$\mathfrak{gl}(1|1)$-Alexander polynomial for 3-manifolds

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Abstract

As an extension of Reshetikhin and Turaev's invariant, Costantino, Geer and Patureau-Mirand constructed 3-manifold invariants in the setting of relative G-modular categories, which include both semisimple and non-semisimple ribbon tensor categories as examples. In this paper, we follow their method to construct a 3-manifold invariant from Viro's $\mathfrak{gl}(1|1)$-Alexander polynomial. We take lens spaces $L(7,1)$ and $L(7,2)$ as examples to show that this invariant can distinguish homotopy equivalent manifolds.

1 Introduction

Given a framed link L in S^3, the integral surgery along L produces an oriented closed 3-manifold. The link L is called a surgery presentation of the resulting manifold. Kirby calculus [3] says that any oriented closed 3-manifold can be obtained in this way. In addition, surgery presentations of the same 3-manifold are related to each other by Kirby moves.

A linear sum of quantum invariants of framed links defines a topological invariant for 3-manifolds, if it is invariant under Kirby moves. Reshetikhin and Turaev [4] gave the first rigorous construction of 3-manifold invariant along this line. Their invariant was defined for a modular category, which is semisimple and all simple objects are required to have non-zero quantum dimensions.

Costantino, Geer and Patureau-Mirand [1] extended Reshetikhin and Turaev's construction to categories which may not be semisimple or may contain objects with zero quantum dimensions. They proposed the concept: relative G-modular category and proved that the quantum invariant of framed links constructed from a relative G-modular category can be used to define a 3-manifold invariant. Let $\mathscr{C}$ be a relative G-modular category. For an oriented closed 3-manifold M, a $\mathscr{C}$-ribbon graph T and a cohomology class $\omega : H_1(M\backslash T, \mathbb{Z}) \to G$ which satisfy some compatible conditions, [1] showed that the quantum invariant of $L \cup T$ after normalization is a topological invariant of (M, T, ω), where L is a surgery presentation of M with color induced from ω.

In this paper, we follow the method in [1] to construct a 3-manifold invariant. The quantum invariant we use is Viro's $\mathfrak{gl}(1|1)$-Alexander polynomial defined in [5]. Consider a 1-palette defined by (B, G) where B is a field of characteristic 0 and $G \subset B$ is an abelian group. There is a category $\mathcal{M}_B$ of finite dimensional modules over a subalgebra U^1 of $U_q(\mathfrak{gl}(1|1))$, the quantum group of the Lie superalgebra $\mathfrak{gl}(1|1)$. The category $\mathcal{M}_B$ is not semisimple and the objects of which have zero quantum dimensions. Viro defined a functor from the category of trivalent graphs to $\mathcal{M}_B$. For a colored graph Λ, the Alexander polynomial $\Delta(\Lambda)$ is defined using this functor.

Now consider a triple (M, Γ, ω), where M is a 3-manifold, Γ is a trivalent graph colored by objects of $\mathcal{M}_B$ and $\omega : H_1(M \backslash \Gamma, \mathbb{Z}) \to G$ is a cohomology class. We assume that (M, Γ, ω) satisfies certain compatible conditions. Here is our main result. The definitions of computable surgery presentation and Kirby color will be given in Section 3.3.

Theorem 1.1. *For the* 1*-palette* (B, G) *where* G *contains* $\mathbb{Z}$ *but no* $\mathbb{Z}/2\mathbb{Z}$ *as a subgroup, let* (M, Γ, ω) *be a compatible triple. Let* L *be a computable surgery presentation of* (M, Γ, ω)*. Then*

$$\Delta(M, \Gamma, \omega) := \frac{\Delta(L \cup \Gamma)}{2^{r(L)}(-1)^{\sigma_+(L)}}$$

is a topological invariant of (M, Γ, ω)*, where* $r(L)$ *is the component number of* L *and* $\sigma_+(L)$ *is the number of positive eigenvalues of the linking matrix of* L*. Here each component* K *of* L *has Kirby color* $\Omega(\omega([m_K]), 1)$*, where* m_K *is the meridian of* K*.*

Our strategy is as follows. Instead of proving that $\mathcal{M}_B$ has a relative G-modular category structure, we show directly that the value $\Delta(M, \Gamma, \omega)$ is invariant under Kirby moves. So the flavor of this paper is quite combinatorial without involving many algebras. However we believe the existence of a relative G-modular category structure on $\mathcal{M}_B$ so that the corresponding invariant is the one given in Theorem 1.1. We hope to discuss this topic in our future work. In the definitions of compatible triple, Kirby color and the proof of Theorem 1.1, we imitate many ideas from [1].

