm{b_m}$



m = 1   m = 2   m = 3 $\cdots$



$\cdots$

$P(m,4)$ $P(m,5)$ $P(m,6)$

Generalize $\mathrm{RII}(P(m,n)) \geq 1$ to $\mathrm{RII}(P(m,n)) \geq m$.

Lemma. $P(m,n) = Q_0 \to Q_1 \to \cdots \to Q_r$

consists of a single negative RII, some RIs, RIIIs.

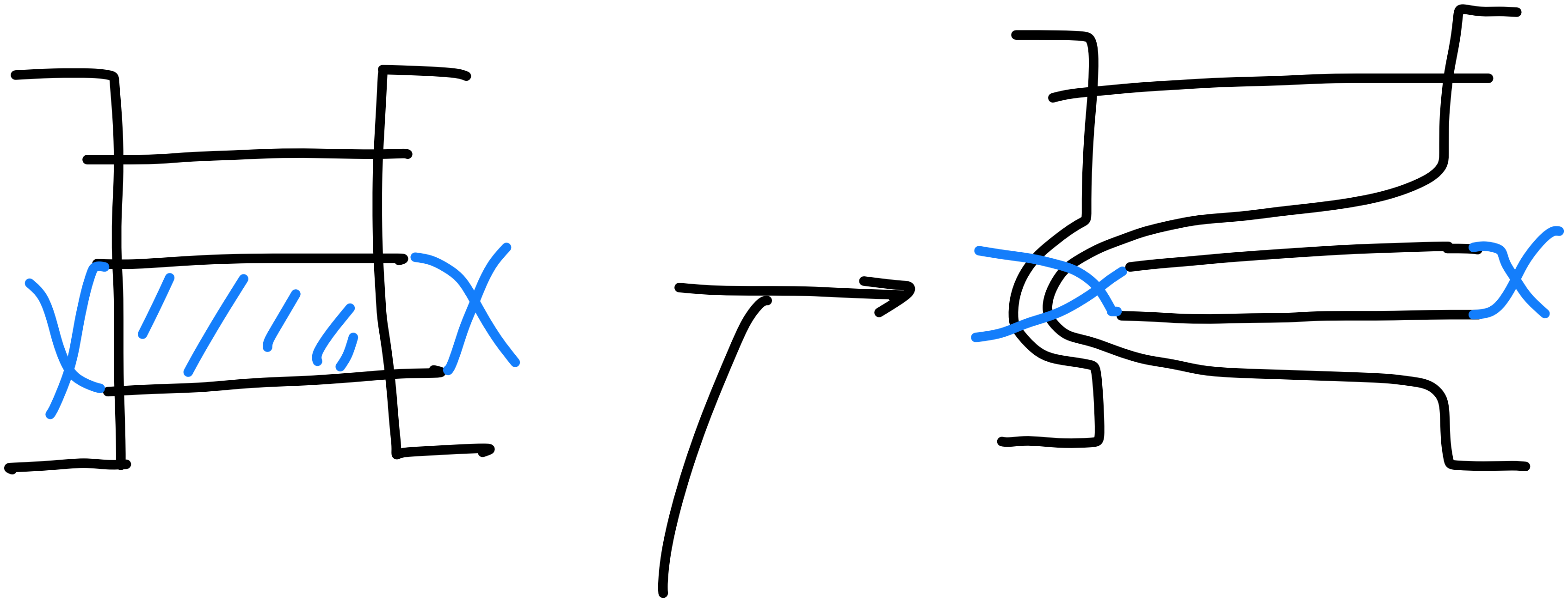
If Q_k is negative RII, Q_i $(0 \leq i \leq k)$ preserves

(m,n) box property and Q_i $(k+1 \leq i \leq r)$

preserves $(m-1,\ n)$ box property.

$\rightarrow$ Case RII is within a box

Case RII is not within a box



retaking a box

□

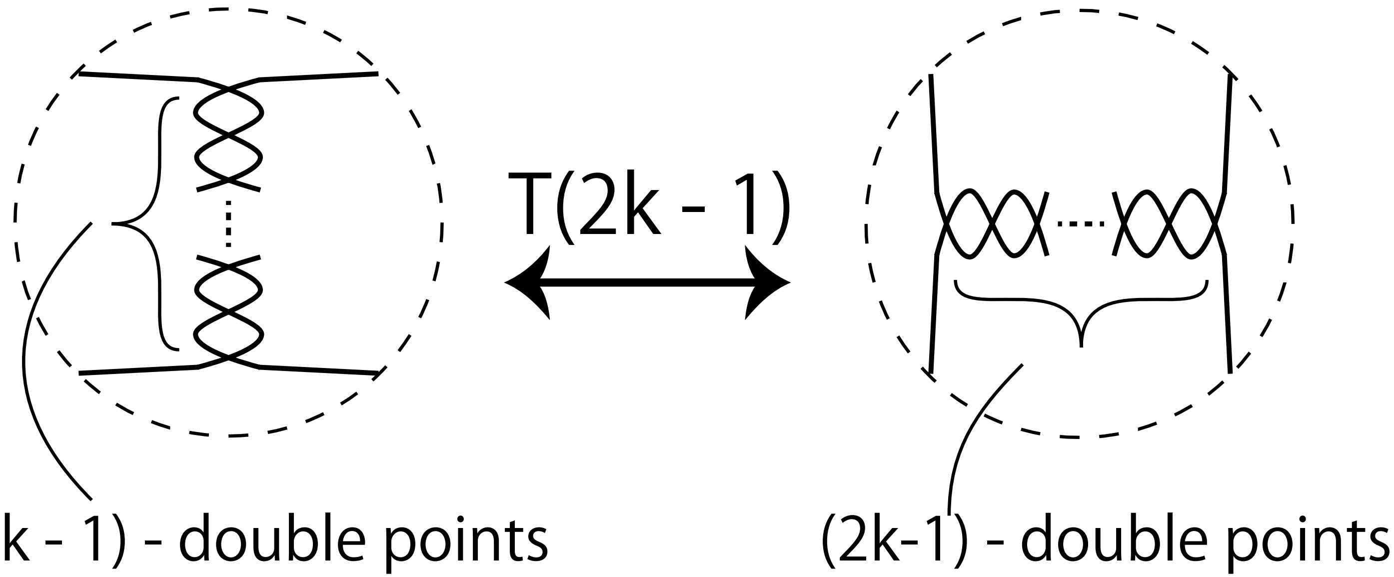
$$\mathrm{RII}(P(m,n)) \leq m\,.$$

We can find a concrete path by at most m negative RII-moves.

