Lifting link invariants by nanophrases

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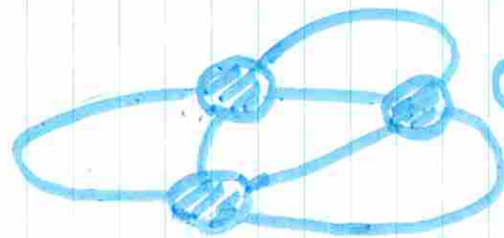
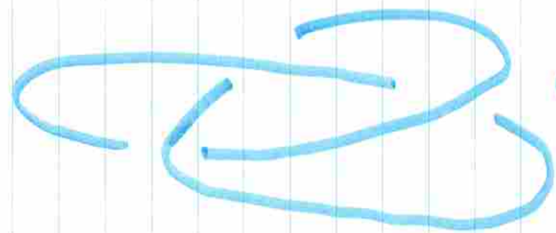
Nanoword (Covering / Covered)

$A \to \alpha$



α : a set

$A \to \alpha_*$

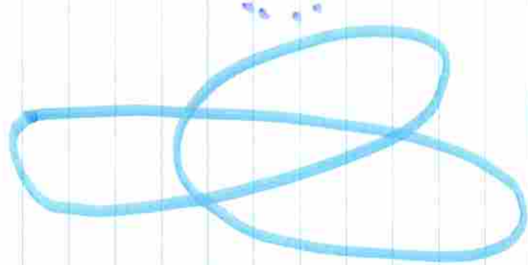


$\alpha_* = \{a_+, a_-, b_+, b_-\}$

forget

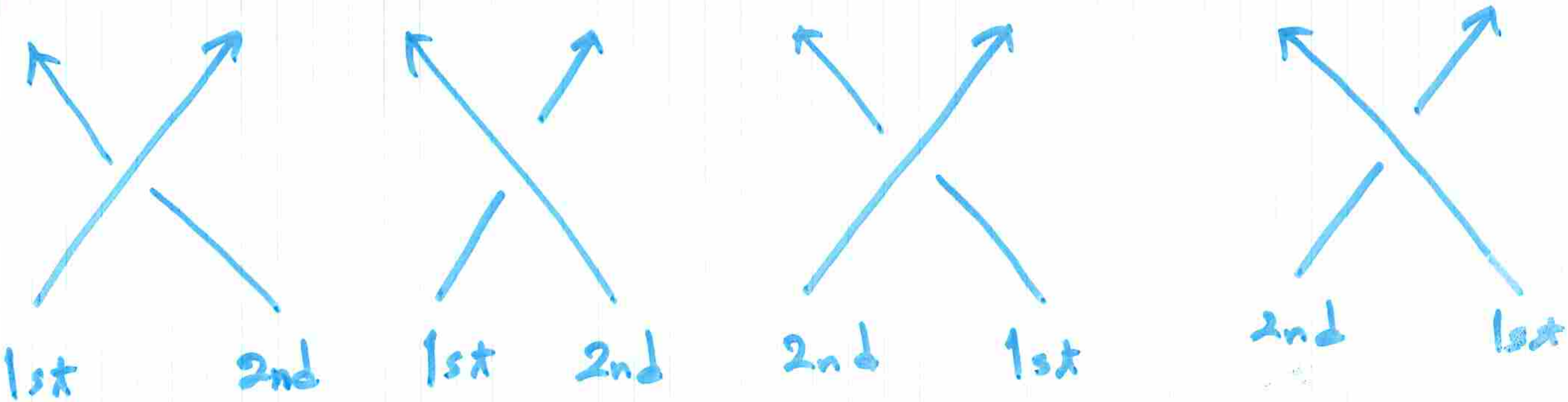
$A \to \alpha_1$

$A \to \alpha_0$



$\alpha_1 = \{+, -\}$ $\alpha_0 = \{a, b\}$

a_+ , a_- , b_+ , b_-



Application

Nanoword (Covering / Covered)

identify

$a_+ \sim b_-$,

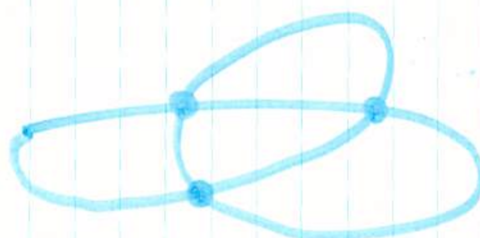
$a_- \sim b_+$

$\alpha_* = \{a_+, a_-, b_+, b_-\}$

$\downarrow$

quasilink

(quandle)



$\alpha_2 = \{c, d\}$

$\downarrow$

free knot

Gauss word $\quad \alpha_G = \{1\}$

Nanowords

Nanophrases (Turaev)

- α : set
- n-component nanophrase over α $(\mathcal{A}, w_1 | w_2 | \cdots | w_n)$

  $w = w_1 w_2 \cdots w_n$ Gauss word, $\mathcal{A}$: set of letters

  proj. $\mathcal{A} \longrightarrow \alpha : A \longmapsto |A|$

- nanoword : 1-component nanophrase
- Empty word $\emptyset$

Homotopy data (α, τ, S)

- τ : involution $\alpha \to \alpha$, i.e. $\tau^2 = id_\alpha$.
- $S \subseteq \alpha \times \alpha \times \alpha$
- H1 $\quad xAAy \longleftrightarrow xy \quad (\text{any } |A|)$
- H2 $\quad xAByBAz \longleftrightarrow xyz \quad (\tau(|A|) = |B|)$
- H3 $\quad xAByACzBCt \longleftrightarrow xBAyCAzCBt$