The authors of [1] discussed in detail how to define the 3-manifold invariant in the context of quantum $\mathfrak{sl}(2)$. For any finite-dimensional simple complex Lie algebra $\mathfrak{g}$, they also showed the existence of relative G-modular category associated with certain version of quantum $\mathfrak{g}$. The representation theory for Lie superalgebras is much more complicated than that of Lie algebras. Based on the concept relative G-modular category, NP Ha [2] constructed 3-manifold invariant from quantum group associated with Lie superalgebra $\mathfrak{sl}(2|1)$. It is not clear to us yet whether $\Delta(M, \Gamma, \omega)$ coincides with any known invariant or not.

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2 Viro's $\mathfrak{gl}(1|1)$-Alexander polynomial

Viro [5] defined a functor from the category of colored framed oriented trivalent graphs to the category of finite dimensional modules over a subalgebra U^1 of the q-deformed universal enveloping superalgebra $U_q(\mathfrak{gl}(1|1))$. Using this functor, in Section 6 of [5], he defined the $\mathfrak{gl}(1|1)$-Alexander polynomial for a trivalent graph. We recall how this polynomial is calculated. For the algebraic structures of U^1 and $U_q(\mathfrak{gl}(1|1))$, please read [5, §11: Appendix].

2.1 Colored framed graphs

A 1-*palette* (see [5, 2.8]) is a quadruple

$$(B, G, W, G \times W \to G),$$

where B is a commutative ring with unit, G is a subgroup of the multiplicative group of B, W is a subgroup of the additive group of B which contains the unit of B, and $G \times W \to G : (t, w) \mapsto t^w$ is a bilinear map satisfying $t^1 = t$ for each $t \in G$. For each $t \in G$ satisfying $t^4 \neq 1$ and $N \in W$, there exist two modules $U(t, N)_+$ and $U(t, N)_-$ of U^1. For the definition of $U(t, N)_+$ and $U(t, N)_-$, see [5, §11: Appendix].

In this paper, we consider the case that B is a field of characteristic 0. Let G is a subgroup of the multiplicative group of B, which is abelian, and let $W = \mathbb{Z}$ and $G \times \mathbb{Z} \to G : (t, n) \mapsto t^n$. Obviously $(B, G, W, G \times W \to G)$ becomes a 1-palette. Since W and $G \times \mathbb{Z} \to G$ have specific definitions, we suppress them and use (B, G) to denote the 1-palette. When we say a 1-palette, we mean a 1-palette defined in this way.

Let T be an oriented trivalent graph, and let E be the set of edges of T. Consider a map which we call a coloring

$$\begin{aligned} c = (\mathrm{mul}, \mathrm{wt}) : E &\to G \backslash \{g \in G \mid g^4 = 1\} \times \mathbb{Z} \\ e &\mapsto (t, N). \end{aligned}$$

The first number $t = \mathrm{mul}(e)$ is called the *multiplicity* and the second number $N = \mathrm{wt}(e)$ is called the *weight*.

Around a vertex, suppose the three edges adjacent to it are colored by (t_1, N_1), (t_2, N_2) and (t_3, N_3). Let $\epsilon_i = -1$ if the i-th edge points toward the vertex and $\epsilon_i = 1$ otherwise. The coloring c needs to satisfy the following conditions, which are called *admissibility conditions* in [5]:

$$\prod_{i=1}^{3} t_i^{\epsilon_i} = 1, \tag{1}$$

$$\sum_{i=1}^{3} \epsilon_i N_i = -\prod_{i=1}^{3} \epsilon_i. \tag{2}$$

A vertex is called *source* (resp. *sink*) if all the adjacent edges have $\epsilon = 1$ (resp. $\epsilon = -1$).

Now consider a proper embedding of T into a 3-manifold M. We still use T to represent the embedded graph. A *framing* of T is an orientable compact surface F embedded in M in which T is sitting as a deformation retract. More precisely, in F each vertex of T is replaced by a disk where the vertex is the center, and each edge of T is replaced by a strip $[0,1] \times [0,1]$ where $[0,1] \times \{0,1\}$ is attached to the boundaries of its adjacent vertex disks and $\{\frac{1}{2}\} \times [0,1]$ is the given edge of T.

A *framed graph* is a graph with a framing. By an *isotopy* of a framed graph we mean an isotopy of the graph in M which extends to an isotopy of the framing.