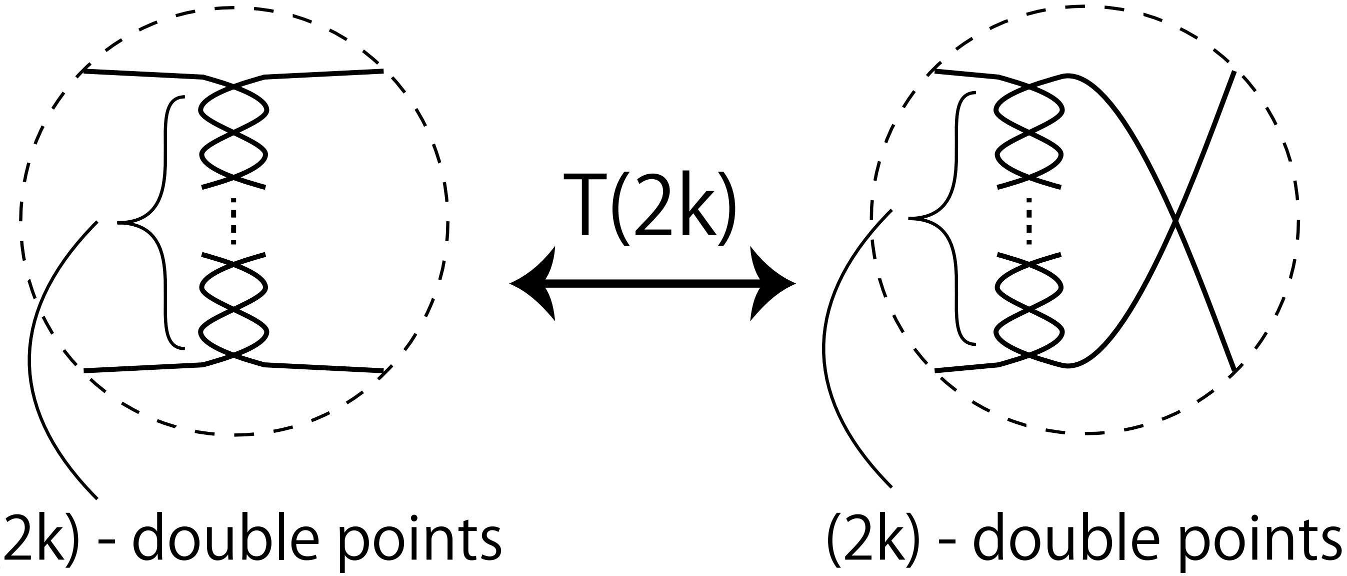
Moves T(2k-1) and T(2k) by RI and RIII

Lemma

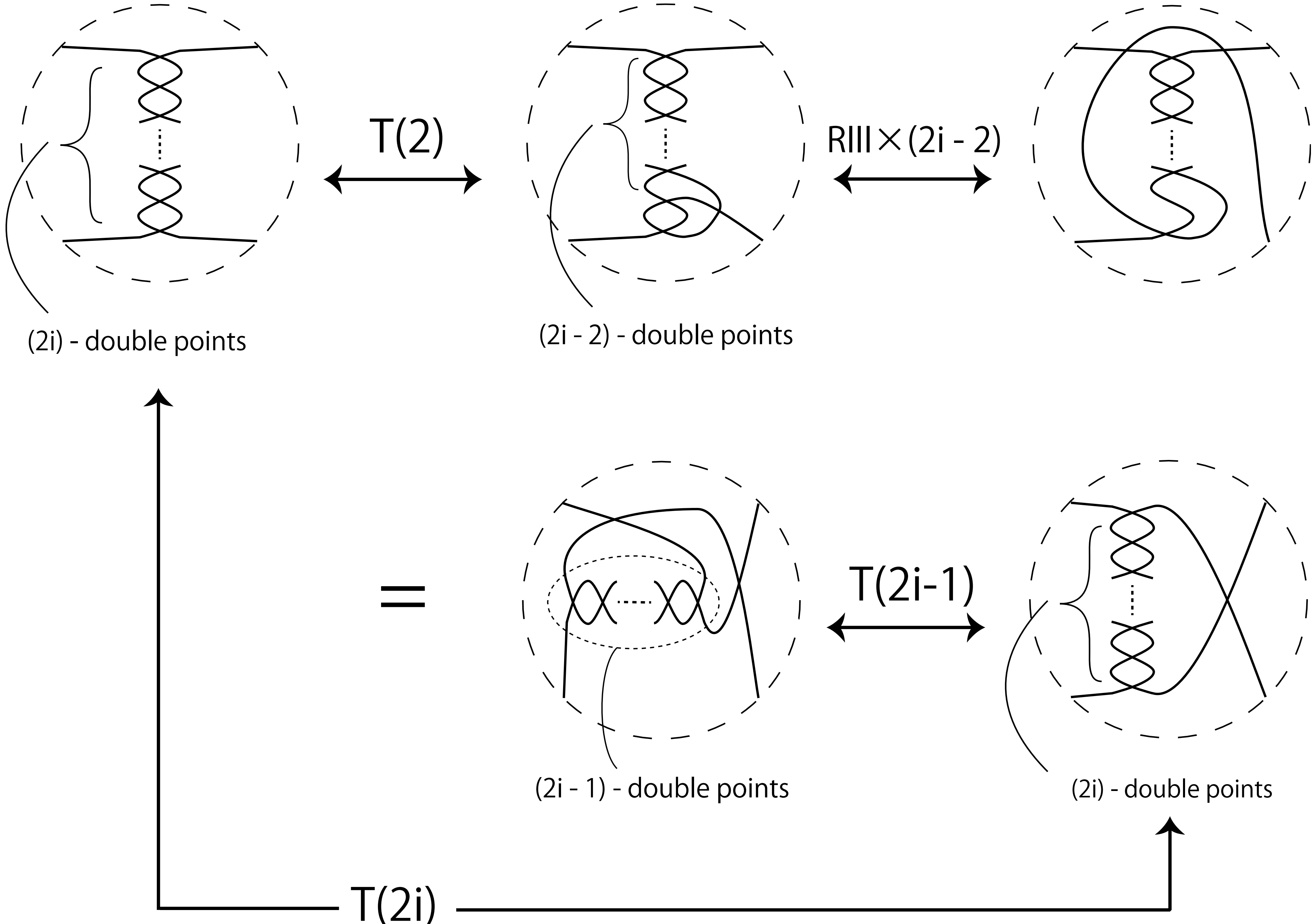
(odd)



