

Goussarov-Polyak-Viro 予想 ($n = 3$) について

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1. Introduction

有限型不変量をガウス図式で組織的に表す方法を記述した Goussarov-Polyak-Viro の論文 ([1]) で次の主張が予想 (Conjecture 3.C) されている。

Conjecture 1 ([1], Conjecture 3.C) Every finite-type invariant of classical knots can be extended to a finite-type invariant of long virtual knots.

この論文では, $n = 2$ については base point に関する議論により, 正しいことが確かめられている。しかしながら, n が 3 以上については, どのように解決するかの方策について明示がなされていない。予想の意味について Viro 氏に直接口頭で確認したところ次の問題を提示された。

Problem 1 ([6]) (1) Clarify a relationship between Gauss diagram formulas as long virtual knot invariants and Gauss diagram formula as classical knot invariants (e.g., Polyak-Viro formula for the degree 3).

(2) How to merge Gauss diagram formulas as long virtual knot invariants into Gauss diagram formula as classical knot invariants (e.g., Polyak-Viro formula for the degree 3)?

(3) How to transform Gauss diagram formulas as classical knot invariants into Gauss diagram formulas as long virtual knot invariants?

予想の $n = 3$ に現れる, 上記の問題を肯定的に解いたので報告する。

Theorem 1 y_i^* ($1 \leq i \leq 168$) を Notation 1 のアロー図式とする。 v_i ($1 \leq i \leq 23$) を Proposition 2 の Gauss diagram formula とし, \tilde{v}_i ($1 \leq i \leq 9$) を Proposition 1 の Gauss diagram formula とする。 \mathbf{v} を v_i の係数からなる 23×168 行列とし, \mathbf{w} を \tilde{v}_i の係数からなる 9×168 行列とする。このとき, 9×23 行列 A が一意的に存在し, $A\mathbf{v} = \mathbf{w}$ となる。

さらに, Proposition 1 の系として, 次数 3 の Polyak-Viro formula [4] に関して次を得た。

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Corollary 1 Long virtual knot の Gauss diagram formula $v_{3,7}$ を $v_{3,7} := \langle f_{3,7}, \cdot \rangle$ により定める. ここで $f_{3,7} := \textcircled{+} + \textcircled{-} + \textcircled{+} + \textcircled{-} + \textcircled{+} + \textcircled{-} + 2\textcircled{+} + 2\textcircled{-}$ とする. このとき, $v_{3,7}(\cdot) = \langle \textcircled{+} + 2\textcircled{-}, \cdot \rangle$ が成り立つ.

2. 不変量の構成法

K を knot とし, D_K を K の knot diagram とする. D_K の交点でない場所に基点を選び, 向きを指定する. 基点から指定した向きに沿って進み, 最初の交点に1を割り当てる. 次の交点が新しい交点の場合は2を割り当てる. この様にして各交点に文字(数字)を割り当てる. また, 交点を通過するとき, 跨いでいるときには \bar{i} とマーキングをし, アローの向きは, i から \bar{i} へ向かうとする. アローの符号は対応する交点の符号を付ける. この様にして, 符号付き基点付きアロー図式を得る. (同様に virtual knot, long virtual knot についても定義される.)

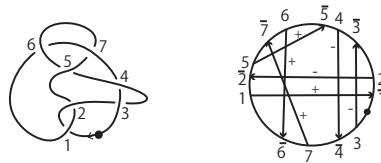


図 1: Knot diagram と 符号付き基点付きアロー図式

\check{G}_∞ を符号付き基点付きアロー図式全体の集合とする¹. x^*, y^* を符号付き基点付きアロー図式とする. x^* の符号 $\text{sign}(x^*)$ を

$$\text{sign}(x^*) := \prod_{\alpha: \text{arrow in } x^*} \text{sign}(\alpha)$$

により定める. x^* に対し, 整数値関数 $\tilde{x}^*: \check{G}_\infty \rightarrow \mathbb{Z}$ を

$$\tilde{x}^*(y^*) = \begin{cases} \text{sign}(x^*) & \text{if } y^* = x^*, \\ 0 & \text{if } y^* \neq x^* \end{cases}$$

により定める. $\text{Sub}(y^*)$ を y^* からいくつかのアローを除いてできるアロー図式の集合とする. x^* に対し, (同じ記号を用いて) 整数値関数 $x^*: \check{G}_\infty \rightarrow \mathbb{Z}$ を

$$x^*(y^*) := \sum_{z^* \in \text{Sub}(y^*)} \tilde{x}^*(z^*)$$

により定める. $x^*(y^*)$ を $\langle x^*, y^* \rangle$ と書く場合もある. 線形に拡張して, Gauss diagram formula と呼ぶ. 符号無し基点付きアロー図式は, 符号付き基点付きアロー図式の和として見る:

$$\textcircled{\pm} = -\textcircled{-} + -\textcircled{+} + \textcircled{-} + \textcircled{+}$$