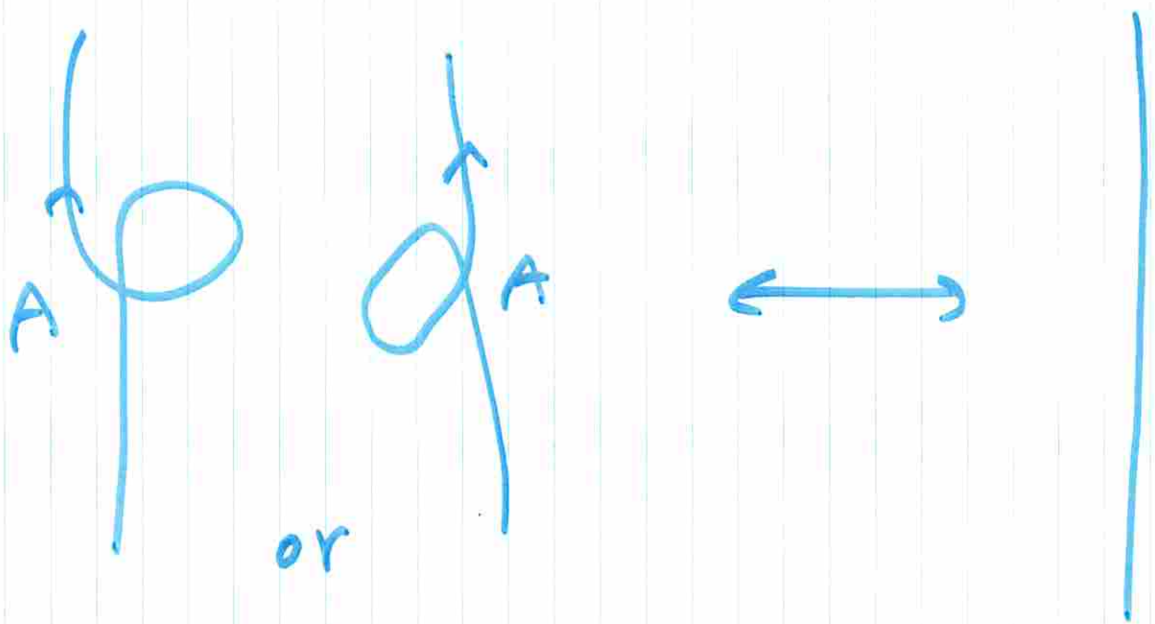
$$((|A|, |B|, |C|) \in S)$$

where x, y, z, t : seq. of letters, A, B, C : letters.

$\rightsquigarrow$ (α, τ, S)-homotopy or S-homotopy

$\alpha = \alpha_0, \ \tau = \tau_0, \ S = S_0$ Case

$xAAy$ $\qquad\qquad$ xy

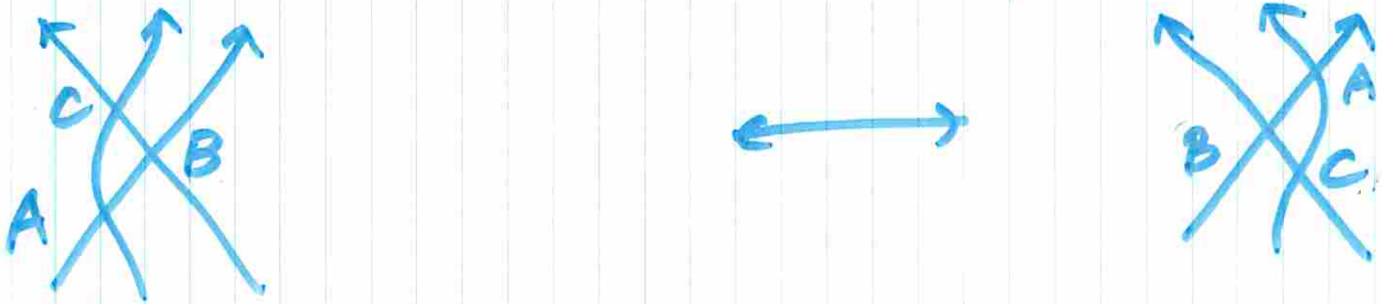


$\alpha = \alpha_0$, $\tau = \tau_0$, $s = s_0$ Case

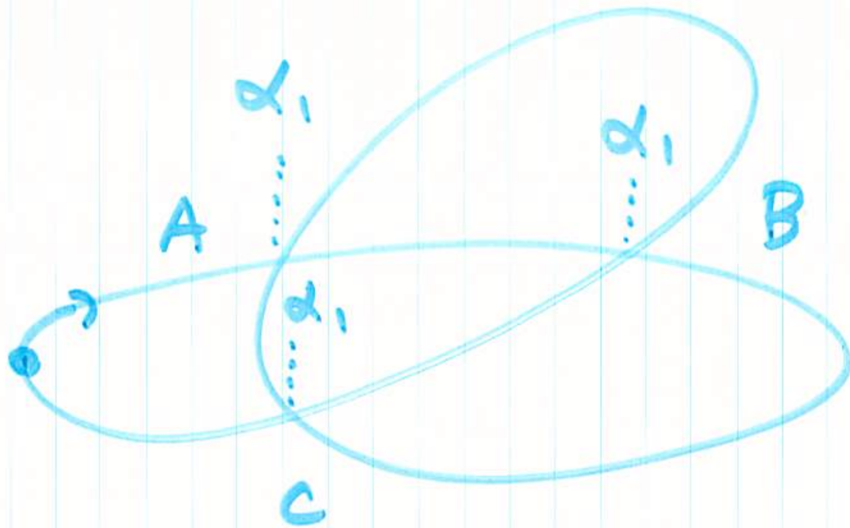
$x\,A\,B\,y\,B\,A\,z$ $\qquad\qquad$ $x\,y\,z$