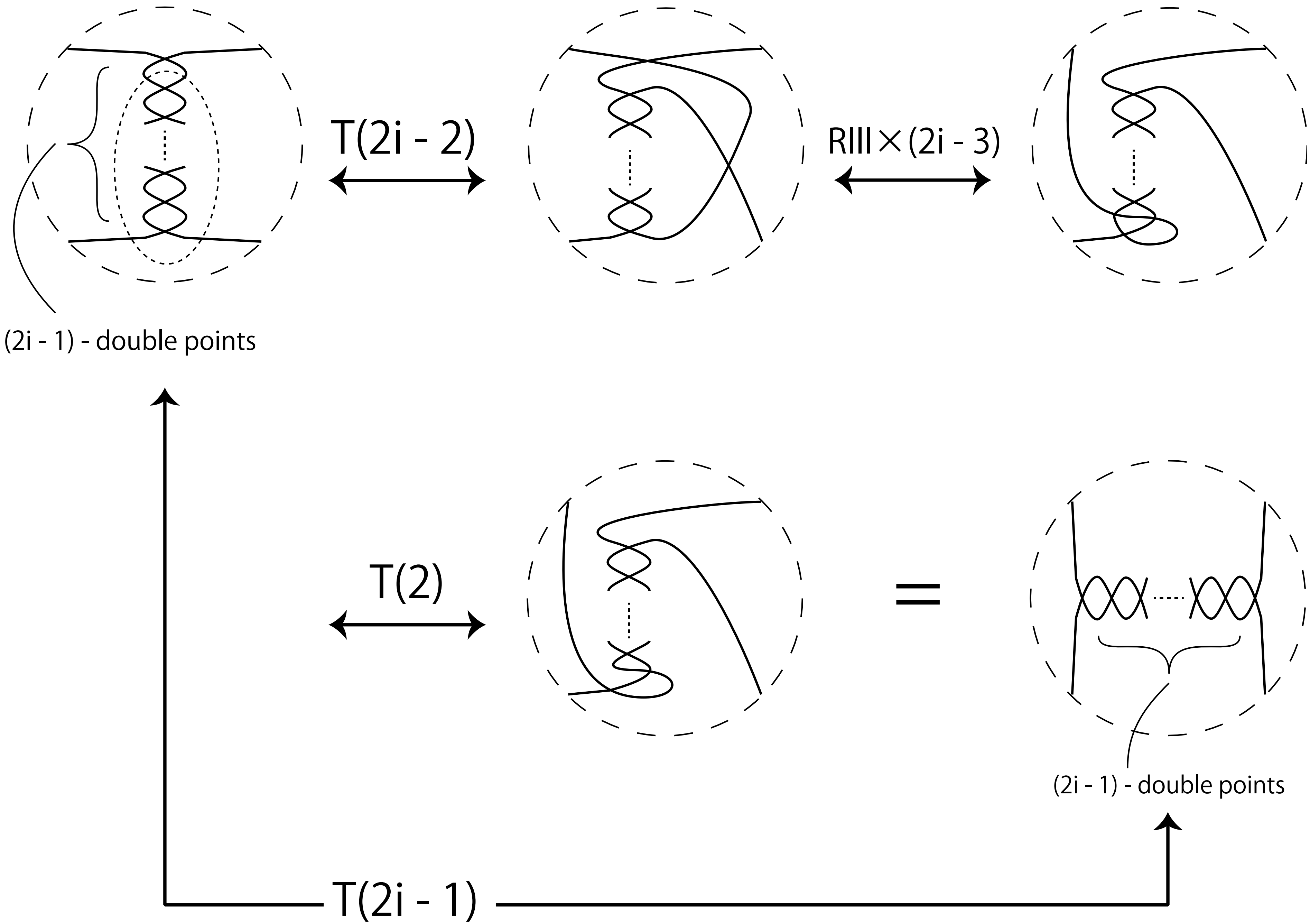
(even)



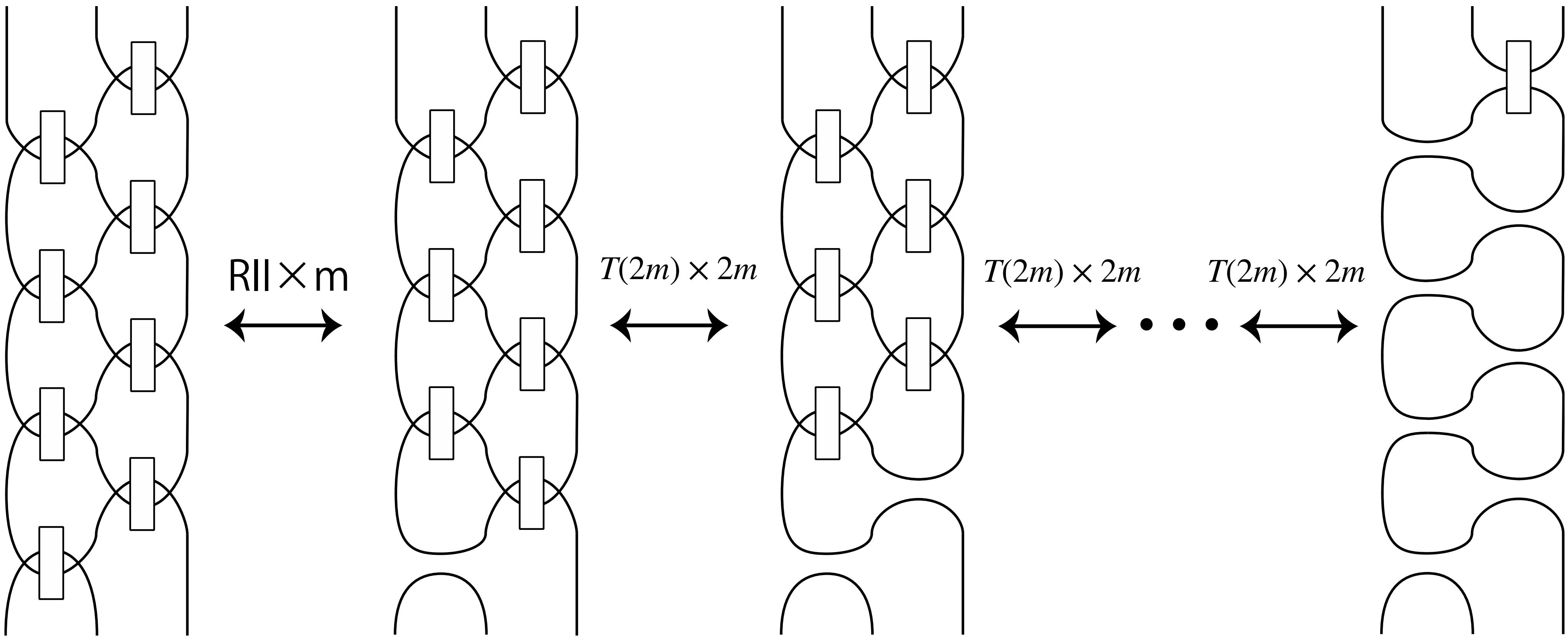
T(2i) from T(2i-1)

