A DISTANCE ON THE EQUIVALENCE CLASSES OF SPHERICAL CURVES GENERATED BY DEFORMATIONS OF TYPE RI

YUKARI FUNAKOSHI, MEGUMI HASHIZUME, NOBORU ITO, TSUYOSHI KOBAYASHI, AND HIROKO MURAI

Abstract. In this paper, we introduce a distance $d_{w3}$ on the equivalence classes of spherical curves under deformations of type RI and ambient isotopies. We obtain an inequality that estimate its lower bound (Theorem 1). In Theorem 2, we show that if for a pair of spherical curves $P$ and $P'$, $d_{w3}([P],[P']) = 1$ and $P$ and $P'$ satisfy a certain technical condition, then $P'$ is obtained from $P$ by a single weak RI I only. In Theorem 3, we show that if $P$ and $P'$ satisfy other conditions, then $P'$ is ambient isotopic to a spherical curve that is obtained from $P$ by a sequence of a particular local deformations, which realizes $d_{w3}([P],[P'])$.

1. Introduction

A spherical curve is the image of a generic immersion of a circle into a 2-sphere. Any two spherical curves can be transformed each other by a finite sequence of deformations, each of which is either one of type RI, type RI I, or type RI I I that is a replacement of a part of the curve contained in a disk as in Figure 1, and ambient isotopies. These deformations are obtained from Reidemeister moves of type $\Omega_1$, type $\Omega_2$, and type $\Omega_3$ on knot diagrams by ignoring over/under information.

Viro [8] suggested to decompose $RIII$ into the following two types: suppose that $P_1$ is transformed into $P_2$ by a deformation of type $RIII$. Note that in $RIII$ of Figure 1, a triangle is observed in each of the disks. We say that $P_1$ and $P_2$ are related by a strong $RIII$ if the orientations on the edges of the triangles induced by an orientation of the spherical curves are coherent. If the orientations are not coherent, then we say that $P_1$ and $P_2$ are related by a weak $RIII$. See Figure 2.

Let $C$ be the set of ambient isotopy classes of spherical curves. We say that two elements $v$ and $v'$ of $C$ are RI-equivalent, denoted by $v \sim_{RI} v'$ if there are representatives $P$, $P'$ of $v$, $v'$ respectively such that $P'$ is obtained from $P$ by a
sequence of deformations of type RI and ambient isotopies. We note that $\sim_{RI}$ is an equivalence relation on $\mathcal{C}$ (see the proof of Proposition 1.1). Then $\hat{\mathcal{C}}$ denotes the quotient set $\mathcal{C}/\sim_{RI}$ and for a spherical curve $P$, $[P]$ denotes the quotient containing $P$. Then we obtain a 1-complex, denoted by $\hat{\mathcal{K}}_{w3}$ by:

- the set of vertices of $\hat{\mathcal{K}}_{w3}$ corresponds to $\hat{\mathcal{C}}$, and
- two vertices $v$ and $v'$ are joined by an edge if there are representatives $P$ and $P'$ of $v$ and $v'$ respectively such that $P'$ is obtained from $P$ by a sequence consisting of one deformation of type weak RI III, and some (possibly, empty) deformation(s) of type RI and ambient isotopies.

Then $\hat{d}_{w3}$ denotes the path-metric distance of $\hat{\mathcal{C}}$ induced by $\hat{\mathcal{K}}_{w3}$. That is, for each pair of vertices $v$ and $v'$, we define $\hat{d}_{w3}$ by:

$$\hat{d}_{w3}(v, v') = \min\{\text{the number of the edges of } J \mid J : \text{path in } \hat{\mathcal{K}}_{w3} \text{ joining } v \text{ and } v'\}$$

Let $P$ and $P'$ be spherical curves. We say that $P$ and $P'$ are equivalent under RI and weak RI III if there are spherical curves $\tilde{P}$ and $\tilde{P}'$ such that $\tilde{P}$ is ambient isotopic to $P$, $\tilde{P}'$ is ambient isotopic to $P'$ and $\tilde{P}'$ is obtained from $\tilde{P}$ by a sequence of deformations each of which is of type RI or type weak RI III. Then, we define $d_{w3}(P, P')$ by:

$$d_{w3}(P, P') = \min \left\{ \begin{array}{l} \text{the number of weak RI III's} \mid \text{in the sequence of} \ P \to \tilde{P} \\ \text{deformations from } \tilde{P} \text{ to } \tilde{P}' \end{array} \right\}$$

If $P$ and $P'$ are not equivalent under RI and weak RI III, let $d_{w3}(P, P') = \infty$. Maybe the next proposition is well-known to the experts, but it will be worth to give a concrete statement for understanding the proofs of the theorems of this paper.

**Proposition 1.1.** For any pair of spherical curves $P$, $P'$ which are equivalent under RI and weak RI III, we have:

$$d_{w3}(P, P') = \hat{d}_{w3}([P], [P'])$$

For a proof of this proposition, see Appendix.