$\alpha = \alpha_0 \qquad \tau = \tau_0 \qquad s = s_0$ Case

$x\,A\,B\,y\,A\,C\,z\,B\,C\,t \qquad\qquad x\,B\,A\,y\,C\,A\,z\,C\,B\,t$



Example. Nanoword (Pseudolink)



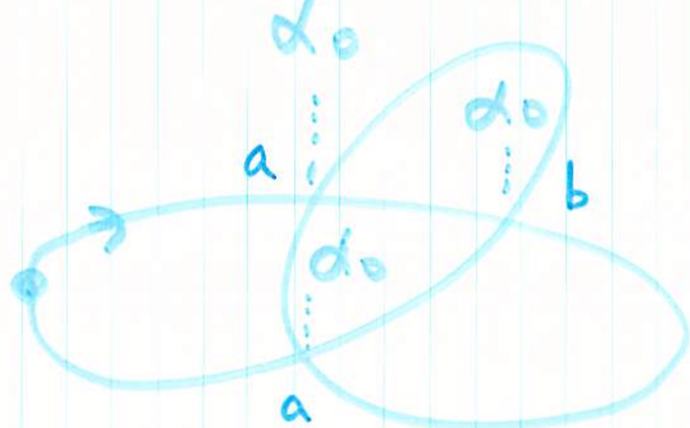
write $\alpha_1 = \{+, -\}$

$$\underset{+\,+\,+\,+\,+\,+}{ABCABC} \Big/ (\alpha_1, \tau_1, S_1)\text{-homotopy}$$

Jones Poly.

$$\langle \text{right trefoil} \rangle = \langle \underset{+\,+\,+\,+\,+\,+}{ABCABC} \rangle$$

Example Nanoword (Virtual String)



intersection sign

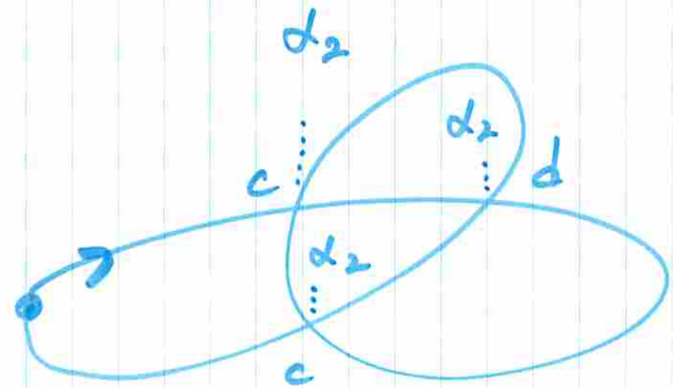
$\alpha_0 = \{a, b\}$

$$\left. \begin{matrix} A & B & C & A & B & C \\ a & b & a & a & b & a \end{matrix} \right/ (\alpha_0, \tau_0, s_0) - \text{homotopy}$$

Turaev cobracket

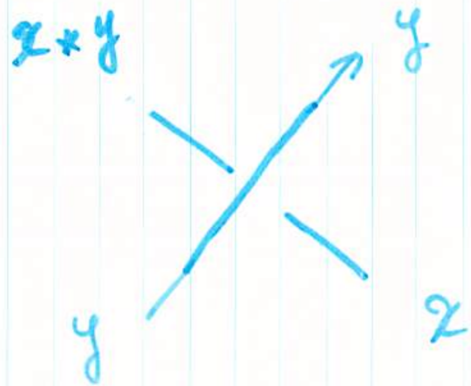
$$\langle C \rangle = \sum_e \left\{ \langle C_e^1 \rangle \otimes \langle C_e^2 \rangle - \langle C_e^2 \rangle \otimes \langle C_e^1 \rangle \right\}$$

Example Nanoword (quasilink)

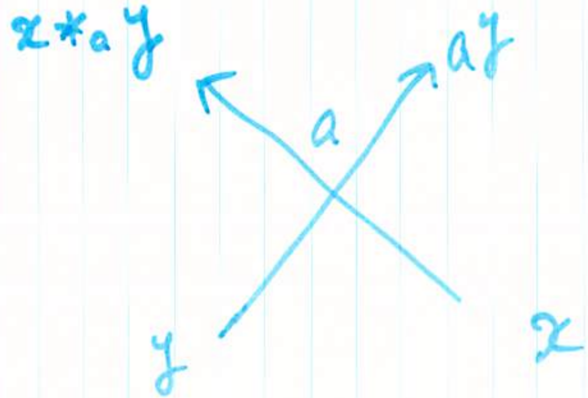


$$\left.\begin{matrix} A\,B\,C\,A\,B\,C \\ c\;d\;c\;c\;d\;c \end{matrix}\right/ (\alpha_2, \tau_2, S_2)\text{-homotopy}$$