For a framed graph, at each source or sink, we can assign an orientation to the boundary of the associated disk, which is regarded as part of the coloring of T. Now we are ready to give the following definition.

Definition 2.1. A *colored framed oriented trivalent graph* Γ in a 3-manifold M is an oriented trivalent graph T embedded in M with the following three structures:

- a framing;
- a coloring on the set of edges which satisfies the admissibility conditions;
- an orientation of the boundary of the associated disk on each source or sink vertex.

In the following sections, a framed graph means a framed oriented trivalent graph, while a colored framed graph means a colored framed oriented trivalent graph.

When Γ is a graph in S^3, we can use a graph diagram to represent Γ, the blackboard framing of which coincides with the framing of Γ. Around a source or sink vertex, the counter-clockwise orientation is chosen unless otherwise stated.

2.2 $\mathfrak{gl}(1|1)$-Alexander polynomial

Let (B, G) be a 1-palette. Suppose Γ is a colored framed graph embedded in S^3 whose coloring is given by the map c. We review the definition of the $\mathfrak{gl}(1|1)$-Alexander polynomial of Γ, which is denoted by $\Delta(\Gamma)$ or $\Delta(\Gamma; c)$.

Note that the pair $(t, N) \in G \backslash \{g \in G \mid g^4 = 1\} \times \mathbb{Z}$ corresponds to two irreducible U^1-modules of dimension $(1|1)$, which are denoted by $U(t,N)_+$ and $U(t,N)_-$. These two modules are dual to each other. The module $U(t,N)_+$ (resp. $U(t,N)_-$) is generated by two elements e_0 (boson) and e_1 (fermion). For details of their definitions please see Appendix 1 of [5].

Choose a graph diagram of Γ in $\mathbb{R}^2$. The diagram divides $\mathbb{R}^2$ into several regions, one of which is unbounded. Choose an edge of Γ on the boundary of the unbounded region and cut the edge at a generic point. Suppose the color of the edge is (t, N). Deform the graph diagram under isotopies of $\mathbb{R}^2$ to make it in a Morse position under a given orthogonal coordinate system of $\mathbb{R}^2$ so that the two endpoints created by cutting have heights zero and one and the critical points, the crossings, and the vertices of the

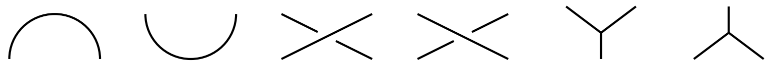


Figure 1: Critical points, crossings, and vertices.

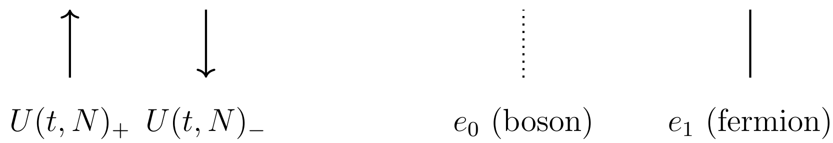


Figure 2: Under the coloring c, each edge corresponds to an irreducible U^1-module. In a state, if an edge is assigned with e_0 (resp. e_1), we represent it by a dotted (resp. solid) arc.

diagram have different heights between zero and one. Namely after deformation the diagram can be divided into several slices by horizontal lines so that each slice is a disjoint union of trivial vertical segments and one of the six elements in Fig. 1. Each slice connects a sequence of endpoints on its bottom to a sequence of endpoints on its top. In Example 2.2, we show how the Hopf link is divided into such slices.

Under Viro's functor, each sequence of endpoints corresponds to the tensor product of irreducible U^1-modules of dimension $(1|1)$. Suppose the sequence of endpoints is $(p_1, \cdots, p_k)$ for $k \geq 1$, where the subindices represent the x-coordinates of the endpoints. Then $(p_1, \cdots, p_k)$ corresponds to the tensor product

$$U(t_1, N_1)_{\epsilon_1} \otimes \cdots \otimes U(t_k, N_k)_{\epsilon_k},$$

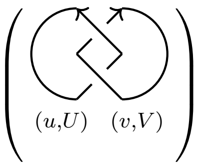
where (t_i, N_i) is the color of the edge containing p_i and $\epsilon_i = +$ when the edge points upward and $\epsilon_i = -$ otherwise for $1 \leq i \leq k$. See Fig 2.

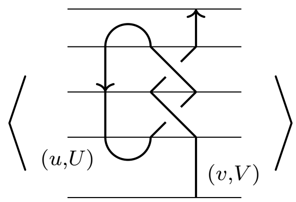
Each slice connects two sequences of endpoints. Under Viro's functor, each slice, read from bottom to top, is mapped to a morphism between the corresponding tensor products of irreducible U^1-modules.