Some equivalence classes under RI and weak RI III are studied by [1, 2, 3, 4]. Let $\mathcal{P}_{\leq 7}$ be the set of the ambient isotopy classes of all of the prime spherical curves with at most seven double points. In Figure 45 of [3], there is a diagram consisting of certain elements of $\mathcal{P}_{\leq 7}$, which describes how the elements are related by a weak or strong RI III. The elements are named after the notations in Rolfsen’s table [7].
However, we should note that the spherical curves of the diagram are treated up to mirror image. (For example, the spherical curve denoted \(7_6\) is transformed into the mirror image of \(6_3\) in the diagram, and not transformed into \(6_3\) itself by a single deformation of type weak RI and some deformations of type RI, nevertheless the diagram in [3] says that \(7_6\) is transformed into \(6_3\) by a single weak RI and some RI’s.) For a spherical curve \(P\), we use \(P^*\) to denote the mirror image of \(P\). Figure 3 is a similar diagram obtained from all of the elements of \(\mathcal{P}_{\leq 7}\). (We note that, by using elementary geometric arguments together with Fact 3.2 in Section 3, it is easy to show that \(6_3\) is not ambient isotopic to \(6_5\), that \(7_6\) is not ambient isotopic to \(7_6^*\), that \(7_7\) is not ambient isotopic to \(7_5\), and that \(\mathcal{P}_{\leq 7}\) consists of the 21 elements in Figure 3. A systematic proof of this fact will be found in [5].) In [1], an idea for detecting spherical curves which are not equivalent under RI and weak RI by using positive knot diagrams is introduced (for details, see [1, Corollary 3.2]). By passing through spherical curves each with eight double points [4, Figure 6], and this shows that the quotient set consists of nine elements, depicted in Figure 3.

In this paper, as a sequel of these researches, we study the distance \(d_{w3}\). We obtain an inequality that estimate its lower bound (Theorem 1). Further, we show that if \(d_{w3}(\{P\}, \{P'\}) = 1\) and \(P\) and \(P'\) satisfy a certain technical condition, then \(P'\) is obtained from \(P\) by a single weak RI only (Theorem 2). Theorem 2 can be regarded as a kind of “rigidity” of the deformations from \(P\) to \(P'\). In Theorem 3, we give a similar result under a different setting. We say that \(P'\) is obtained from \(P\) by a deformation of type \(\alpha\), if \(P'\) is obtained by replacing a part of \(P\) contained in a disk as in Figure 4. Then we show that if the pair \(P, P'\) satisfies other conditions, then \(P'\) is obtained from \(P\) by applying a sequence of deformations of type \(\alpha\) and ambient isotopies, and this sequence realizes \(d_{w3}(\{P\}, \{P'\})\) (Theorem 3).

Further by using the results, we study the distance of each pair of elements in the equivalence classes of Figure 3. We note that we can show, by Remark 3.9 of Section 3, that the spherical curves in Figure 3 are not mutually RI-equivalent. Since each of the equivalence classes (1), (4), (5), (7), (9) has only one element, this problem is trivial for these equivalence classes. Since each of the equivalence classes (2), (3) consists of two elements and they are related by a single weak RI and RI, this problem is solved for these equivalence classes. For the equivalence class (8), we will see that \(d_{w3}([7_5], [7_5]) > 1\) (Example 2.16). On the other hand, by [3, Figure 45] we see that \(d_{w3}([7_5], [7_5]) \leq 2\). Hence we have \(d_{w3}([7_5], [7_5]) = 2\). For the equivalence class (6), we show: For each pair \(P, P'\) of elements in the equivalence class (6), \(d_{w3}(\{P\}, \{P'\})\) is realized by the minimal number of arrows in the paths joining \(P\) and \(P'\) in the diagram (Theorem 4).

2. Preliminaries

**Definition 2.1 (Gauss word)**. Let \(\hat{n} = \{1, 2, 3, \ldots, n\}\). A word \(w\) of length \(n\) is a map from \(\hat{n}\) to \(\mathbb{N}\). The word is represented by \(w(1)w(2)w(3)\cdots w(n)\). Then, we call each element of \(w(\hat{n})\) a letter. A Gauss word of length \(2n\) is a word \(w\) of length \(2n\) satisfying that each letter in \(w(2n)\) appears exactly twice in \(w(1)w(2)w(3)\cdots w(2n)\).
Figure 3. The elements of $\mathcal{P}_{\leq 7}$. Fourteen of them are obtained from the knot table of Rolfsen [7] by ignoring over/under informations of the crossings. We assign them the same symbols as Rolfsen’s. The spherical curve $7_A$ ($7_B$, $7_C$ resp.) is obtained from $7_6$ ($7_7$, $7_5$ resp.) by a single flype. The arrow from a spherical curve $P$ to a spherical curve $P'$ means that $P$ is transformed into $P'$ by a transformation consisting of one deformation of type positive weak $\text{RI}$ (see Definition 2.14) and some deformation(s) of type $\text{RI}$.

![Diagram](image)

Figure 4
Let \( \text{cyc} \) and \( \text{rev} \) be maps \( 2^n \to 2^n \) where \( \text{cyc}(p) \equiv p + 1 \mod 2n \) and \( \text{rev}(p) \equiv -p + 1 \mod 2n \). Two Gauss words, \( v \) and \( w \), of length \( 2n \) are isomorphic if there exists a bijection \( f : v(2n) \to w(2n) \) satisfying that there exists \( t \in \mathbb{Z} \) such that \( w \circ (\text{cyc})^t \circ (\text{rev})^t = f \circ v \) (\( \epsilon = 0 \) or \( 1 \)). The isomorphisms give an equivalence relation on the Gauss words. For a Gauss word \( v \) of length \( 2n \), the equivalence class containing \( v \) is denoted by \([v]\). A Gauss word \( v' \) is called a sub-Gauss word of the Gauss word \( v \) if \( v' \) is obtained from \( v \) by ignoring some letters of \( v \). Then, the set of sub-Gauss words of \( v \) is denoted by \( \text{Sub}(v) \).

**Definition 2.2** (chord diagram). A chord diagram is a configuration of \( n \) pair(s) of points on a circle up to ambient isotopy and reflection of the circle. Traditionally, two points of each pair are connected by a (straight) arc. This arc is called a chord. We say that a chord in a chord diagram is isolated if there is no chord transversely intersecting the chord.

We note that the equivalence classes of the Gauss words of length \( 2n \) have one to one correspondence with the chord diagrams consisting of \( n \) chords as in Figure 5. In this paper, we identify these four expressions in Figure 5, and freely use either

![Figure 5. Four expressions.](image)

one of them depending on situations.