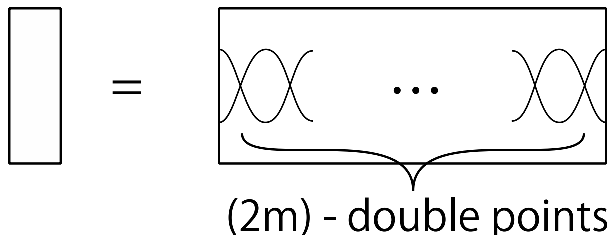


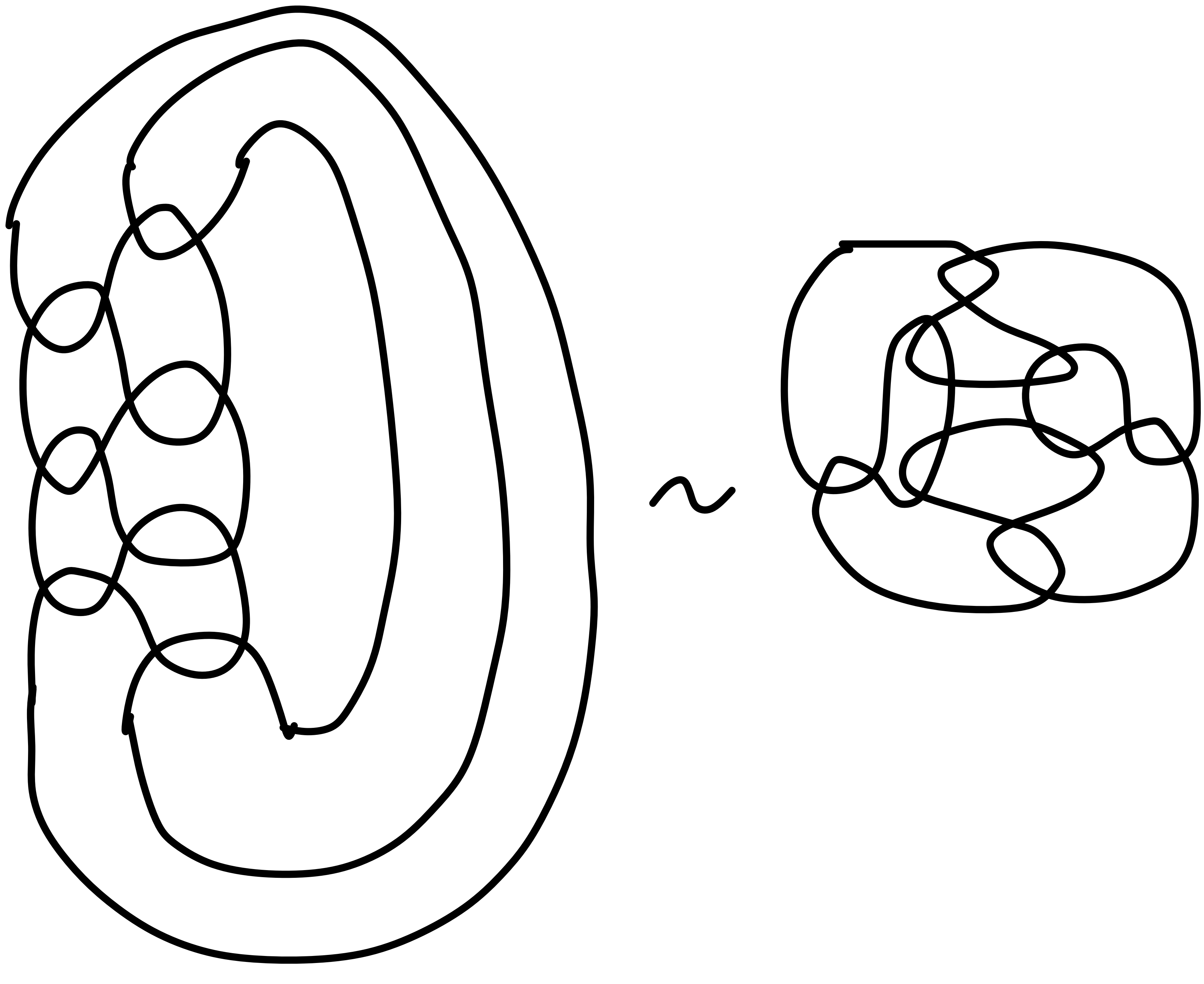
T(2i-1) from T(2i-2)