Quandle



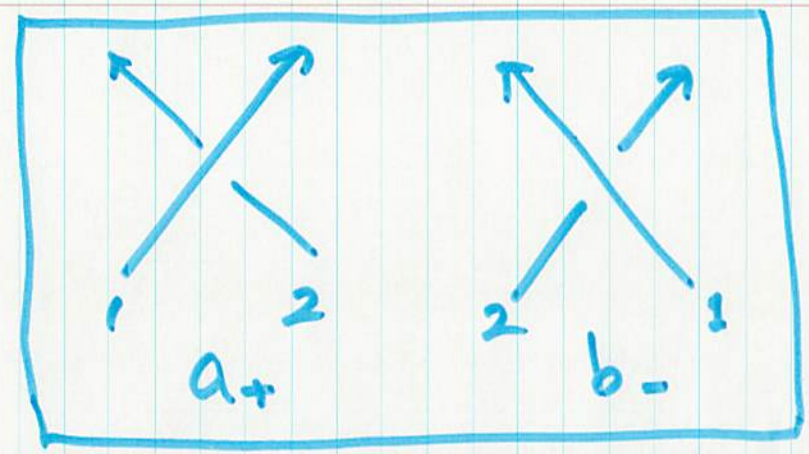
knot

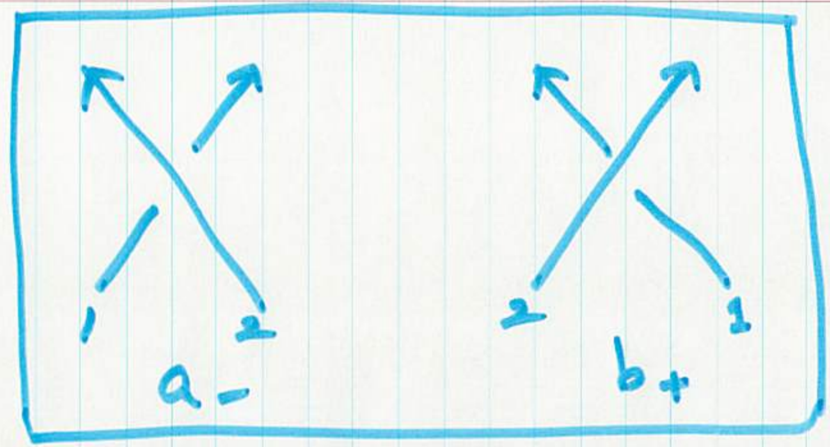


Nanoword

$P(\alpha_*, \tau_*, S_*)$ vs quandle.