**Definition 2.3** (a chord diagram \( CD_P \) of a spherical curve \( P \)). Let \( P \) be a spherical curve. Then, there is a generic immersion \( g : S^1 \to S^2 \) such that \( g(S^1) = P \). We define a chord diagram of \( P \) (e.g., Figure 6) as follows: Let \( k \) be the number of the double points of \( P \), and \( m_1, m_2, \ldots, m_k \) mutually distinct positive integers. For

![Figure 6. A chord diagram \( CD_P \) of a spherical curve \( P \).](image)
Let $A$ be a spherical curve. For a spherical curve $P$, the quotient set $\mathcal{P}$ of its set of double points is nontrivial and is not a connected sum of two nontrivial spherical curves. $P$ is a double point (or a double point assigned $i$) if the connected components of the union of the chords of $P$ is denoted by $CD_P$. Following the same procedure, we finally label all the double points of $P$. We note that $g^{-1}(\text{double point assigned } m_i)$ consists of two points on $S^1$ and we shall assign $m_i$ to them. The chord diagram represented by $g^{-1}(\text{double point assigned } m_1)$, $g^{-1}(\text{double point assigned } m_2)$, ..., and $g^{-1}(\text{double point assigned } m_k)$ on $S^1$ is denoted by $CD_P$, and is called a chord diagram of the spherical curve $P$. Clearly if $P'$ is a spherical curve that is ambient isotopic to $P$, then $CD_P = CD_{P'}$ as chord diagrams.

We note that $P \rightarrow CD_P$ induces a map from $\mathcal{C}$ to chord diagrams. Recall that $CD_P$ is identified with an equivalence class of Gauss words, say $[v_P]$. By these facts, we see that there is a map from $\mathcal{C}$ to the set of the equivalence classes of the Gauss words $P \rightarrow [v_P]$.

**Definition 2.4.** Let $CD$ be a chord diagram. Then $f_c(CD)$ denotes the number of the connected components of the union of the chords of $CD$. For a spherical curve $P$, $f_c(P)$ denotes $f_c(CD_P)$.

**Definition 2.5** (connected sum). Let $P_i$ ($i = 1, 2$) be a spherical curve. Suppose that the ambient 2-spheres are oriented. Let $p_i$ be a point on $P_i$ where $p_i$ is not a double point ($i = 1, 2$). Let $d_i$ be a sufficiently small disk with center $p_i$ ($i = 1, 2$) where $d_i \cap P_i$ consists of an arc properly embedded in $d_i$. Let $d_i = cl(S^2 \setminus d_i)$ and $P_i = P_i \cap d_i$. Let $h : \partial d_1 \rightarrow \partial d_2$ be an orientation reversing homeomorphism such that $h(\partial P_1) = \partial P_2$. Then, $P_1 \cup_h P_2$ is a spherical curve in the oriented 2-sphere $d_1 \cup_h d_2$. The spherical curve $P_1 \cup_h P_2$ in the oriented 2-sphere is denoted by $P_1 \#_{(p_1, p_2), h} P_2$ (or, simply $P_1 \# P_2$). This spherical curve $P_1 \#_{(p_1, p_2), h} P_2$ is called a connected sum of the spherical curves $P_1$ and $P_2$ at the pair of points $p_1$ and $p_2$ (see Figure 7).

![Figure 7](image-url)  

**Figure 7.** Two spherical curves $P_1$ and $P_2$ and a connected sum $P_1 \#_{(p_1, p_2), h} P_2$.

**Definition 2.6** (trivial spherical curve, prime spherical curve). A spherical curve $P$ is **trivial** if $P$ is a simple closed curve. A spherical curve $P$ is **prime** if $P$ is nontrivial and is not a connected sum of two nontrivial spherical curves.

**Remark 2.7.** For a spherical curve $P$, it is easy to see that $f_c(P)$ is the number of prime factors of $P$.

Recall that $\mathcal{C}$ denotes the ambient isotopy classes of spherical curves and $\tilde{\mathcal{C}}$ the quotient set $\mathcal{C}/\sim_{RI}$, where $[P]$ denotes the element of $\tilde{\mathcal{C}}$ represented by the spherical curve $P$.

**Definition 2.8** ($x(CD)$). Let $x$ be a chord diagram. For a chord diagram $CD$, fix a Gauss word $G$ representing $CD$. Then let $\text{Sub}_x(G) = \{H \mid H \in \text{Sub}(G), \ [H] = x \}$, where $\text{Sub}(G)$ denotes the set of the sub-Gauss words of $G$, defined in Definition 2.1. The cardinality of this subset is denoted by $x(G)$, that is, $x(G) = \sharp \text{Sub}_x(G)$. By the definition of the isomorphic Gauss words, for another Gauss
word $G'$ representing $CD$, it is easy to see $x(G') = x(G)$. Hence, we shall denote the number by $x(CD)$. If $CD$ is a chord diagram of a spherical curve $P$, then $x(P)$ denotes $x(CD)$. Clearly if $P$ is ambient isotopic to $P'$, then $x(P) = x(P')$. This show that $x$ induces a map from $\mathcal{C}$ to $\mathbb{Z}_{\geq 0}$, which will be also denoted by $x(\cdot)$.

Because each equivalence class of the Gauss words is identified with a chord diagram, we can calculate the number $x(CD)$ by using geometric observations. We explain this philosophy in Example 2.9.

**Example 2.9.** We consider the chord diagram $CD$ in Figure 8 (Note that $CD = CD_P$ in Figure 6), and we label the chords of $CD$ by $\alpha_i$ $(1 \leq i \leq 6)$ as in Figure 8. We consider the subset of the power set of $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$, each element of which represents a chord diagram isomorphic to $\otimes$. It is elementary to see that this subset consists of ten elements, those are, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_1, \alpha_5\}$, $\{\alpha_2, \alpha_3\}$, $\{\alpha_2, \alpha_4\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_3, \alpha_6\}$, and $\{\alpha_4, \alpha_6\}$, and this fact shows that $\otimes(CD) = 10$. Similarly, we can show that $\otimes(CD) = 6$ and $\overline{\otimes}(CD) = 8$.