Tangle presentation



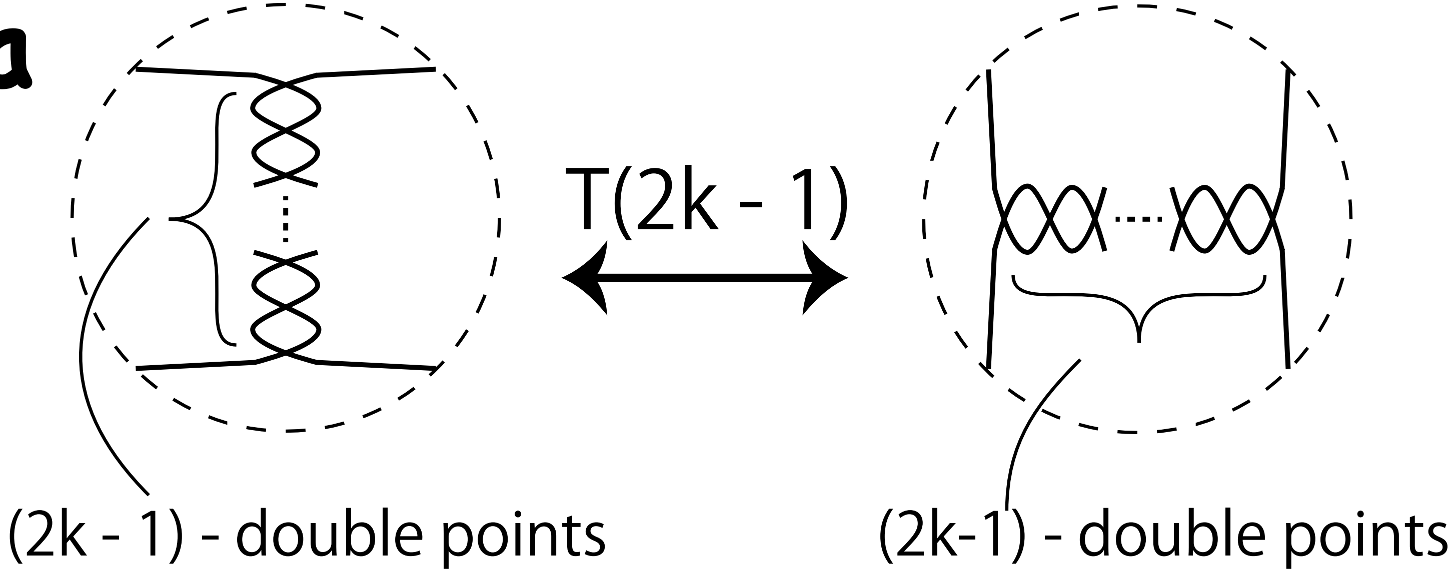




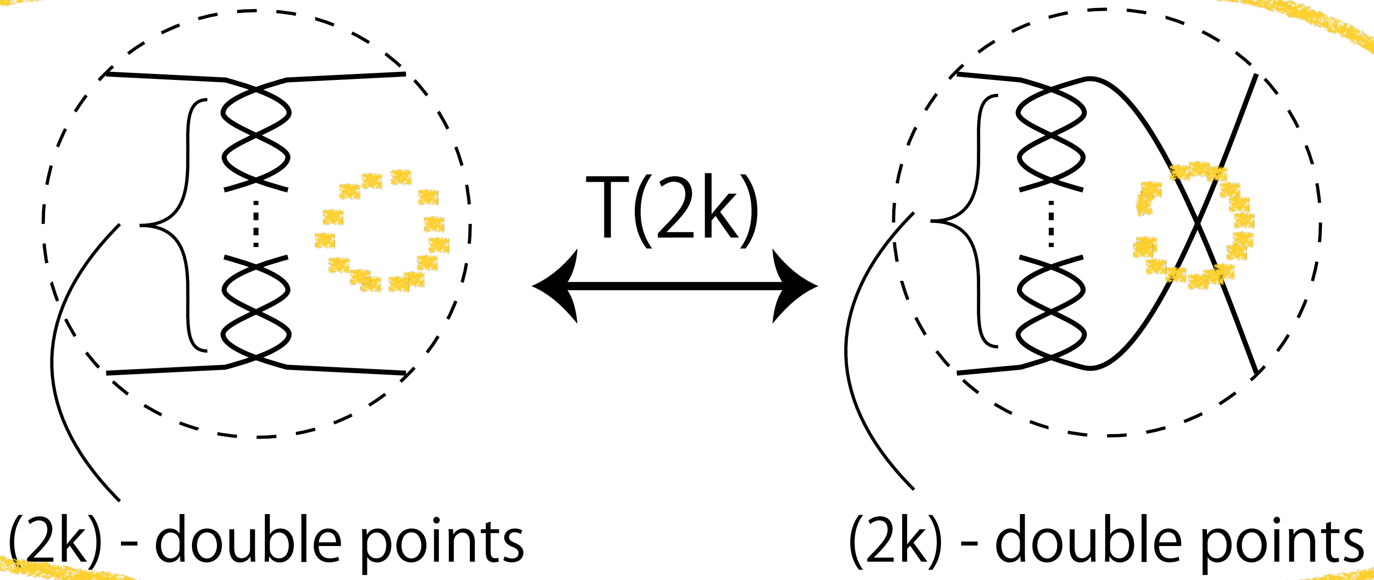
Moves T(2k-1) and T(2k) by RI and RIII

Lemma

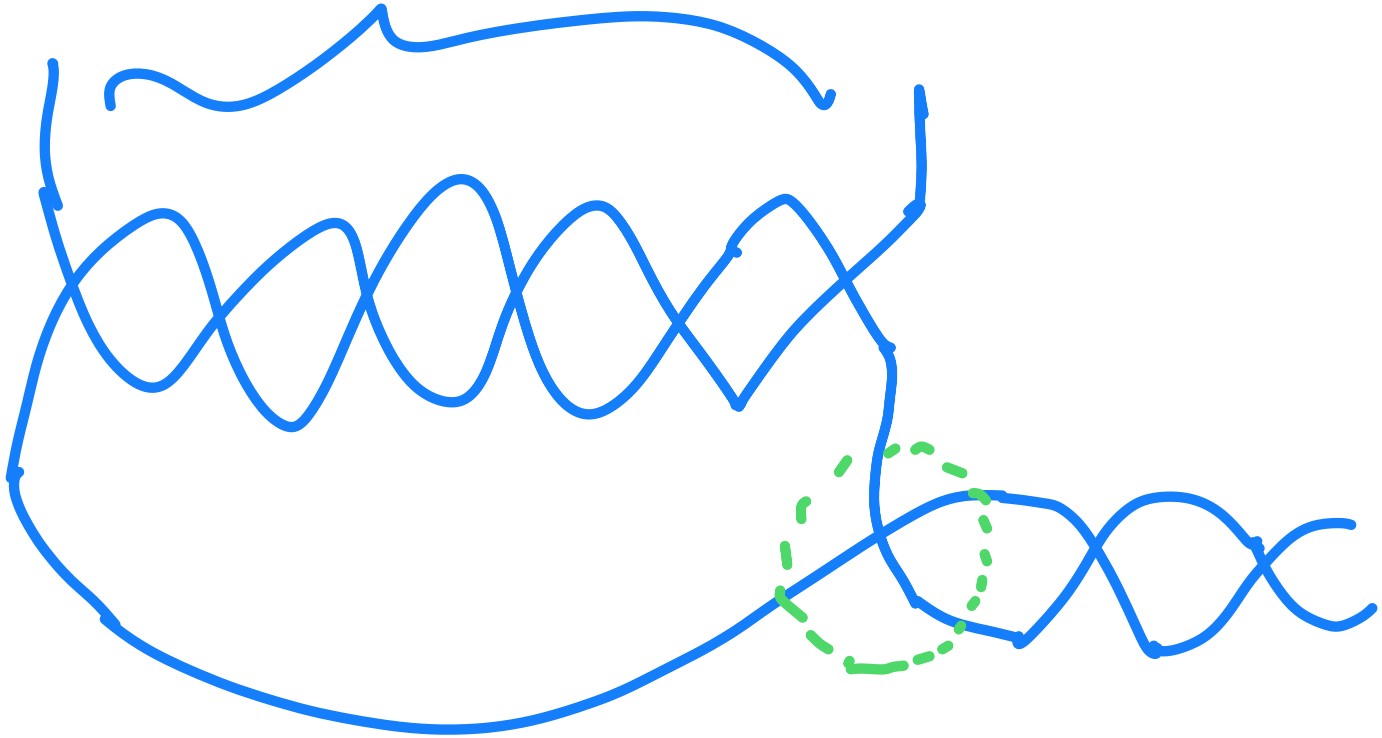
(odd)