The morphism is defined in the language of Boltzmann weights. Simply speaking, each module $U(t, N)_+$ or $U(t, N)_-$ has two generators e_0 (boson) and e_1 (fermion), and therefore $U(t_1, N_1)_{\epsilon_1} \otimes \cdots \otimes U(t_k, N_k)_{\epsilon_k}$ is generated by $e_{\delta_1} \otimes e_{\delta_2} \otimes \cdots \otimes e_{\delta_k}$ for $\delta_i = 0$ or 1. The morphism is represented by a matrix under the above choice of generators, and the Boltzmann weights are the entries of the matrix. For the full table of Boltzmann weights, see Tables 3 and 4 of [5].

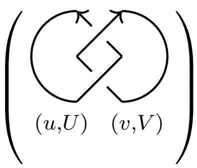
The composition of two slices (attaching them by identifying the top of the first slice with the bottom of the second slice) corresponds to the composition of their morphisms for U^1-modules. As a consequence, the graph diagram in Morse position with two endpoints of heights zero and one is mapped to a morphism from $U(t, N)_+$ to $U(t, N)_+$ (or $U(t, N)_-$ to $U(t, N)_-$ depending the orientation of Γ at the endpoints), which is a scalar of identity ([5, 6.2.A]). Recall that (t, N) is the color of the edge which was cut. Then multiplying the scalar by the inverse of $t^2 - t^{-2}$ we get $\Delta(\Gamma)$. In the following paragraphs, we use (Γ) to represent the Alexander polynomial of Γ when Γ is a colored framed graph.

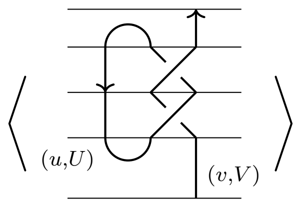
Example 2.2. *For $u, v \in G \backslash \{g \in G | g^4 = 1\}$, we have*



$= \frac{1}{v^2 - v^{-2}}$


$= \frac{-u^{2V}v^{2U}(v^2 - v^{-2})}{v^2 - v^{-2}} = -u^{2V}v^{2U}.$



$= \frac{1}{v^2 - v^{-2}}$


$= \frac{u^{-2V}v^{-2U}(v^2 - v^{-2})}{v^2 - v^{-2}} = u^{-2V}v^{-2U}.$

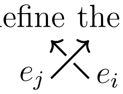
Here $\langle D \rangle$ denotes the scalar defined the by the tangle D.

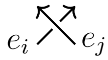
3 On the Proof of Main Result

3.1 Cohomology classes

We review a characterization of cohomology classes given in [1, Sect. 2.3]. Let M be a closed 3-manifold, let T be a framed graph in M. Suppose L is an oriented framed link in S^3 which is a surgery presentation for M. Since T is disjoint from L, we also view T as a graph in S^3 before the surgery.

Now we consider diagrams of L and T, which are still denoted by L and T. Let $e_1, e_2, \cdots, e_r$ be the components of L, and $e_{r+1}, e_{r+2}, \cdots, e_{r+s}$ be the oriented edges of T. For two different components e_i and e_j in L ($1 \leq i, j \leq r$), let $lk_{ij} = lk(e_i, e_j)$ denote the linking number of e_i and e_j. Namely, it is half of the sum of signs of all the crossings between e_i and e_j. Let $lk_{ii} = lk(e_i, e_i)$ ($1 \leq i \leq r$) be the framing of e_i. Namely it is the sum of signs of self-crossings of e_i (since we use blackboard framing). It is well-known that lk_{ij} does not depend on the diagram we choose. The matrix $(lk_{ij})_{1 \leq i,j \leq r}$ is called the *linking matrix* of L.

For a component e_i of L and an edge e_j of T, we define the linking number $lk_{ij} = lk(e_i, e_j)$ to be the number of all the crossings of type 

minus the number of crossings of type 

between e_i and e_j. Note that this number depends on the diagrams of L and T.