**Figure 8. CD.**

**Remark 2.10.** The integer assigned to each spherical curve $P$ in Figure 3 denotes $\otimes(P)$.

**Figure 9. Reidemeister moves.**

**Notation 2.11.** For a chord diagram $CD$, $n(CD)$ denotes $\bigoplus(CD)$, that is the number of the chords of $CD$. For a spherical curve $P$, $n(CD_P)$ is denoted by $n(P)$, that is the number of the double points of $P$.

A calculation of the number $x(CD)$ by using geometric observations as in Example 2.9 together with Figure 9, we have Facts 2.12 and 2.13 below.
**Fact 2.12** ([3], Theorem 2(1)). Let $c, c' \in C$. Suppose that there are representatives $P$ and $P'$ of $c$ and $c'$ respectively such that $P'$ is obtained from $P$ by a deformation of type RI. Then, $\bigotimes \langle c \rangle = \bigotimes \langle c' \rangle$, $n(c') = n(c) + 1$ or $n(c') = n(c) - 1$.

**Fact 2.13** ([3], Theorem 2(3)). Let $c, c' \in C$. Suppose that there are representatives $P$ and $P'$ of $c$ and $c'$ respectively such that $P'$ is obtained from $P$ by a deformation of type weak RI III. Then, $\bigotimes \langle c \rangle = \bigotimes \langle c' \rangle - 1$ or $\bigotimes \langle c \rangle = \bigotimes \langle c' \rangle + 1$.

**Definition 2.14.** Let $c, c' \in C$. Suppose that there are representatives $P$ and $P'$ of $c$ and $c'$ respectively such that $P'$ is obtained from $P$ by a deformation of type RI. If $n(c') = n(c) + 1$ ($n(c') = n(c) - 1$ resp.), then we call such RI a positive (negative resp.) RI.

Suppose that there are representatives $P$ and $P'$ of $c$ and $c'$ respectively such that $P'$ is obtained from $P$ by a deformation of type weak RI III. If $\bigotimes \langle P' \rangle = \bigotimes \langle P \rangle - 1$ ($\bigotimes \langle P' \rangle = \bigotimes \langle P \rangle + 1$ resp.), then we call such weak RI III a positive (negative resp.) weak RI III.

By Facts 2.12 and 2.13, it is easy to show Proposition 2.15 below.

**Proposition 2.15.** Let $P, P'$ be spherical curves such that $d_{w3}(P, P') < \infty$. Then $\tilde{d}_{w3}([P], [P']) = d_{w3}(P, P') \equiv 0 \pmod{2}$ if and only if $\bigotimes \langle P \rangle \equiv \bigotimes \langle P' \rangle \pmod{2}$.

**Example 2.16.** Let $7_5, 7_6, 7_7$ be the spherical curves in Figure 3. In Figure 45 of [3], it is shown that $\tilde{d}_{w3}([7_5], [7_6]) \leq 2$. On the other hand, since $\bigotimes \langle 7_5 \rangle = \bigotimes \langle 7_6 \rangle = 14$ (Remark 2.10), Proposition 2.15 shows that $\tilde{d}_{w3}([7_5], [7_6]) \equiv 0 \pmod{2}$. These show that $\tilde{d}_{w3}([7_5], [7_6]) = 2$.

**Example 2.17.** In the diagram of the equivalence class (6) in Figure 3, for any two spherical curves $P$ and $P'$, which are joined by a path consisting of two arrows in the diagram, we can show $\tilde{d}_{w3}([P], [P']) = 2$. For example, take $7_6$ and $7_A$ in Figure 3. Since they are joined by a path consisting of two arrows, we have $\tilde{d}_{w3}([7_6], [7_A]) \leq 2$. On the other hand, since $\bigotimes \langle 7_6 \rangle = \bigotimes \langle 7_A \rangle = 11$ (Remark 2.10), Proposition 2.15 shows that $\tilde{d}_{w3}([7_6], [7_A]) \equiv 0 \pmod{2}$. These shows that $\tilde{d}_{w3}([7_6], [7_A]) = 2$. Similar arguments works for all of the pairs each of which is joined by a path consisting of two arrows, those are $(5_1, 6_3), (5_1, 7_7), (5_1, 7_B), (6_2, 7_6), (6_2, 7_A), (6_3, 6_3), (6_3, 7_7), (6_3, 7_B), (6_3, 7_7), (6_3, 7_B), (7_6, 7_6), (7_6, 7_A), (7_6, 7_A), (7_7, 7_B), (7_7, 7_B), (7_7, 7_B)$. Details are left to the reader.

### 3. Main results

For spherical curves $P, P'$, let $n(P), f_c(P), \tilde{d}_{w3}([P], [P'])$ and $d_{w3}(P, P')$ be as in Section 2.

**Theorem 1.** Let $P, P'$ be spherical curves. Suppose that $n(P) > n(P')$, and that $f_c(P) = f_c(P')$. Then we have:

$$\tilde{d}_{w3}([P], [P']) \geq n(P) - n(P').$$

**Example 3.1.** For each $N \in \mathbb{N}$, there exist $P$ and $P'$ such that $f_c(P) = f_c(P')$, and that $\tilde{d}_{w3}([P], [P']) = n(P) - n(P') = N$. In fact, let $P_N, P'_N$ be spherical curves as in Figure 10. It is easy to see that $d_{w3}(P_N, P'_N)(= d_{w3}([P_N], [P'_N])) \leq N$. On the other hand, since $n(P_N) - n(P'_N) = 4N - 3N = N$ and $f_c(P_N) = f_c(P'_N) = N$, we have $\tilde{d}_{w3}([P_N], [P'_N]) \geq n(P_N) - n(P'_N) = N$ by Theorem 1.
We say that a spherical curve $P$ contains a 1-gon if there is an open disk component of $S^2 \setminus P$ that contains exactly one corner. A spherical curve $P$ is called RI-minimal if $P$ does not contain a 1-gon. It is clear that any spherical curve is transformed to a minimal one by successively applying deformations of type negative RI. The following fact is known.