Gauss diagram formula が Reidemeister move で不変となるための十分条件を書くために, relator を用意する ([2] Definition 8を参照). 例として

¹ 気持ちが悪ければ十分高い次数の有限生成加群を考えても差し障りない.

$$\begin{array}{cccccccc} \begin{array}{c} + \\ \oplus \\ \ominus \\ + \end{array} & + & \begin{array}{c} + \\ \oplus \\ \oplus \\ + \end{array} & + & \begin{array}{c} + \\ \oplus \\ \ominus \\ + \end{array} & + & \begin{array}{c} - \\ \oplus \\ \oplus \\ - \end{array} & - & \begin{array}{c} - \\ \oplus \\ \oplus \\ - \end{array} & - & \begin{array}{c} - \\ \oplus \\ \oplus \\ - \end{array} & - & \begin{array}{c} - \\ \oplus \\ \oplus \\ - \end{array} & - & \begin{array}{c} - \\ \oplus \\ \oplus \\ - \end{array} \end{array}$$

などがある.

Theorem 2 b, d ($2 \leq b \leq d$) を整数とする. $\check{G}_{\leq d}, \{x_i^*\}_{i \in \mathbb{N}}, \mathbb{Z}[\check{G}_{\leq d}], \check{n}_d = |\check{G}_{\leq d}|, \check{G}_{b,d} = \{x_i^*\}_{\check{n}_{b-1}+1 \leq i \leq \check{n}_d}, \sum_{\check{n}_{b-1}+1 \leq i \leq \check{n}_d} \alpha_i x_i^*, \sum_{\check{n}_{b-1}+1 \leq i \leq \check{n}_d} \alpha_i \tilde{x}_i^*, \check{O}_{b,d}(\check{R}_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5}(b, d))$ の定義は [2] を参照していただきたい.

32通りの選択肢から $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \in \{0, 1\}^5$ を任意の一つ選ぶ. このとき, 任意の $r^* \in \check{O}_{b,d}(\check{R}_{\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5}(b, d))$ に対して, $\sum_{\check{n}_{b-1}+1 \leq i \leq \check{n}_d} \alpha_i \tilde{x}_i^*(r^*) = 0$ ならば, 対応する各 Reidemeister move に対して $\sum_{\check{n}_{b-1}+1 \leq i \leq \check{n}_d} \alpha_i x_i^*$ は整数値不変量である.

Classical knot のみについて, アロー図式のペア間に自然に成り立つ関係式がある ([2], Definition 17). 最大次数が 4 の Type (SIII) relator の一部をこの関係式に取り替えた relator を考える. 次数が 3 以下の Gauss diagram formula に対し, Theorem 2 と同様の結果を得る.

3. コンピュータによる計算

ガウスワードの同値類全体の集合, 向き付きガウスワードの同値類全体の集合を求めるコンピュータ・プログラムの実装は [3] で完成している. 今回, このプログラムに基点と符号の情報を追加する改良を行った. 次数 2 の符号付き基点付きアロー図式と次数 3 の基点付きアロー図式のリストは次になる.

Notation 1 (y_i^* ($1 \leq i \leq 168$) の定義) $y_1^* = \begin{array}{c} \ominus \\ \oplus \\ \oplus \\ \ominus \end{array}, y_2^* = \begin{array}{c} \ominus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_3^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_4^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_5^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_6^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_7^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_8^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_9^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{10}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{11}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{12}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{13}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{14}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{15}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{16}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{17}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{18}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{19}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{20}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{21}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{22}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{23}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{24}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{25}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{26}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{27}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{28}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{29}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{30}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{31}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{32}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{33}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{34}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{35}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{36}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{37}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{38}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{39}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{40}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{41}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{42}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{43}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{44}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{45}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{46}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{47}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{48}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{49}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{50}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{51}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{52}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{53}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{54}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{55}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{56}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{57}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{58}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{59}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{60}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{61}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{62}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{63}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{64}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{65}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{66}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{67}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{68}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{69}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{70}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{71}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{72}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{73}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{74}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{75}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{76}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{77}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{78}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{79}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{80}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{81}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{82}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{83}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{84}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{85}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{86}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{87}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{88}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{89}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{90}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{91}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{92}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{93}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{94}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{95}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{96}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{97}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{98}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{99}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{100}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{101}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{102}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{103}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{104}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{105}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{106}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{107}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{108}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{109}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{110}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{111}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{112}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{113}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{114}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{115}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{116}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{117}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{118}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$y_{119}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{120}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{121}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{122}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{123}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{124}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{125}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array}, y_{126}^* = \begin{array}{c} \oplus \\ \oplus \\ \oplus \\ \oplus \end{array},$