β





$\underline{\alpha_*}$ - kei $\hat{K}_\beta(P)$

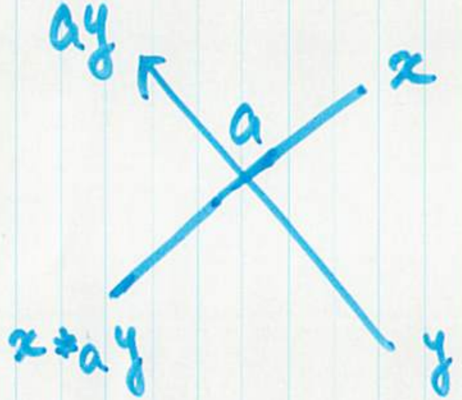
Memo

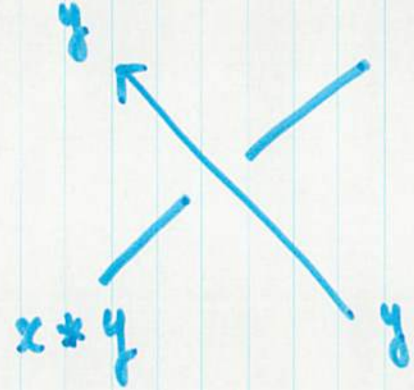
$\alpha_* = \{a\pm, b\pm\}$

$\beta = \{a_+, b_-\}$

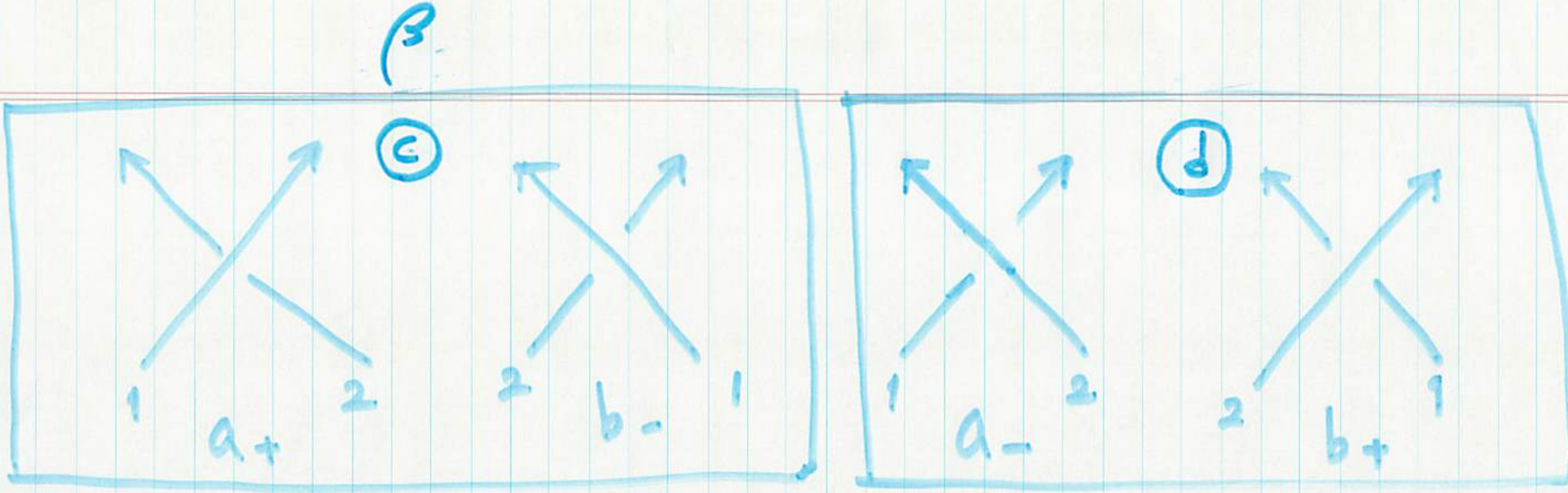
$a_+ \sim b_+, \quad a_- \sim b_-$

$\tau_* : a_\pm \mapsto b_\mp$





$P(\alpha_2, \tau_2, S_2)$ vs quandle



$\underline{\alpha_2}$ - kei $\hat{K}_\beta(P)$

Memo

$\alpha_2 = \{c, d\}$

$\beta = \{c\}$

$c \sim d$

$\tau = id$

Question

- Find two links that are detected by $\underline{\alpha_*}$-kei but not by $\underline{\alpha_2}$-kei.
- Is this pair detected by Alexander poly.?

(Turaev)

(α_0, τ_0, S_0) vs (α_1, τ_1, S_1)

$$\begin{tikzcd}
\alpha_0 = \{a, b\} \arrow[r, "\tau_0"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \alpha_0 \arrow[d] \\
\alpha_1 = \{-, +\} \arrow[r, "\tau_1"] & \alpha_1
\end{tikzcd}$$

$\tau_0(a) = b$

$\tau_0(b) = a$

$\tau_1(+) = -$

$\tau_1(-) = +$

$$S_0 \xhookrightarrow{i} S_1$$

$S_0 = \{(a,a,a), (b,b,b)\}$

$S_1 = \{(+,+,+), (+,+,-), (-,+,+), (-,-,-), (-,-,+), (+,-,-)\}$

- $p \simeq_{S_0} p' \Longrightarrow p \simeq_{S_1} p'$

Fukunaga functor <u>idea</u>

$$\begin{array}{ccc} \alpha & \text{fix point}/\tau & \Delta_\alpha = \{(a,a,a) \mid a \in \alpha\} \\ \downarrow & \downarrow & \downarrow \\ \alpha_0 = \{a, b\} & 0 & S_0 = \{(a,a,a), (b,b,b)\} \end{array}$$

<u>Application</u> Invariant for $(\alpha_0, \tau_0, S_0) \rightsquigarrow$ Inv. for $(\alpha, \tau, \Delta\alpha)$

Further, $\left.\begin{array}{l}\text{Jones polynomial} \\ \text{Khovanov homology}\end{array}\right\}$ for $(\alpha, \tau, \Delta\alpha)$ are given.

Fukunaga functor <u>idea</u>

$$\begin{array}{ccccc} \alpha & & \text{fix point}/\tau & & \Delta_\alpha = \{(a,a,a) \mid a \in \alpha\} \\ \downarrow & & \downarrow & & \downarrow \\ \alpha_0 = \{1,-1\} & & 0 & & S_0 = \{(1,1,1),(-1,-1,-1)\} \end{array}$$

<u>Application</u> Invariant for $(\alpha_0, \tau_0, S_0) \rightsquigarrow$ Inv. for $(\alpha, \tau, \Delta_\alpha)$

Further, $\left.\begin{array}{l}\text{Jones polynomial} \\ \text{Khovanov homology}\end{array}\right\}$ for $(\alpha, \tau, \Delta_\alpha)$ are given.

Fukunaga Functor (from Δ_α to S_0)

Orbit $\alpha/\tau = \{a_1, a_2, \cdots, a_m\}$, Fix $L \subset \alpha/\tau$.

$$\mathrm{sign}_L(A) = \begin{cases} 1 & |A| \in L \ : \text{free} \\ -1 & |A| \in \tau(L) \ : \text{free} \\ 0 & \text{otherwise} \end{cases}$$

1°. Remove $\mathrm{sign}_L = 0$ letter.

2°. Replace A with B $|B| = \mathrm{sign}_L(A)$.

$\rightsquigarrow$ Phrase $P \mapsto \mathcal{U}_L(P)$

Theorem A (Fukunaga)

$$(A_1, P_1) \simeq_{\Delta\alpha} (A_2, P_2)$$

$$\Rightarrow \mathcal{U}_L(A_1, P_1) \simeq_{S_0} \mathcal{U}_L(A_2, P_2).$$

Proof.

(H1)

$$\begin{tikzcd} xAAy \arrow[r, "\mathrm{sign}_L A = 0"] \arrow[dr, "\mathrm{sign}_L A \neq 0"'] & xy \\ & xAAy \simeq xy \end{tikzcd}$$

(H2)

$$\begin{tikzcd} xAByBAz \arrow[r, "\mathrm{sign}_L A = 0"] \arrow[dr, "\mathrm{sign}_L A \neq 0"'] & xyz \\ & xAByBAz \simeq xyz \end{tikzcd}$$

$\tau(|A|) = |B|$

(H3)

$$\begin{tikzcd} xAByAczBct \arrow[r, "\mathrm{sign}_L A = 0"] \arrow[d, "\mathrm{sign}_L A \neq 0"] & xyz & xBAycAzCBt \arrow[l, "\mathrm{sign}_L A = 0"'] \arrow[d, "\mathrm{sign}_L A \neq 0"] \\ xAByAczBCt & \simeq & xBAyCAzCBt \end{tikzcd}$$