**Fact 3.2** ([1] (cf. [6])). For any spherical curve $P$, the RI-minimal spherical curves so obtained from $P$ are mutually ambient isotopic.

**Theorem 2.** Let $P, P'$ be RI-minimal spherical curves such that $f_c(P) - n(P) = f_c(P') - n(P')$. Suppose that $\tilde{d}_{w3}([P],[P']) = 1$. Then there is a spherical curve $\bar{P}'$ such that $\bar{P}'$ is ambient isotopic to $P'$, and that $\bar{P}'$ is obtained from $P$ by applying a deformation of type weak RI-I only, where no RI is required.

Recall that the transformation of spherical curves depicted in Figure 4 is called a deformation of type $\alpha$. Here we note that each deformation of type $\alpha$ is represented by successively applying a deformation of type RI-I, and a deformation of type RI (Figure 11). Then we call it a deformation of type weak (strong resp.) $\alpha$ if the type of the RI-I deformation is weak (strong resp.). By Figure 9 and Figure 11, we have:

**Claim 3.3.** If the deformation of type $\alpha$ is of type weak $\alpha$, then the deformation of type RI-I relevant to it is always positive.

**Theorem 3.** Let $P, P'$ be RI-minimal spherical curves such that $f_c(P) = f_c(P') = 1$, and that $\tilde{d}_{\mathbb{W}}([P],[P']) = q < \infty$. Suppose that $n(P) - n(P') = q$, and that $\mathcal{O}(P) - \mathcal{O}(P') = q$. Then there is a spherical curve $\bar{P}'$ such that $\bar{P}'$ is ambient isotopic to $P'$, and that $\bar{P}'$ is obtained from $P$ by applying deformations of type weak $\alpha$ successively $q$ times and this sequence realizes $\tilde{d}_{\mathbb{W}}([P],[P'])$.

**Example 3.4.** For each $N \in \mathbb{N}$, let $Q_N, Q'_N$ be spherical curves as in Figure 12. It is easy to see that $f_c(Q_N) = f_c(Q'_N) = 1$, $n(Q_N) = 3N + 1$, $n(Q'_N) = 2N + 1$.
hence \( n(Q_N) - n(Q'_N) = N \), and \( \otimes(Q_N) = 2N^2 + 2N \), \( \otimes(Q'_N) = 2N^2 + N \), hence \( \otimes(Q_N) = \otimes(Q'_N) = N \). By Theorem 1, we see that \( d_{w3}([Q_N], [Q'_N]) \geq N \). On the other hand, it is easy to show that \( Q'_N \) is obtained from \( Q_N \) by applying deformations of type weak \( N \) times. This shows that \( d_{w3}([Q_N], [Q'_N]) = N \). These show that the statement in Theorem 3 is exact.

**Figure 12**

For the proofs of Theorems 1, 2 and 3, we prepare some notations and lemmas.

**Notation 3.5.** Let \( P \) and \( P' \) be two spherical curves that are equivalent under RI and weak \( RIII \), i.e., there exists a finite sequence of spherical curves \( P = P_0, P_1, \ldots, P_m = P' \), where \( P_i \) is obtained from \( P_{i-1} \) by a deformation of type RI or type weak \( RIII \). Then, \( O_{p_i} \) denotes the deformation from \( P_{i-1} \) to \( P_i \), and these settings are expressed by using the notation:

\[
P = P_0 \xrightarrow{O_{p_1}} P_1 \xrightarrow{O_{p_2}} \ldots \xrightarrow{O_{p_m}} P_m = P'.
\]

Since every deformation of type \( RIII \) does not change the number of double points of \( P (= n(P)) \), we have the next lemma.

**Lemma 3.6.** \( n(P') - n(P) = \sharp\{i \mid O_{p_i} : positive \ RI\} - \sharp\{j \mid O_{p_j} : negative \ RI\} \).

By Figure 9, we immediately have:

**Lemma 3.7.** If \( O_{p_i} \) is of type positive RI, then \( f_c(\bar{P}_i) - f_c(\bar{P}_{i-1}) = 1 \).

Let \( P \) and \( P' \) be spherical curves. Suppose that \( P' \) is obtained from \( P \) by a deformation of type weak \( RIII \). Then, Figure 13 describes the corresponding transformation on chord diagrams. The chords \( i, j, \) and \( k \) are called the triple relevant to the weak \( RIII \). We call the chord \( j \) in the right chord diagram in Figure 13 the isolated chord in the triple relevant to the weak \( RIII \). Then we define the value, denoted \( \mu_{w3}(P, P') \) as follows.

\[
\mu_{w3}(P, P') = \sharp\{\text{chord in } CD_{P'} \text{ intersecting the chord } j \text{ transversely}\}.
\]

By geometric observations of Figure 14, we have Lemma 3.8 below.
DISTANCE ON SPHERICAL CURVES

Lemma 3.8. If $O_{P_i}$ is of type positive weak $RI_{III}$, then $f_c(P_i) - f_c(P_{i-1}) = 0$ or 1. Further, $f_c(P_{i-1}) - f_c(P_i) = 0$ if and only if $\mu_{w3}(P_{i-1}, P_i) \neq 0$.