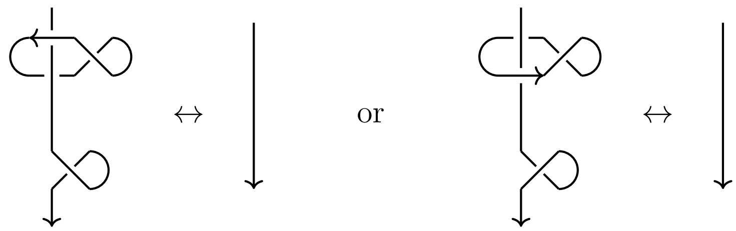


Figure 3: Blow up/down moves

Let $M\backslash T$ be the complement of T in M. The first homology group $H_1(M\backslash T, \mathbb{Z})$ has a presentation

$$H_1(M\backslash T, \mathbb{Z}) = \left\langle \ \{[m_i]\}_{1\leq i\leq r+s} \ \middle| \begin{array}{l} \forall 1 \leq i \leq r, \sum_{j=1}^{r+s} lk_{ij}[m_j] = 0; \\ \forall v : \text{vertex of } T, r_v = 0; \\ \forall 1 \leq i, j \leq r+s, [m_i] + [m_j] = [m_j] + [m_i] \end{array} \right\rangle,$$

where m_i is the oriented meridian of e_i, and for a vertex v of T, r_v is the sum of meridians of the edges entering v minus the sum of meridians of the edges outgoing from v. Note that for each $1 \leq i \leq r$, $\sum_{j=r+1}^{r+s} lk_{ij}[m_j]$ does not depend on the choice of diagram of L and T and is a well-defined value.

Let G be an abelian group. Then the cohomology class

$$\omega \in H^1(M\backslash T, G) \cong \text{Hom}(H_1(M\backslash T, \mathbb{Z}), G)$$

is uniquely determined by the images of $[m_i]'s$ under ω.

3.2 Kirby calculus

We review basic facts about Kirby calculus, which can be found, for instance in [1, 5.1]. Kirby [3] showed that any compact connected oriented closed 3-manifold can be obtained by doing surgeries along a framed link in S^3. Such a link is called the surgery presentation of the given 3-manifold. There are two types of moves connecting surgery presentations, which are called blow up/down moves and handle-slide move. See Fig 3 and Fig 4.

Theorem 3.1 (Theorem 5.2 in [1]). *Let M_1 and M_2 be compact connected oriented closed 3-manifolds and $T_1 \subset M_1$ and $T_2 \subset M_2$ be embedded framed graphs. Let $f : M_1 \to M_2$ be an orientation preserving diffeomorphism such that $f(T_1) = T_2$. Let $L_i \subset S^3$ be a surgery presentation of M_i which is disjoint from T_i for $i = 1, 2$. Then f is isotopic to the diffeomorphisms induced by a finite sequence of moves:*

$$L_1 = L^0 \xrightarrow{k_1} L^2 \xrightarrow{k_2} \cdots \xrightarrow{k_r} = L_2,$$

where each k_i $(1 \leq i \leq r)$ is one of the following moves.

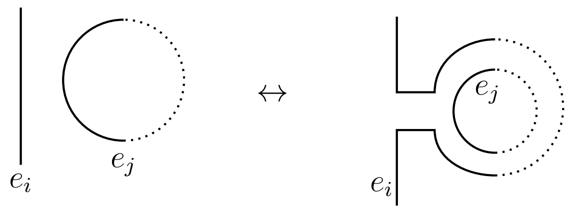


Figure 4: Handle-slide e_i along e_j.

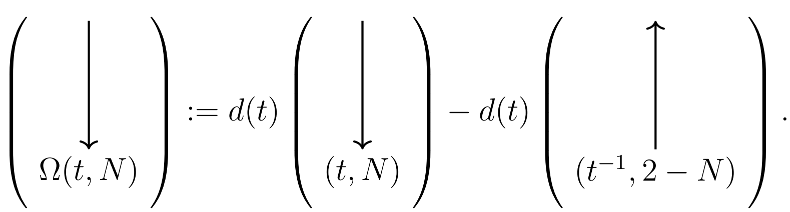
i. handle-slide move of a component/edge of $L^{i-1} \cup T_1$ *along a component of* L^{i-1}*;*

ii. blow up/down move along a component/edge of $L^{i-1} \cup T_1$*, where the circle component which appears or disappears during this move must be a component of the surgery presentation.*

3.3 Sketch of proof of Theorem 1.1

The proof of Theorem 1.1 is essentially the same as the proof of Theorem 4.7 in [1]. In the following two lemmas, we discuss how the Alexander polynomial changes under Kirby moves.

Definition 3.2. For $(t, N) \in G\backslash\{g \in G \mid g^4 = 1\} \times \mathbb{Z}$, if one component of a link has *Kirby color* $\Omega(t, N)$, the Alexander polynomial is calculated as follows:



For a strand with Kirby color $\Omega(t, N)$, its *multiplicity* is defined to be t.