$$\begin{aligned}
y_{127}^* &= \text{diagram}, y_{128}^* = \text{diagram}, y_{129}^* = \text{diagram}, y_{130}^* = \text{diagram}, y_{131}^* = \text{diagram}, y_{132}^* = \text{diagram}, y_{133}^* = \text{diagram}, y_{134}^* = \text{diagram}, \\
y_{135}^* &= \text{diagram}, y_{136}^* = \text{diagram}, y_{137}^* = \text{diagram}, y_{138}^* = \text{diagram}, y_{139}^* = \text{diagram}, y_{140}^* = \text{diagram}, y_{141}^* = \text{diagram}, y_{142}^* = \text{diagram}, \\
y_{143}^* &= \text{diagram}, y_{144}^* = \text{diagram}, y_{145}^* = \text{diagram}, y_{146}^* = \text{diagram}, y_{147}^* = \text{diagram}, y_{148}^* = \text{diagram}, y_{149}^* = \text{diagram}, y_{150}^* = \text{diagram}, \\
y_{151}^* &= \text{diagram}, y_{152}^* = \text{diagram}, y_{153}^* = \text{diagram}, y_{154}^* = \text{diagram}, y_{155}^* = \text{diagram}, y_{156}^* = \text{diagram}, y_{157}^* = \text{diagram}, y_{158}^* = \text{diagram}, \\
y_{159}^* &= \text{diagram}, y_{160}^* = \text{diagram}, y_{161}^* = \text{diagram}, y_{162}^* = \text{diagram}, y_{163}^* = \text{diagram}, y_{164}^* = \text{diagram}, y_{165}^* = \text{diagram}, y_{166}^* = \text{diagram}, \\
y_{167}^* &= \text{diagram}, y_{168}^* = \text{diagram}.
\end{aligned}$$

Notation 2 Type (I) relator を122個, Type (WII) relator を96個, Type (SIII) relator を246個定める. 具体的な表示は [5] をご参照いただきたい.

Notation 1 と Notation 2 に対して, Theorem 2 を適用すると, long virtual knot の不変量が得られる.

Proposition 1 9個の Gauss diagram formula

$$\tilde{v}_{3,i}(\cdot) = \langle \tilde{f}_{3,i}, \cdot \rangle (1 \leq i \leq 7), \quad \tilde{v}_{2,i}(\cdot) = \langle \tilde{f}_{2,i}, \cdot \rangle (i = 1, 2)$$

は独立な次数3以下の long virtual knot の Goussarov-Polyak-Viro 有限型不変量となる. ここで, $\tilde{f}_{3,i}$ ($1 \leq i \leq 7$) と $\tilde{f}_{2,i}$ ($i = 1, 2$) は次で定義される:

$$\begin{aligned}
\tilde{f}_{3,1} &:= \text{diagram} + 2 \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} - \text{diagram} + 3 \text{diagram} + 2 \text{diagram}, \\
\tilde{f}_{3,2} &:= \text{diagram} + \text{diagram} - \text{diagram} + \text{diagram} - \text{diagram} + \text{diagram} - \text{diagram}, \\
\tilde{f}_{3,3} &:= \text{diagram} + \text{diagram} - \text{diagram} + \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram}, \\
\tilde{f}_{3,4} &:= - \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} + \text{diagram}, \\
\tilde{f}_{3,5} &:= \text{diagram} - \text{diagram} + \text{diagram} + \text{diagram} + 2 \text{diagram} + \text{diagram} + \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram}, \\
\tilde{f}_{3,6} &:= 4 \text{diagram} + 4 \text{diagram} + \text{diagram} - \text{diagram} - \text{diagram} - \text{diagram} - 2 \text{diagram} + \text{diagram} + \text{diagram} - 4 \text{diagram} - \text{diagram} \\
&\quad - 2 \text{diagram} - \text{diagram} - 3 \text{diagram} - 4 \text{diagram} - 2 \text{diagram} - 2 \text{diagram}, \\
\tilde{f}_{3,7} &:= \text{diagram} - \text{diagram} - \text{diagram} + \text{diagram} - \text{diagram} - \text{diagram} + \text{diagram} + \text{diagram}, \\
\tilde{f}_{2,1} &:= \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}, \text{ and} \\
\tilde{f}_{2,2} &:= \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}.
\end{aligned}$$

Notation 3 Type (SIII) relator の一部を取り替えた186個定める. 具体的な表示は [5] をご参照いただきたい.

Notation 1 と Notation 2 の Type (I) relator, Type (WII) relator と Notation 3 の Type (SIII) relator に対して, 計算を行うと classical knot の不変量が得られる.