By Lemma 3.7 and Lemma 3.8, we have the following equality.

\[
(1) \quad f_c(P') = f_c(P) + \varepsilon \{ i \mid O_{P_i} : \text{positive RI} \} - \varepsilon \{ j \mid O_{P_j} : \text{negative RI} \} + \varepsilon \{ k \mid O_{P_k} : \text{positive weak } RI_{III}, \mu_{w3}(P_{k-1}, P_k) = 0 \} - \varepsilon \{ l \mid O_{P_l} : \text{negative weak } RI_{III}, \mu_{w3}(P_l, P_{l-1}) = 0 \}.
\]

Let $\text{reduced}(P)$ be an RI-minimal spherical curve obtained from $P$ by successively applying deformations of type negative RI. Note that $CD_{\text{reduced}(P)}$ is obtained from $CD_P$ by successively removing outermost isolated chords (e.g., Figure 15).

Remark 3.9. For spherical curves $P$ and $P'$, we see, by Fact 3.2, that $[P] = [P']$ if and only if $\text{reduced}(P)$ is ambient isotopic to $\text{reduced}(P')$. 

Figure 13. A single positive weak $RI_{III}$.

Figure 14. $CD_P$ and $CD_{P'}$ under a positive weak $RI_{III}$ sending $P$ to $P'$.

Figure 15. $(P, \text{reduced}(P))$ and $(CD_P, CD_{\text{reduced}(P)})$. 

Remark 3.9. For spherical curves $P$ and $P'$, we see, by Fact 3.2, that $[P] = [P']$ if and only if $\text{reduced}(P)$ is ambient isotopic to $\text{reduced}(P')$. 

\[
\]
Lemma 3.10. Suppose that \( f_c(\text{reduced}(P_{i-1})) = 1 \). If \( O_{P_i} \) is a positive weak RI \( R \mathbb{I} \), then \( f_c(\text{reduced}(P_i)) = 1 \), and \( n(\text{reduced}(P_{i-1})) = n(\text{reduced}(P_i)) = 0 \) or 1. Further \( n(\text{reduced}(P_{i-1})) = n(\text{reduced}(P_i)) = 0 \) if and only if \( \mu_{w3}(P_{i-1}, P_i) \neq 0 \).

Proof. Since \( O_{P_i} \) is of type positive weak RI \( R \mathbb{I} \), the chords relevant to the weak RI \( O_{P_i} \) is not isolated in \( CD_{P_{i-1}} \). It is observed by Figure 16 that \( n(\text{reduced}(P_{i-1})) \) or 1, and if \( \mu_{w3}(P_{i-1}, P_i) \neq 0 \), (hence, the isolated chord in the triple relevant to the weak RI \( O_{P_i} \) is not isolated in \( CD_{P_i} \)) then \( f_c(\text{reduced}(P_i)) = 1 \), and \( n(\text{reduced}(P_{i-1})) = 0 \). Suppose that \( \mu_{w3}(P_{i-1}, P_i) = 0 \). Then

\[
\mu \neq 0
\]

![Figure 16](image)

we see by Figure 16 that \( \text{the set of the isolated chords of } CD_{P_{i-1}} \) = \( \text{the set of the isolated chords of } CD_{P_i} \) \( \cup \{\beta\} \), where \( \beta \) is the isolated chord in the triple relevant to the weak RI \( O_{P_i} \). Since \( f_c(\text{reduced}(P_{i-1})) = 1 \), the double points of \( P_{i-1} \) corresponding to the isolated chords are removed by successively applying deformation of type negative RI. Note that by this transformation all of the isolated chords of \( CD_{P_{i-1}} \) is removed. This together with Figure 16 shows that \( f_c(\text{reduced}(P_i)) = 1 \), and \( n(\text{reduced}(P_{i-1})) = n(\text{reduced}(P_i)) = 1 \). These complete the proof of the lemma. \( \square \)

Proof of Theorem 1. Let

\[ P = P_0 \xrightarrow{O_{P_1}} P_1 \xrightarrow{O_{P_2}} \ldots \xrightarrow{O_{P_{m-1}}} P_m, \]

where \( P_m \) is ambient isotopic to \( P' \), be a sequence realizing \( d_{w3}(P, P') \) (= \( d_{w3}(P, P') \)).

By (1) above,

\[
f_c(P') - f_c(P) = \# \{i \mid O_{P_i} : \text{positive RI}\} - \# \{j \mid O_{P_j} : \text{negative RI}\}
+ \# \{k \mid O_{P_k} : \text{positive weak } R \mathbb{I}, \mu_{w3}(P_{k-1}, P_k) = 0\}
- \# \{l \mid O_{P_l} : \text{negative weak } R \mathbb{I}, \mu_{w3}(P_l, P_{l-1}) = 0\}.
\]

This together with Lemma 3.6 shows

\[
f_c(P') - f_c(P) = n(P') - n(P) + \# \{k \mid O_{P_k} : \text{positive weak } R \mathbb{I}, \mu_{w3}(P_{k-1}, P_k) = 0\}
- \# \{l \mid O_{P_l} : \text{negative weak } R \mathbb{I}, \mu_{w3}(P_l, P_{l-1}) = 0\}.
\]
By the assumption of Theorem 1 \((f_c(P) = f_c(P'))\), this implies

\[(2) \quad n(P) - n(P') = z\{k \mid Op_k : \text{positive weak } R III, \mu_{w3}(P_{k-1}, P_k) = 0\} - z\{l \mid Op_l : \text{negative weak } R III, \mu_{w3}(P_l, P_{l-1}) = 0\}.
\]

By the definition of \(d_{w3}(P, P')\), we have

\[d_{w3}(P, P') = z\{k \mid Op_k : \text{positive weak } R III\} + z\{l \mid Op_l : \text{negative weak } R III\}.
\]