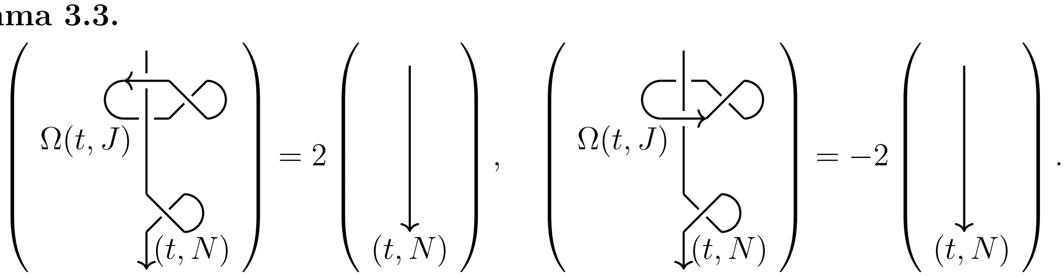
It is easy to see that if a knot $K \subset S^3$ has Kirby color $\Omega(t, N)$, we have

$$\Delta(K; \Omega(t, N)) = \Delta(-K; \Omega(t^{-1}, 2 - N)),$$

where $-K$ is the same knot K with opposite orientation.

Now, we discuss how the Alexander polynomial changes under blow-up/down moves when the circle component has a Kirby color.

Lemma 3.3.



For a colored framed graph $\Lambda \subset S^3$ and a knot component $K \subset \Lambda$, we define *the colored linking number* of K with Λ as

$$clk(K, \Lambda) := \prod_{e\text{: edge of }\Lambda} t_e^{lk(K,e)},$$

where $lk(K, e)$ is the linking number as defined in Section 3.1, and t_e is the multiplicity of e. Due to the admissibility condition (1) of multiplicities, $clk(K, \Lambda)$ is well-defined.

Next, we study how the Alexander polynomial changes under a handle-slide move. We have the following lemma.

Lemma 3.4. *Suppose Λ is a colored framed graph, and K is a knot component of Λ with Kirby color $\Omega(s, S)$. Let e be an oriented edge of $\Lambda \backslash K$ with color (t, N). Let Λ' be a graph obtained from Λ by a handle-slide move of e along K, and K has the new Kirby color $\Lambda(ts, N + S - 1)$. If $clk(K, \Lambda) = 1$ and $(ts)^4 \neq 1$, we have $\Delta(\Lambda) = \Delta(\Lambda')$.*

Now we consider a 1-palette for which G is finitely generated abelian group containing at least one $\mathbb{Z}$ summand and satisfies $t^4 = 1 \iff t = 1$. Namely G contains $\mathbb{Z}$ but no $\mathbb{Z}/2\mathbb{Z}$ as a subgroup, as required by Theorem 1.1. It is not hard to find such 1-palettes as we can see in the following examples.

Example 3.5. *The* 1*-palettes defined by the following data meet our requirements.*

i. Let $B = \mathbb{Q}(t)$, the field of rational functions of t, and let $G = \mathbb{Z}\langle t\rangle$, the cyclic group generated by t.

ii. Let ξ_l be the l-th primitive root of unity for a prime number $l \geq 3$. Let $B = \mathbb{Q}(\pi, \xi_l)$, the extension field of $\mathbb{Q}$ generated by π and ξ_l. Let $G = \mathbb{Z}\langle \pi, \xi_l\rangle$, the abelian group generated by π and ξ_l.

Let M be a 3-manifold, let Γ be a colored framed graph in M colored by a 1-palette (B, G) where G contains $\mathbb{Z}$ but no $\mathbb{Z}/2\mathbb{Z}$ as a subgroup. Consider a cohomology class $\omega : H_1(M\backslash\Gamma, \mathbb{Z}) \to G$. We say that (M, Γ, ω) is a *compatible triple* if for each edge e of Γ, the multiplicity of e is equal to $\omega([m])$, where m is the meridian of e. Let L be a surgery presentation of M. We say that L is *computable* for a compatible triple (M, Γ, ω) if $L \cup \Gamma \neq \emptyset$ and $\omega([m]) \neq 1 \in G$ for any meridian m of L. We can show the existence of a computable surgery presentation of (M, Γ, ω) if ω is non-trivial.