Proposition 2 21個の Gauss diagram formulas

$$v_{3,i}(\cdot) = \langle f_{3,i}, \cdot \rangle (1 \leq i \leq 21), v_{2,i}(\cdot) = \langle f_{2,i}, \cdot \rangle (i = 1, 2)$$

は classical knot 不変量となる。ここで $f_{3,i}$ ($1 \leq i \leq 21$) と $f_{2,i}$ ($i = 1, 2$) は次で定義される：

$$\begin{aligned}
 f_{3,1} &:= -\text{diag}_1 - \text{diag}_2 - \text{diag}_3 - \text{diag}_4 - \text{diag}_5 - \text{diag}_6 - \text{diag}_7, \\
 f_{3,2} &:= -\text{diag}_1 - \text{diag}_2 - \text{diag}_3 - \text{diag}_4 - \text{diag}_5 - \text{diag}_6 + \text{diag}_7 - \text{diag}_8 - \text{diag}_9, \\
 f_{3,3} &:= -\text{diag}_1 - \text{diag}_2 + \text{diag}_3 - \text{diag}_4 + \text{diag}_5 + \text{diag}_6 - \text{diag}_7 - \text{diag}_8 + \text{diag}_9 - \text{diag}_{10}, \\
 f_{3,4} &:= -\text{diag}_1 - \text{diag}_2 + \text{diag}_3 + \text{diag}_4 - \text{diag}_5 - \text{diag}_6 - \text{diag}_7 + \text{diag}_8, \\
 f_{3,5} &:= -\text{diag}_1 - \text{diag}_2 - \text{diag}_3 + \text{diag}_4 - \text{diag}_5 - \text{diag}_6 - \text{diag}_7 - \text{diag}_8 - \text{diag}_9, \\
 f_{3,6} &:= -\text{diag}_1 - \text{diag}_2 - \text{diag}_3 + \text{diag}_4 - \text{diag}_5 + \text{diag}_6 + \text{diag}_7 - \text{diag}_8, \\
 f_{3,7} &:= \text{diag}_9 + \text{diag}_{10} + \text{diag}_{11} + \text{diag}_{12} + \text{diag}_{13} + \text{diag}_{14} + 2\text{diag}_{15} + 2\text{diag}_{16}, \\
 f_{3,8} &:= -\text{diag}_9 - \text{diag}_{10} - \text{diag}_{11} - \text{diag}_{12} - \text{diag}_{13} - \text{diag}_{14} + \text{diag}_{15} + \text{diag}_{16} + \text{diag}_{17} + \text{diag}_{18} + \text{diag}_{19} + \text{diag}_{20}, \\
 f_{3,9} &:= -\text{diag}_9 - \text{diag}_{10} - \text{diag}_{11} - \text{diag}_{12} - \text{diag}_{13} - \text{diag}_{14} + \text{diag}_{15} + \text{diag}_{16} + \text{diag}_{17} + \text{diag}_{18} + \text{diag}_{19} + \text{diag}_{20}, \\
 f_{3,10} &:= \text{diag}_9 + \text{diag}_{10} + \text{diag}_{11} - \text{diag}_{12} - \text{diag}_{13} - \text{diag}_{14}, \\
 f_{3,11} &:= \text{diag}_9 - \text{diag}_{10} - \text{diag}_{11} - \text{diag}_{12} + \text{diag}_{13} + \text{diag}_{14} - \text{diag}_{15} + \text{diag}_{16}, \\
 f_{3,12} &:= -\text{diag}_9 + \text{diag}_{10}, \\
 f_{3,13} &:= -\text{diag}_9 + \text{diag}_{10} - \text{diag}_{11} + \text{diag}_{12}, \\
 f_{3,14} &:= -\text{diag}_9 + \text{diag}_{10}, \\
 f_{3,15} &:= \text{diag}_9 - \text{diag}_{10} + \text{diag}_{11} - \text{diag}_{12}, \\
 f_{3,16} &:= -\text{diag}_9 - \text{diag}_{10} + \text{diag}_{11} + \text{diag}_{12}, \\
 f_{3,17} &:= -\text{diag}_9 + \text{diag}_{10}, \\
 f_{3,18} &:= \text{diag}_9 - \text{diag}_{10} - \text{diag}_{11} + \text{diag}_{12}, \\
 f_{3,19} &:= \text{diag}_9 + \text{diag}_{10} + \text{diag}_{11} - \text{diag}_{12} - \text{diag}_{13} - \text{diag}_{14}, \\
 f_{3,20} &:= \text{diag}_{15} - \text{diag}_{16}, \\
 f_{3,21} &:= \text{diag}_{15} - \text{diag}_{16}, \\
 f_{2,1} &:= -\text{diag}_{17} + -\text{diag}_{18} + \text{diag}_{19} + \text{diag}_{20}, \text{ and} \\
 f_{2,2} &:= -\text{diag}_{17} + -\text{diag}_{18} + \text{diag}_{19} + \text{diag}_{20}.
 \end{aligned}$$

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