Then we note that

\[
z\{k \mid Op_k : \text{positive weak } R III\} + z\{l \mid Op_l : \text{negative weak } R III\}
\]

\[
\geq z\{k \mid Op_k : \text{positive weak } R III, \mu_{w3}(P_{k-1}, P_k) = 0\}
\]

\[
\geq z\{k \mid Op_k : \text{positive weak } R III, \mu_{w3}(P_{k-1}, P_k) = 0\} - z\{l \mid Op_l : \text{negative weak } R III, \mu_{w3}(P_l, P_{l-1}) = 0\}
\]

\[
= n(P) - n(P') \quad \text{(by (2))}.
\]

These together with Proposition 1.1 imply that

\[
\tilde{d}_{w3}([P], [P']) = d_{w3}(P, P') \geq n(P) - n(P').
\]

\[\square\]

**Proof of Theorem 2.** Let

\[P = P_0 \xrightarrow{Op_1} P_1 \xrightarrow{Op_2} \cdots \xrightarrow{Op_m} P_m,\]

where \(P_m\) is ambient isotopic to \(P'\) be a sequence realizing \(\tilde{d}_{w3}([P], [P']) = 1\) (see Proposition 1.1), hence there exists unique \(s (1 \leq s \leq m)\) such that \(Op_s\) is of type weak \(R III\). By exchanging \(P\) and \(P'\), if necessary, we may suppose that \(Op_s\) is of type positive weak \(R III\).

**Claim 3.1.**

\[\mu_{w3}(P_{s-1}, P_s) \neq 0.\]

**Proof.** Suppose, for a contradiction, that \(\mu_{w3}(P_{s-1}, P_s) = 0\).

By Lemma 3.6 and (1) above,

\[f_c(P) - f_c(P') = n(P') - n(P)
\]

\[+ z\{k \mid Op_k : \text{positive weak } R III, \mu_{w3}(P_{k-1}, P_k) = 0\}
\]

\[- z\{l \mid Op_l : \text{negative weak } R III, \mu_{w3}(P_l, P_{l-1}) = 0\}.
\]

This together with the assumption of Theorem 2 \((f_c(P) - n(P) = f_c(P') - n(P'))\) shows

\[z\{k \mid Op_k : \text{positive weak } R III, \mu_{w3}(P_{k-1}, P_k) = 0\}
\]

\[= z\{l \mid Op_l : \text{negative weak } R III, \mu_{w3}(P_l, P_{l-1}) = 0\}.
\]

Since \(\mu_{w3}(P_{s-1}, P_s) = 0\), this shows that there exists another deformation of type weak \(R III\) in the sequence, contradicting the fact that \(P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_m\) realizes \(\tilde{d}_{w3}([P], [P']) = 1\).

Recall that \(Op_1, Op_2, \ldots, Op_{s-1}\) are of type RI. Since \(P\) contains no 1-gon, by Fact 3.2 we obtain the next claim.
We first consider the pair \( \text{reduced}(P_{s-1}) \), in particular \( CD_P \) is obtained from \( CD_{P_{s-1}} \) by successively removing outermost isolated chords, say \( a_1, a_2, \ldots, a_t \).

Since each member of the triple in \( CD_{P_{s-1}} \) relevant to the weak RI \( O_{P_s} \) is not isolated in \( CD_{P_{s-1}} \), we see that each \( a_i \) \((1 \leq i \leq t)\) is not a member of the triple relevant to the weak RI \( O_{P_s} \). Hence \( a_1, a_2, \ldots, a_t \) survive in \( CD_{P_s} \). By Claim 3.11, we see that the isolated chord in the triple relevant to the weak RI \( O_{P_s} \) is not isolated in \( CD_{P_s} \). These show that \( CD_{\text{reduced}(P_{s-1})} \) is obtained from \( CD_{P_s} \) by successively removing the chords \( a_1, a_2, \ldots, a_t \) that are outermost at each step. Since \( O_{P_{s+1}}, \ldots, O_{P_m} \) are of type RI, and \( P' \) is RI-minimal, by Fact 3.2, this shows:

**Claim 3.13.** \( \text{reduced}(P_s) \) is ambient isotopic to \( P' \), and that \( CD_{P'} \) is obtained from \( CD_{P_s} \) by removing \( a_1, a_2, \ldots, a_t \).

Claim 3.12 and Claim 3.13 show that \( \text{reduced}(P_{s-1}) \) and \( \text{reduced}(P_s) \) are related by a single weak RI, and, hence \( P \) and \( P' \) are related by a single weak RI. This completes the proof of Theorem 2. \( \square \)

**Proof of Theorem 3.** Let \( P = P_0 O_{P_1} P_1 O_{P_2} \ldots O_{P_n} P_m \), where \( P_m \) is ambient isotopic to \( P' \), be the sequence realizing \( d_{w3}([P],[P']) \) \((= d_{w3}(P,P')) \). Then let \( \{i_1,i_2,\ldots,i_q\} \) \((1 \leq i_1 < i_2 < \cdots < i_q \leq m)\) be the set of the numbers such that \( O_{P_{i_1}}, O_{P_{i_2}}, \ldots, O_{P_{i_q}} \) are of type RI. By the assumption of Theorem 3 \((\bigotimes(P) - \bigotimes(P') = q)\), and Facts 2.12 and 2.13, we see that each \( O_{P_{i_j}} \) \((j = 1, 2, \ldots, q)\) is of type positive weak RI.

