On a question of Östlund —
Diagrammatic Vassiliev invariant and
RII number

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A numerical knot diagram invariant extends to knots in the codimension one stratum by the formula:

\[ v(K) = v(K_+) - v(K_-) \]

where \( K_+ \), \( K_- \) are created by resolving projection-degenerate knot (= non projection-generic knot) in positive respectively negative direction.
Revisiting Vassiliev invariant

The invariant $v$ is said to be of finite degree if there is a number $n$ such that $v(K) = 0$ whenever $K$ has more than $n$ 1-degenerate germs. The smallest such $n$ is the degree of $v$. 
Def (projection-degenerate germs) [Östlund PhD thesis, 2001]

\(\Omega_1\): standard cusp; the first derivative of a projection of a knot vanishes, while 2\(^{nd}\) and 3\(^{rd}\) derivatives are linearly independent,

\(\Omega_2\): double point with first order self tangency

\(\Omega_3\): triple point with pairwise transversal crossings
Östlund wrote: “the concept of degree can be refined by considering the different types”

A knot diagram invariant of finite degree \( n \) in \( \Omega_i \) (\( i = 1,2,3 \)) if it takes value zero on any projection-degenerate knot with more than \( n \) 1-degenerate germs of \( \Omega_i \).
Theorem (Östlund PhD thesis, 2001)

Let $\nu$ be a knot diagram invariant that is unchanged under $\Omega_1$- and $\Omega_3$-moves, and of finite degree in $\Omega_2$. Then $\nu$ is a knot invariant.
Östlund’s Theorem (Theorem 6 in thesis)

Let \( \nu \) be a knot diagram invariant that is unchanged under \( \Omega_1 \)- and \( \Omega_3 \)-moves, and of finite degree in \( \Omega_2 \). Then \( \nu \) is a knot invariant.

In the view of finite degree invariants, \( \Omega_2 \)-move is superfluous.
In the view of finite degree invariants, $\Omega_2$-move is superfluous. Östlund wrote:

If a non-trivial invariant that jumps only under $\Omega_2$-moves is found, ... ... it disproves the knot diagram counterpart of Vassiliev’s conjecture for finite degree invariants of knots.
Östlund’s Question

Östlund wrote:

Whether all self-tangency moves of plane curves can be replaced by cusp- and triple point moves is a different question: The author knows of no potential counterexample to this statement.
Östlund further wrote:
Can knot diagram invariants based on state sum models distinguish path-components of \( K \setminus \{ \text{knots with projection self-tangency} \} \)?

(So far, all such “quantum” invariants of knots and plane curves has been showed to be expressible in invariants of finite degree.)
Let's search for non-trivial knot projection under RI and RIII!
w \text{RI} \downarrow \text{wR} \uparrow \\
\text{w: RI} \downarrow \text{wR} \uparrow \\

s \text{RI} \downarrow \text{sR} \uparrow \\
\text{s: RI} \downarrow \text{sR} \uparrow
By applying $T(2)$, we have ($\ast$):
Non-trivial curve given by Hagge-Yazinski (arXiv: 0812.1241)
There exists a knot diagram of the unknot that needs RII to be transformed into the trivial as follows:
\( P_{HY} \) can be generalized:

\[ P(1, m, 4) \quad b_m \quad m = 1 \]
$P_{HY}$ can be generalized:

$P(1, m, 4)$

$b_m$

$m = 2$
$P_{HY}$ can be generalized:

$P(1, m, 4)$  

$b_m$  

$m = 3$
$P_{HY}$ can be generalized:

$b_m$

$m = 1$

$m = 2$

$m = 3$

$P(1, m, 4)$

$P(1, m, 5)$

$P(1, m, 6)$

$\ldots$

$\ldots$
$P_{HY}$ can be generalized:

$b_m$

$m = 1$

$m = 2$

$m = 3$

$\ldots$

$P(1, m, 4)$

$P(2, m, 4)$

$P(3, m, 4)$

$\ldots$
$P_{HY}$ can be generalized:
Invariant of knot projections under RI and RIII.

Def (Takimura-I. arXiv:2010.10793). The RII number is the minimum number of deformations of negative RII in sequences to obtain the standard embedding of the circle from a knot projection.

$$\text{RII}(P(m, n)) = m$$ for the following:

$$b_m$$

$m = 1$

$m = 2$

$m = 3$

$$P(m,4)$$

$$P(m,5)$$

$$P(m,6)$$
RII($P(m, n)$) $\geq 1$.