$|A| = |B| = |C|$

Lifting invariants

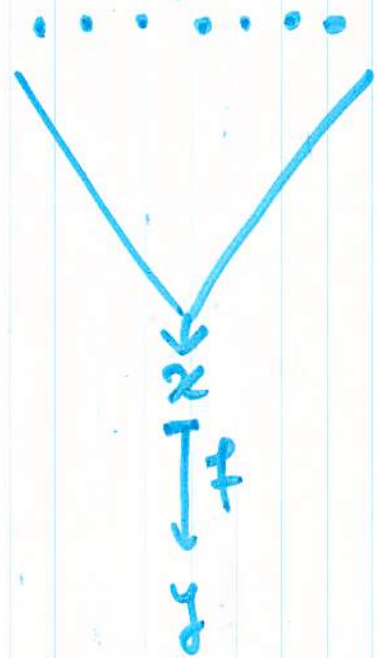
i, p (Turaev)

$$
\begin{tikzcd}
\text{general objects} \arrow[d, "\mathcal{U}"', "\text{Thm A (Fukunaga)}"'{xshift=-30pt}] & & \text{objects} \arrow[d, "p"] \arrow[dd, bend left=60, "\text{Jones poly.}"] \\
\text{objects} \arrow[d, "\bar{f}"] \arrow[rr, hook, "\bar{i}"] & & \text{objects} \arrow[d, "f"] \\
\text{poly.} & & \text{poly.}
\end{tikzcd}
$$

Lifting invariants

Motivation

$$
\begin{array}{rc}
 & \text{general objects} \\
\text{ThmB} & \Big\downarrow g \\
\text{Fukunaga-I.} & \text{objects} \\
 & \Big\downarrow f \\
 & \text{poly.}
\end{array}
$$



Application

Jones poly. for nanophrases over any set α

Nanophrase construction

$$\begin{tikzcd}
\text{Jones poly. (links)} \arrow[d] & \\
\text{Jones poly. (virtual links)} \arrow[r, phantom, "\overset{\text{Turaev}}{=}"] & \boxed{\text{Jones poly. (pseudo links)}} \arrow[d, squiggly] \\
\text{Jones poly. } (\alpha, \tau, \Delta_\alpha) & \text{Jones poly. (virtual strings)} \arrow[l, "U_L"']
\end{tikzcd}$$

Prop. J : Jones poly., p, p' : nanophrase over $(\alpha, \tau, \Delta_\alpha)$.

$$p \simeq_{\Delta_\alpha} p' \Longrightarrow J(U_L(p)) = J(U_L(p')).$$

Rem. This approach works for Khovanov homology.

Cor.

Let I be S_0-homotopy inv. over α_0

$\Longrightarrow$ $I(U_L(P))$ is Δ_α-homotopy inv.
over any α.

In particular,

$J(U_L(P))$ is Δ_α-homotopy inv.
over any α.

- $(\alpha, \tau, \Delta a)$ - homotopy ... achieved. ✓

  ↳ A good <u>generalization beyond</u> that?

Such U_L implies questions.

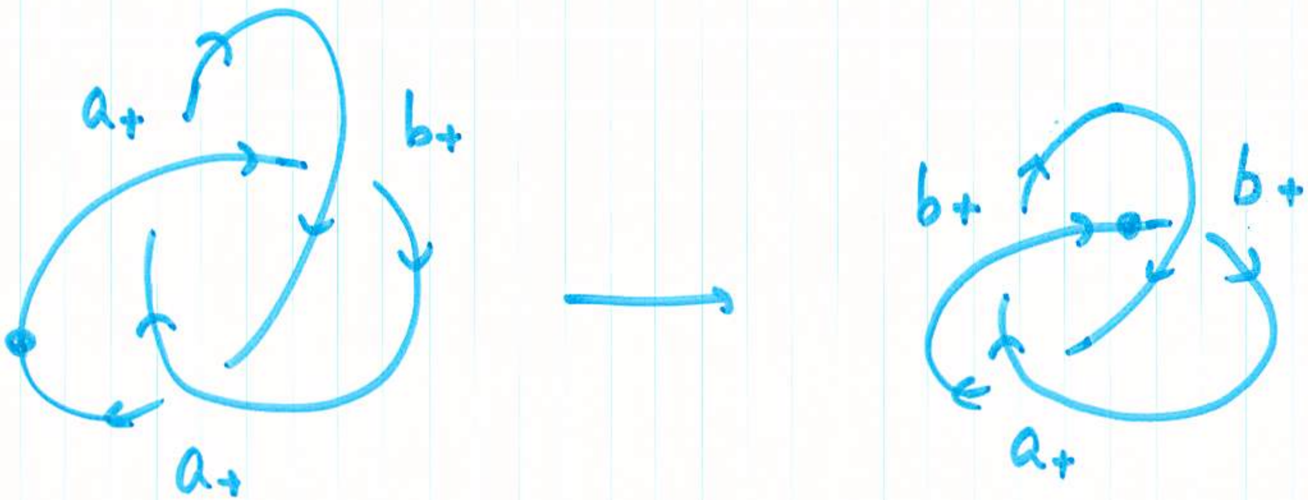
Q1. What nanophrase makes Jones polynomial trivial?

Q2. $\underbrace{\text{Q1}}$ is generalised to (f, y).