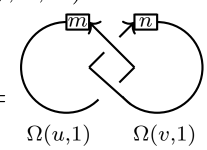
Sketch of proof of Theorem 1.1. Let L and L' be two computable surgery presentations of (M, Γ, ω). By Theorem 3.1, there is a sequence of handle-slide moves, blow-up/down moves connecting $L \cup \Gamma$ and $L' \cup \Gamma$ and the induced diffeomorphism $f : M \to M$ satisfies $f(\Gamma) = \Gamma$ and $f^*(\omega) = \omega$. We want to show that $\frac{\Delta(L\cup\Gamma)}{2^{r(L)}(-1)^{\sigma_+(L)}} = \frac{\Delta(L'\cup\Gamma)}{2^{r(L')}(-1)^{\sigma_+(L')}}$.

If all the intermediate presentations between L and L' are all computable, the equality follows from Lemmas 3.3 and 3.4. Now for the case that some surgery presentations between L_k and L_{k+l} are not computable, we construct a new graph $\tilde{\Gamma}$ and therefore a new $\tilde{\omega}$ so that $L \cup \tilde{\Gamma}$ and $L' \cup \tilde{\Gamma}$ can be connected by computable surgery presentations. By comparing $\Delta(M, \Gamma, \omega)$ and $\Delta(M, \tilde{\Gamma}, \tilde{\omega})$ we can finish the proof. During the construction of $\tilde{\Gamma}$, we need the condition on G.

□

4 Examples and calculations

4.1 A general Formula for a class of Lens spaces

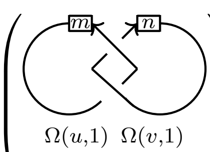
In this section, we compute $\Delta(L(mn-1,n),\omega) := \Delta(L(mn-1,n),\emptyset,\omega)$ for the lens space $L(mn-1,n)$ and $\Gamma = \emptyset$. We use the surgery presentation $L =$ 

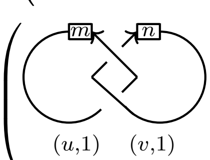
for $L(mn-1,n)$, where m or n inside a square represents the number of positive full twists. Here $\Omega(u,1)$ ($\Omega(v,1)$, resp.) is a Kirby color with $U = 1$ ($V = 1$, resp.); $u = \omega([m_1])$ and $v = \omega([m_2]) \in G$ are given by choosing

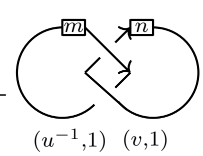
$$\omega : H_1(M,\mathbb{Z}) = \left\langle [m_1],[m_2] \,\middle|\, \begin{pmatrix} m & -1 \\ -1 & n \end{pmatrix} \begin{pmatrix} [m_1] \\ [m_2] \end{pmatrix} = 0 \right\rangle \to G. \tag{3}$$

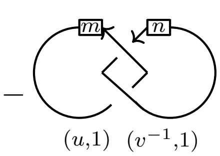
The group operation of G is described by multiplication. In the following calculations, a diagram inside round brackets represents the Alexander polynomial of the diagram. We have

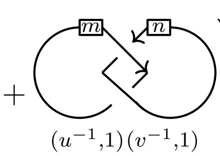
$$\Delta(L(mn-1,n),\omega) = \frac{\Delta(L)}{2^r(-1)^{\sigma_+(L)}}$$

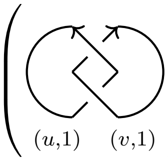
$$= \frac{1}{2^2(-1)^{\sigma_+(L)}}$$ 

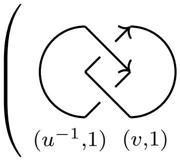
$$= \frac{d(u)d(v)}{4(-1)^{\sigma_+(L)}}$$ 

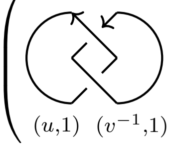
$-$ 

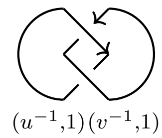
$-$ 

$+$ 

$$= \frac{d(u)d(v)}{4(-1)^{\sigma_+(L)}}\Big[u^{-2m}v^{-2n}$$ 

$- u^{2m}v^{-2n}$ 

$$- u^{-2m}v^{2n} \Bigg(\quad$$ 

$$\Bigg) + u^{2m}v^{2n} \Bigg(\quad$$ 

$$\Bigg) \Bigg]$$

$$= -\frac{d(u)d(v)}{4(-1)^{\sigma_+(L)}} \left(u^{2-2m}v^{2-2n} + u^{2+2m}v^{-2-2n} + u^{-2-2m}v^{2+2n} + u^{-2+2m}v^{-2+2n} \right)$$
$$= -\frac{d(u)d(v)}{4(-1)^{\sigma_+(L)}} (u^2v^{-2n} + u^{-2}v^{2n})(u^{2m}v^{-2} + u^{-2m}v^2).$$