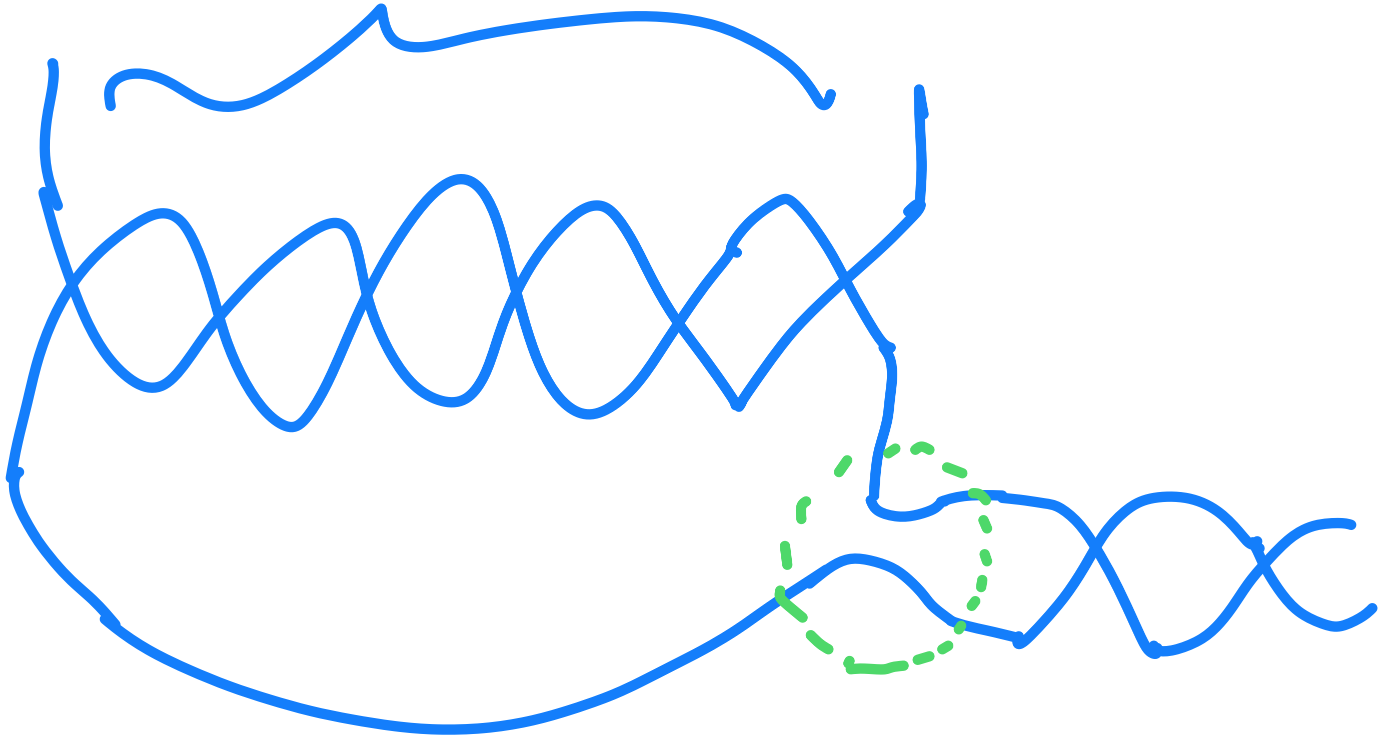
(even)