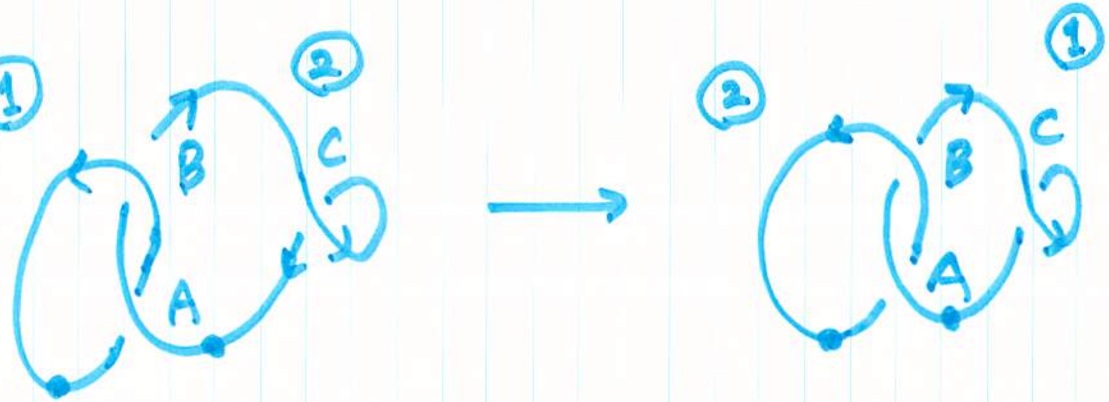
(f = Jones J, Value $y = 1$)

ν - involution

e.g. $\qquad \nu(a_\pm) = b_\pm .$



ν - permutation (ν: involution)



$P_1 = (\{A, B\},\ AB \mid ABCC)$ $\qquad$ $P_2 = (\{A, B\},\ ABCC \mid AB)$

$$|A| = a_- \overset{\nu}{\longmapsto} |A| = b_-$$

$$|B| = b_- \longmapsto |B| = a_-$$

$$|C| = b_- \longmapsto |C| = b_-$$

Lifting invariants

i, p (Turaev)

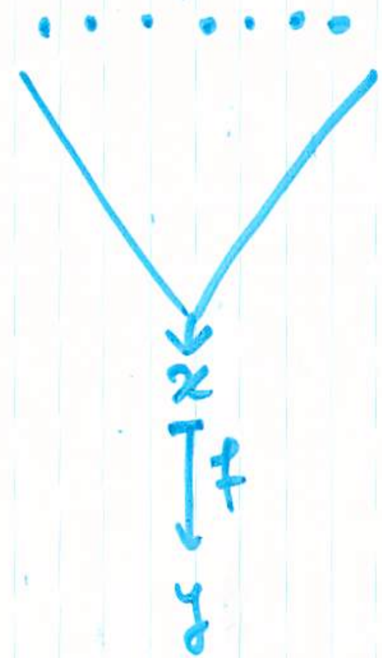
$$\begin{tikzcd} \text{general objects} \arrow[d, "\mathcal{U}", "\text{Thm A (Fukunaga)}"'] & & \text{objects} \arrow[d, "p"] \arrow[dd, bend left=60, "\text{Jones poly.}"] \\ \text{objects} \arrow[rr, hook, "\bar{i}"] \arrow[d, "\bar{f}"] & & \text{objects} \arrow[d, "f"] \\ \text{poly.} & & \text{poly.} \end{tikzcd}$$

Lifting invariants

ThmB
Fukunaga-I.

$$\begin{tikzcd}
\text{general objects} \arrow[d, "g"'] \\
\text{objects} \arrow[d, "f"'] \\
\text{poly.}
\end{tikzcd}$$

Motivation



Knotlike homotopy $S_\#$

α : set , ν, τ : involutions

$$S_\# := \{ (a,a,a),\ (a,a,\nu\tau(a)),\ (a,\nu\tau(a),\nu\tau(a)) \mid a \in \alpha \}$$

e.g.1 link, virtual link

e.g.2 pseudolink

e.g.3. quasilink

e.g.4 virtual string

e.g.5. Gauss phrase

Phrase set $P(\alpha_\diamond, \tau_\diamond, S_\diamond, \nu_\diamond)$

$\diamond \in \{*, 0, 1, 2, G\}$. ν, τ : involution, given.

Let

$$L_* = \{a \in L \mid a \neq \tau(a), a \neq \nu(a), \tau(a) \neq \nu(a)\},$$

$$L_0 = \{a \in L \mid a \neq \tau(a), a \neq \nu(a), \tau(a) = \nu(a)\}$$

$$L_1 = \{a \in L \mid a \neq \tau(a), a = \nu(a)\},$$

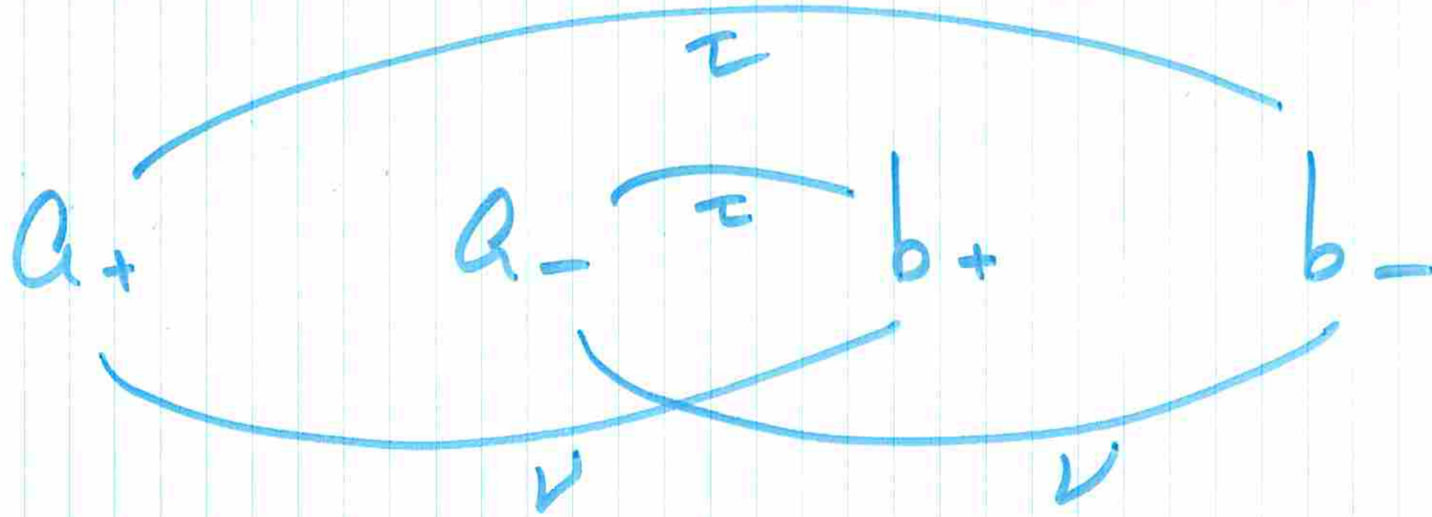
$$L_2 = \{a \in L \mid a = \tau(a), a \neq \nu(a)\},$$