Here note that (3) implies $u^m v^{-1} = 1$, $u^{-1}v^n = 1$. Thus

$$\begin{aligned}
\Delta(L(mn-1,n),\omega) &= -\frac{d(u)d(v)}{4(-1)^{\sigma_+(L)}} (u^2v^{-2n} + u^{-2}v^{2n})(u^{2m}v^{-2} + u^{-2m}v^2) \\
&= -\frac{d(u)d(v)}{4(-1)^{\sigma_+(L)}} ((uv^{-n})^2 + (u^{-1}v^n)^2)((u^m v^{-1})^2 + (u^{-m}v)^2) \\
&= (-1)^{\sigma_+(L)+1} d(u)d(v).
\end{aligned}$$

Then we have

Proposition 4.1.

$$\Delta(L(mn-1,n),\omega) = (-1)^{\sigma_+(L)+1} d(u)d(v).$$

4.2 $L(7,1)$ and $L(7,2)$

It is known that lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent but not homeomorphic. We show that our invariant can distinguish them.

Let $\xi = \exp(\frac{2\pi i}{7})$, $B = \mathbb{Q}(\pi,\xi)$ the field extension of $\mathbb{Q}$ generated by π and ξ, and $G = \mathbb{Z}\langle \pi, \xi \rangle$ the abelian group generated by π and ξ. We consider $\Delta(L(7,1),\omega)$ and $\Delta(L(7,1),\omega)$ for this 1-palette (B,G).

Proposition 4.2. *The invariant $\Delta(M,\omega)$ corresponding to the 1-palette (B,G) where $B = \mathbb{Q}(\pi,\xi)$ and $G = \mathbb{Z}\langle \pi,\xi \rangle$ distinguishes $L(7,1)$ and $L(7,2)$. More concretely, there exists a cohomology class ω_0 for $L(7,1)$ such that for any cohomology class ω for $L(7,2)$, we have*

$$\Delta(L(7,1),\omega_0) \neq \Delta(L(7,2),\omega).$$

Proof. Note that $L(7,1) = L(mn-1,n)$ for $m = 8, n = 1$, and $L(7,2) = L(mn-1,n)$ for $m = 4, n = 2$. So we can apply the discussion we did in Section 5.1.1. A cohomology class

$$\omega : H_1(L(7,2),\mathbb{Z}) \cong \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}\langle \pi,\xi \rangle$$

is determined by $\omega([m_1])$ and $\omega([m_1])$, which satisfy

$$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \omega[m_1] \\ \omega[m_2] \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So we have totally six non-trivial cohomology classes which are given by

$$\omega_1 : \begin{pmatrix} \xi^2 \\ \xi \end{pmatrix}, \omega_2 : \begin{pmatrix} \xi^4 \\ \xi^2 \end{pmatrix}, \omega_3 : \begin{pmatrix} \xi^6 \\ \xi^3 \end{pmatrix}, \omega_4 : \begin{pmatrix} \xi \\ \xi^4 \end{pmatrix}, \omega_5 : \begin{pmatrix} \xi^3 \\ \xi^5 \end{pmatrix}, \omega_6 : \begin{pmatrix} \xi^5 \\ \xi^6 \end{pmatrix}.$$

Let $u_i = \omega_i([m_1])$ and $v_i = \omega_i([m_2])$. By Prop. 4.1 we have

$$\Delta(L(7,2), \omega_i) = -d(u_i)d(v_i).$$

Similarly we can consider the non-trivial cohomology classes for $L(7,1)$. We see that $\omega_0 = \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ is one of them. The corresponding invariant is

$$\Delta(L(7,1), \omega_0) = -d(\xi)d(\xi).$$

We claim that $\Delta(L(7,2), \omega_i) \neq \Delta(L(7,1), \omega_0)$ for $1 \leq i \leq 6$, which can be confirmed by directly calculations. For instance $\Delta(L(7,2), \omega_1) = \Delta(L(7,1), \omega_0) \iff d(\xi^2) = d(\xi) \iff \xi^4 - \xi^{-4} = \xi^2 - \xi^{-2} \iff \xi^2 + \xi^{-2} = 1$, which is impossible since the minimal polynomial of ξ is $\sum_{k=0}^{6} \xi^k = 0$. □

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