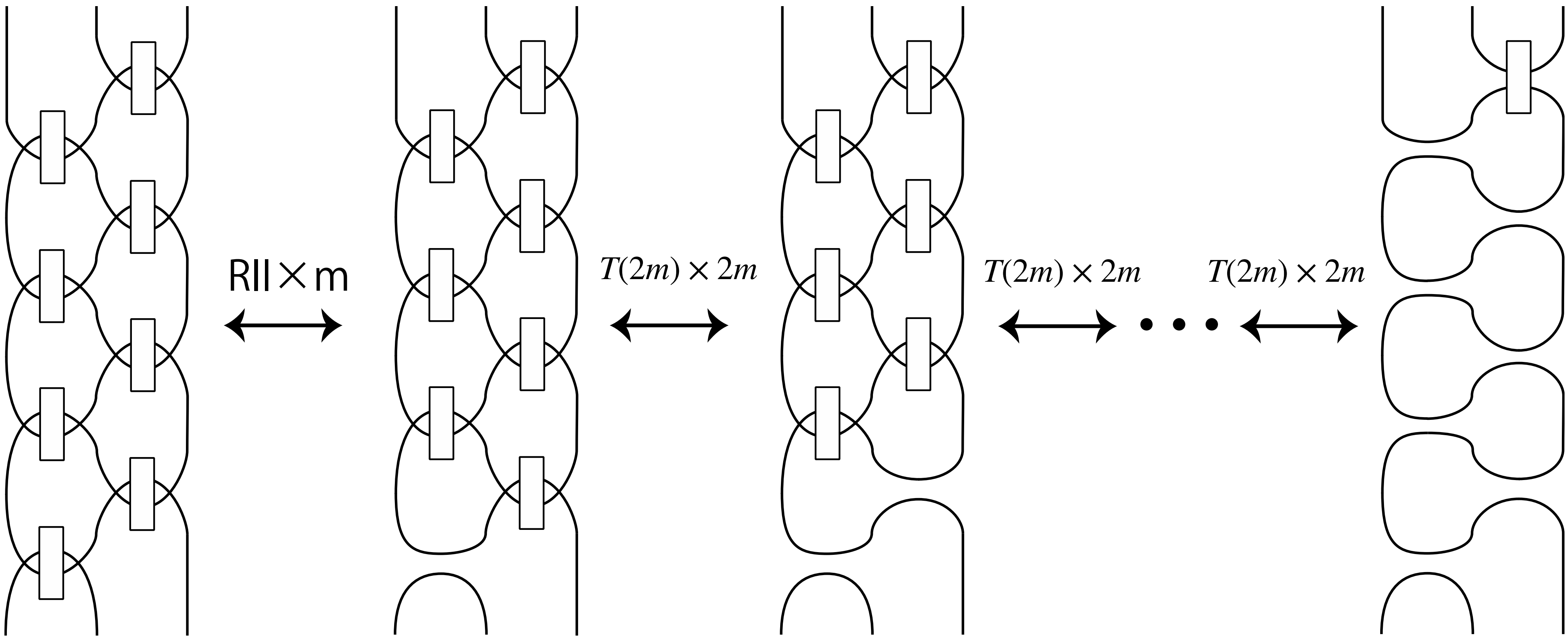
even crossings

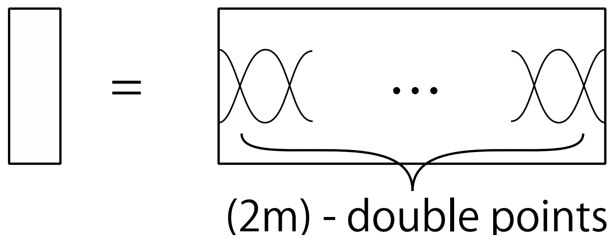


even crossings

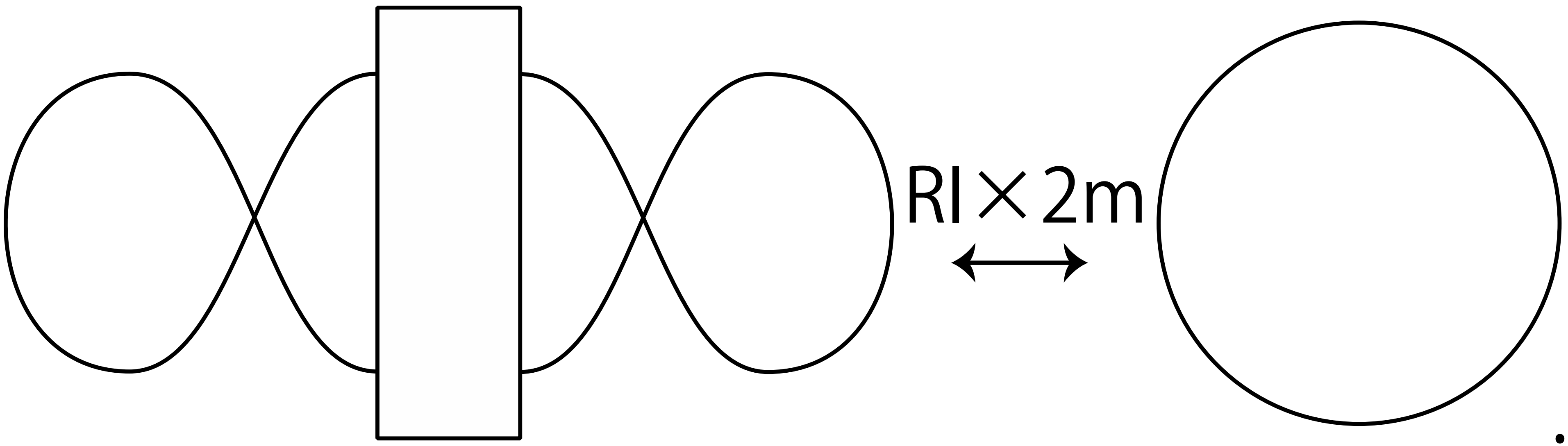


Tangle presentation





Finally,

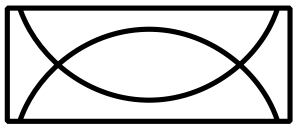


Thus, $\mathrm{RII}(P(m,n)) \leq m$. $\square$

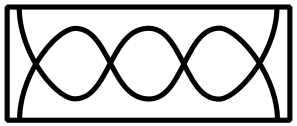
Theorem (Takimura-I.) $\mathrm{RII}(P(m,n)) = m$