$$L_G = \{a \in L \mid a = \tau(a), a = \nu(a)\}.$$

$\forall A \in \mathcal{A}$, $P_*(A) = \begin{cases} a+ & |A| \in L_* \\ a- & |A| \in \nu(L_*) \\ b+ & |A| \in \tau\nu(L_*) \\ b- & |A| \in \tau(L_*) \end{cases}$, $P_0(A) = \begin{cases} a & |A| \in L_0 \\ b & |A| \in \tau(L_0) \end{cases}$

$P_1(A) = \begin{cases} +1 & |A| \in L_1 \\ -1 & |A| \in \tau(L_1) \end{cases}$, $P_2(A) = \begin{cases} c & |A| \in L_2 \\ d & |A| \in \nu(L_2) \end{cases}$, $P_G(A) = a \ (|A| \in L_G)$.

$\Diamond = *$ Type e.g. link



cf.

$$L_* = \{ a \in L \mid a \neq \tau(a),\ a \neq \nu(a),\ \tau(a) \neq \nu(a) \}$$

$\Diamond = O$ Type   e.g. virtual string

$$\begin{tikzcd} a \arrow[r, bend left, leftrightarrow, "\tau"] \arrow[r, bend right, leftrightarrow, "\nu"'] & b \end{tikzcd}$$

cf. $L_0 = \{ a \in L \mid a \neq \tau(a),\ a \neq \nu(a),\ \tau(a) = \nu(a) \}$

$\diamondsuit = 1$ Type   e.g. pseudolink

$$
\begin{tikzcd}
+ \arrow[rr, bend left, leftrightarrow, "\tau"] \arrow[loop below, "\nu"] & & - \arrow[loop below, "\nu"]
\end{tikzcd}
$$

cf.
$$L_1 = \{ a \in L \mid a \neq \tau(a),\ a = \nu(a) \}$$

$\Diamond = 2$ Type e.g. quasi link

$$
\begin{tikzcd}
c \arrow[loop above, "\tau"] \arrow[rr, leftrightarrow, bend right, "\nu"'] & & d \arrow[loop above, "\tau"]
\end{tikzcd}
$$

cf.

$$L_2 = \{ a \in L \mid a = \tau(a),\ a \neq \nu(a) \}$$

$\diamondsuit = G$ Type   e.g. Gauss phrase

$$\begin{tikzcd} a \arrow[loop above, "\tau"] \arrow[loop below] \end{tikzcd}$$

cf. $L_G = \{ a \in L \mid a = \tau(a),\ a = \nu(a) \}$

Jones extented to $P(\alpha, \tau, S_\#, \nu)$.

1°. Remove A s.t. $|A| \notin \eta_\diamond = L_\diamond \cup \tau(L_\diamond) \cup \nu(L_\diamond) \cup \tau\nu(L_\diamond)$

2°. $P_\diamond$ defines the projection $|F_*(A)|$.

<u>Theorem B</u> $F_\diamond : P(\alpha, \tau) \longrightarrow P(\alpha_\diamond, \tau_\diamond)$

induces $F_{\diamond\bullet} : P(\alpha, \tau, S_\#, \nu) \longrightarrow P(\alpha, \tau_\diamond, S_\diamond, \nu_\diamond)$

for $\diamond \in \{*, 0, 1, 2, G\}$

<u>Cor.</u> $J \circ F_*$ is $S_\#$-homotopy inv. for any α.

(Case $\diamond = *$) (extended Jones poly.)

Future work

$$
\begin{tikzcd}
{} \arrow[d, "\tilde{U}_{\tilde{L}}"'] & & {} \arrow[dd, "{}^{\exists}F"'] \\
\cdot \arrow[d, "U_L"'] & \overset{?}{\rightsquigarrow} & \\
\mathrm{object} \arrow[d, "f"'] & & \mathrm{object} \arrow[d, "f"'] \\
\cdot & & \cdot
\end{tikzcd}
$$

Future work

$$
\begin{tikzcd}
{} \arrow[d, "\tilde{U}_{\tilde{L}}"'] & & {} \arrow[dd, "\overset{?}{\exists} F"'] \\
\cdot \arrow[d, "U_L"'] \arrow[rr, squiggly, "?"] & & {} \\
\text{object} \arrow[d, "f"'] & & \text{object} \arrow[d, "f"'] \\
\cdot & & \cdot
\end{tikzcd}
$$

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