The number of intersections between “string 1” and “string 2” are unchanged under RI and RIII (completely) including Box.

Polygon “*” has at least four sides.
The number of intersections between “string 1” and “string 2” are unchanged under RI and RIII (completely) including Box.
For positive RI not within Box,

retaking a box
For negative RI not within Box,
String 1 or 2 has at least two crossings.

String 3 connects string 1 or 2.
If there is a 1-gon not within Box, which implies contradiction.
RIII not within a box
Part of 3-gon
Possible two cases:

case 1

case 2
Retaking Box

case 1

\[ \text{Diagram showing case 1 transformation} \]

\[ \rightarrow \]

\[ \text{Diagram showing case 2 transformation} \]

case 2
Sketch of Proof: $\text{RII}(P(m, n)) \geq 1$.

Induction of the number of RI and RIII. Assumption of induction implies:

1. RI and RIII within Box $\rightarrow$ Least Intersections hold.

2. Positive RI not within Box $\rightarrow$ Retaking Box.

   Negative RI not within Box $\rightarrow$ Non Existence.

RIII not within Box $\rightarrow$ Retaking Box.
Generalize $\text{RII}(P(m, n)) \geq 1$ to $\text{RII}(P(m, n)) \geq m$.

Lemma. $P(m, n) = Q_0 \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_r$

consists of a single negative RII, some RIs, RIIIs.
If $Q_k$ is negative RII, $Q_i$ ($0 \leq i \leq k$) preserves $(m, n)$ box property and $Q_i$ ($k + 1 \leq i \leq r$)

preserves $(m - 1, n)$ box property.
Box Property

The number of intersections between "string 1" and "string 2" are unchanged under RI and RIII (completely) including Box.

Polygon "*" has at least four sides.

\[ \text{RII}(P(m, n)) = m \text{ for the following:} \]

\[ b_m \quad P(m,4) \quad P(m,5) \quad P(m,6) \]
Generalize $\text{RII}(P(m, n)) \geq 1$ to $\text{RII}(P(m, n)) \geq m$.

Lemma. $P(m, n) = Q_0 \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_r$

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Case RII is within a box

Case RII is not within a box
retaking a box
RII\( (P(m,n)) \leq m \).

We can find a concrete path by at most \( m \) negative RII-moves.
Moves $T(2k-1)$ and $T(2k)$ by RI and RIII

Lemma

(odd)

T(2k - 1)

(2k - 1) - double points

(even)

T(2k)

(2k) - double points
$T(2i)$ from $T(2i-1)$

(2i) - double points

$T(2)$

(2i - 2) - double points

$R_{III} \times (2i - 2)$

(2i - 1) - double points

$T(2i-1)$

(2i) - double points

$T(2i)$
$T(2i-1)$ from $T(2i-2)$

$T(2i-2) \leftrightarrow \text{(2i-1) - double points} \leftrightarrow \text{RIII} \times (2i-3) \leftrightarrow T(2i-1)$

$T(2) \leftrightarrow \text{RIII} \times (2i-3) = \text{(2i-1) - double points}$
Tangle presentation

Rll $\times$ m $\leftrightarrow T(2m) \times 2m$ $\leftrightarrow T(2m) \times 2m$ $\leftrightarrow \cdots$ $\leftrightarrow T(2m) \times 2m$

\[\begin{array}{c}
\text{=}
\end{array}\]

(2m) - double points
Moves $T(2k-1)$ and $T(2k)$ by RI and RIII

Lemma

(odd)

(2k - 1) - double points

T(2k - 1)

T(2k) - double points

(even)

(2k) - double points

(2k) - double points
even crossings
even crossings
Tangle presentation

\[ T(2m) \times 2m \]

\[ R II \times m \]

(2m) - double points
Finally,

Thus, $\text{RII}(P(m, n)) \leq m$. □
Theorem (Takimura-I.) $\text{RII}(P(m, n)) = m$

for

$b_m$

$m = 1$

$m = 2$

$m = 3$

$\ldots$

$P(m, 4)$

$P(m, 5)$

$P(m, 6)$

$\ldots$
We found example of 15 crossings.
What is an $n$-crossing example with $n$ less than 15?
Theorem (Takimura-I.) \[ \text{RII}(P(m, n)) = m \]

for

\[ b_m \]

\[ m = 1 \]

\[ m = 2 \]

\[ m = 3 \]

\[ \ldots \]

\[ P(m,4) \]

\[ P(m,5) \]

\[ P(m,6) \]

\[ \ldots \]