for

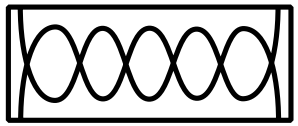
b_m



m = 1

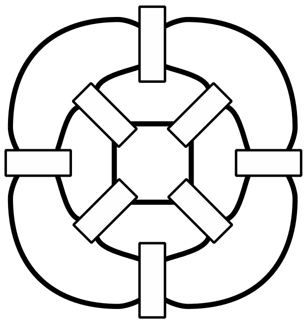


m = 2

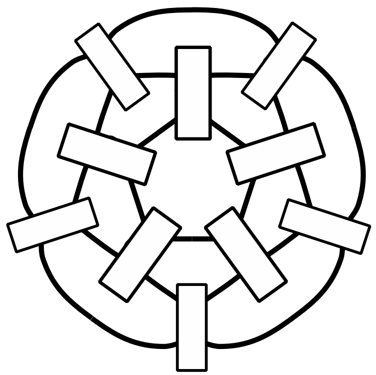


m = 3

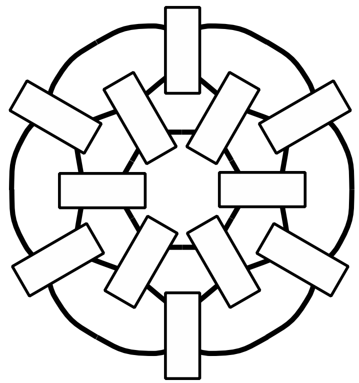
$\cdots$



$P(m,4)$



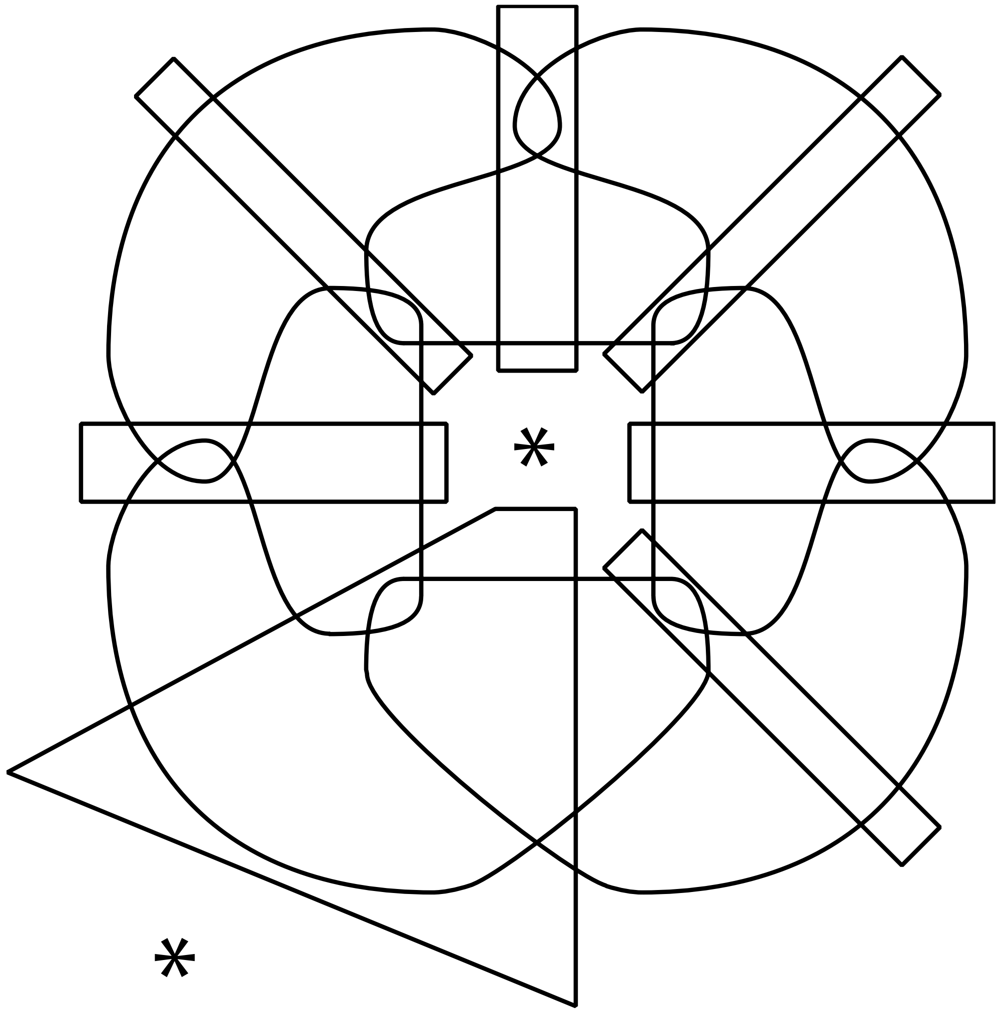
$P(m,5)$



$P(m,6)$

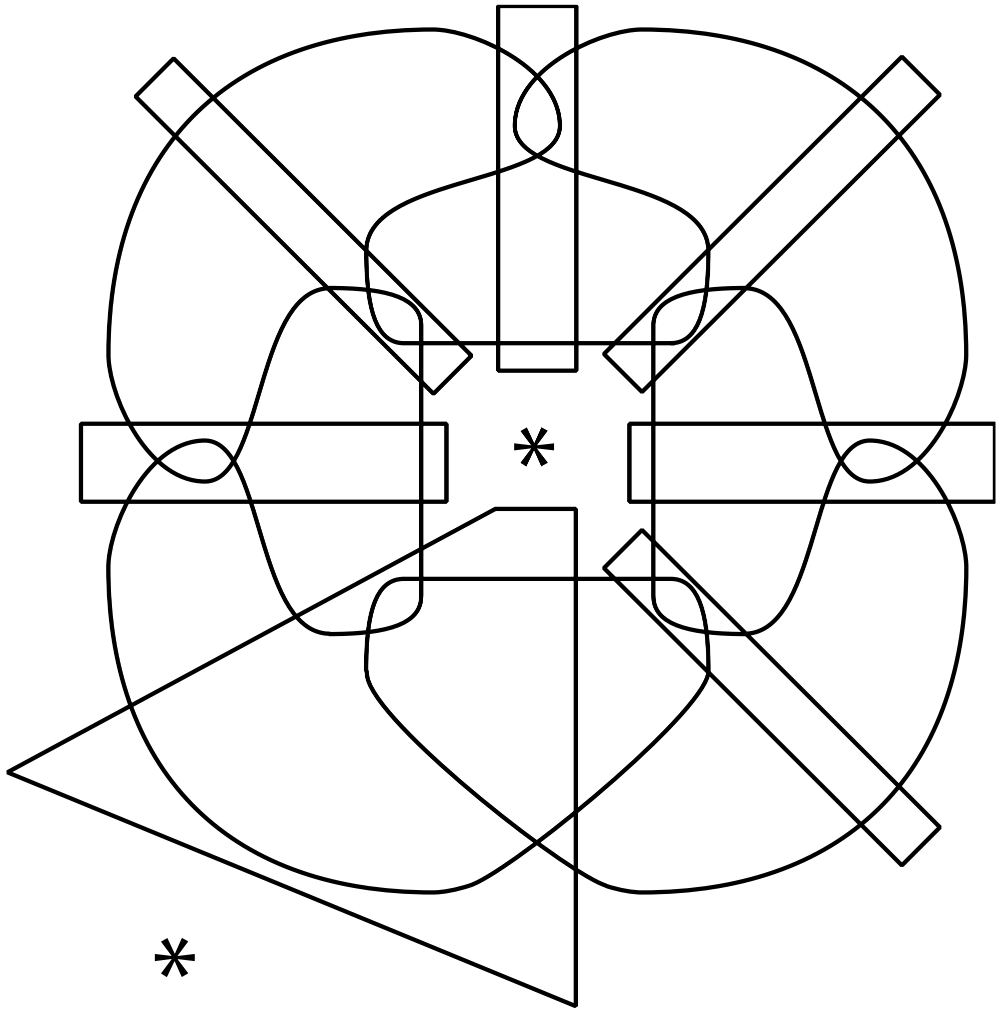
$\cdots$

We found example of 15 crossings.



What is an n-crossing

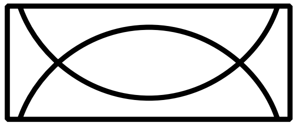
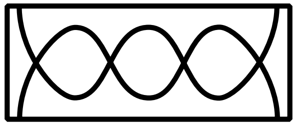
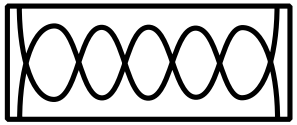
example with n less than 15?

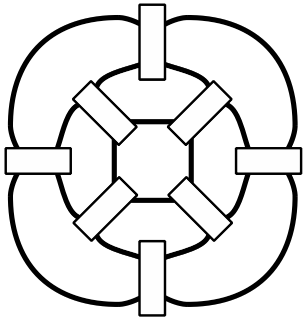


Thank you for your attention!

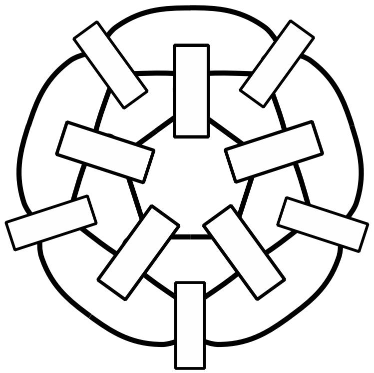
Theorem (Takimura-I.) $\mathrm{RII}(P(m,n)) = m$

for

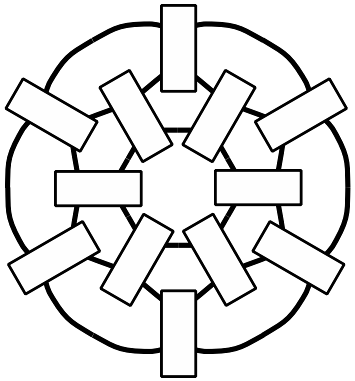
b_m  m = 1  m = 2  m = 3 $\cdots$



$P(m,4)$



$P(m,5)$



$P(m,6)$

$\cdots$