**Claim 3.14.** \( \mu(P_{i_{j-1}}, P_{i_j}) = 0 \) \((j = 1, 2, \ldots, q)\).

**Proof.** We first consider the pair \( P_{i_{j-1}}, P_{i_j} \). Since \( O_{P_{i_1}}, O_{P_{i_2}}, \ldots, O_{P_{i_{j-1}}} \) are of type RI, we immediately have

\[
n(\text{reduced}(P_0)) = n(\text{reduced}(P_1)) = \cdots = n(\text{reduced}(P_{i_{j-1}})).
\]

Then by Lemma 3.10, we have that \( n(\text{reduced}(P_{i_{j-1}})) - n(\text{reduced}(P_{i_j})) = 0 \) or 1, and that \( n(\text{reduced}(P_{i_{j-1}})) - n(\text{reduced}(P_{i_j})) = 1 \) if and only if \( \mu_{w3}(P_{i_{j-1}}, P_{i_j}) = 0 \). By using the same arguments inductively, we have that for each \( j \),

\[
n(\text{reduced}(P_{i_j})) = n(\text{reduced}(P_{i_{j+1}})) = \cdots = n(\text{reduced}(P_{i_{j+1}})),
\]

where \( P_{i_{j+1}} = P_m \), \( n(\text{reduced}(P_{i_{j-1}})) - n(\text{reduced}(P_{i_j})) = 0 \) or 1, and that \( n(\text{reduced}(P_{i_{j-1}})) - n(\text{reduced}(P_{i_j})) = 1 \) if and only if \( \mu_{w3}(P_{i_{j-1}}, P_{i_j}) = 0 \). Since \( P_0 = P \), and \( P_m \) is ambient isotopic to \( P' \), these together with the assumptions of Theorem 3 \((\text{RI minimality of } P \text{ and } P')\), and the equality \( n(P) - n(P') = q \) show that \( \mu_{w3}(P_{i_{j-1}}, P_{i_j}) = 0 \) \((j = 1, 2, \ldots, q)\). \( \square \)
Let \( \{a_1^{(0)}, \ldots, a_{l_0}^{(0)}\} \) be the set of isolated chords in \( CD_{p_{i-1}} \). Since 
\( f_c(\text{reduced}(p_{i-1})) = f_c(P) = 1 \) (by the assumption of Theorem 3), \( CD_{p_0} \) is obtained 
from \( CD_{p_{i-1}} \) by successively removing the chords \( a_1^{(0)}, \ldots, a_{l_0}^{(0)} \) that are 
outermost at each step. Since \( \mu_{\text{w3}}(p_{i-1}, p_i) = 0 \) by Claim 3.14, we see that 
the isolated chord relevant to the weak RI I, say \( \beta_1 \), is isolated in \( CD_{p_{i-1}} \). These 
show that \( CD_{\text{reduced}(p_{i-1})} \) is obtained from \( CD_{p_{i-1}} \) by removing \( a_1^{(0)}, \ldots, a_{l_0}^{(0)} \), \( \beta_1 \).

Let \( P_1^{(1)} = \text{reduced}(P_{i-1}) \). Since \( Op_i \) is positive weak RI, each \( a_k^{(0)} \) is not a member 
of the triple relevant to the weak RI I, say \( \beta_1 \). On the other hand, we note that 
\( \text{reduced}(p_{i-1}) \) is ambient isotopic to \( P_0 \). Hence we can apply a deformation of type 
RI, say \( Op^{(1)}_1 \), corresponding to \( Op_i \) on \( P_0 \) to obtain a spherical curve \( P_1^{(1)} \) such 
that the chord diagram \( CD_{p^{(1)}_1} \) contains exactly one isolated chord corresponding 
to \( \beta_1 \). Since \( f_c(P_0) = 1 \) by the assumption, removing \( \beta_1 \) is realized by a deformation 
of type negative RI, say \( Op^{(1)}_1 \). As a conclusion we have obtained a sequence 
\( P = P_0 \xrightarrow{Op^{(1)}_1} P_1^{(1)} \rightarrow \cdots \rightarrow P_{i-1}^{(1)} = P_i \), where \( Op^{(1)}_1 \) is 
of type positive weak RI, \( Op^{(1)}_2 \) is of type negative RI with \( Op^{(1)}_1 \) is RI-minimal, 
\( n(P_1^{(1)}) = n(P_0) - 1, f_c(P_1^{(1)}) = 1 \).

Then we can retake the sequence realizing \( d_{\text{w3}}(P, P') \) as follows.

\[
P = P_0 \xrightarrow{Op^{(1)}_1} P_1^{(1)} \xrightarrow{Op^{(1)}_2} P_1^{(1)} \xrightarrow{Op^{(1)}_3} \cdots \xrightarrow{Op^{(1)}_{l_0+1}} P_0^{(1)} = P_i
\]

(, where \( Op^{(1)}_1, \ldots, Op^{(1)}_{l_0} \) are of type positive RI that produce isolated chords in 
\( CD_{p_{i-1}} \), which correspond to \( a_1^{(0)}, \ldots, a_{l_0}^{(0)} \) followed by the subsequence

\[
P_i \xrightarrow{Op_{i+1}} P_{i+1} \xrightarrow{Op_{i+2}} \cdots \xrightarrow{Op_{l_0}} P_{m}
\]
of the original sequence. Then by Claim 3.15 we can apply the above arguments to 
the sequence from \( P_1^{(1)} \) to \( P_m = P \) to obtain a sequence such that

\[
(P = P_0 \xrightarrow{Op^{(1)}_1} P_1^{(1)} \xrightarrow{Op^{(1)}_2} P_1^{(1)} \xrightarrow{Op^{(1)}_3} \cdots \xrightarrow{Op^{(1)}_{l_0}} P_0^{(1)} = P_i)
\]

(, where \( Op^{(2)}_1 \) is positive weak RI, \( Op^{(2)}_2 \) is negative RI) followed by some positive 
RI’s, from \( P^{(2)}_1 \) to \( P_2 \), then further followed by the subsequence

\[
P_2 \xrightarrow{P_2^{(2)}} P_1 \xrightarrow{P_2^{(2)}} \cdots \xrightarrow{P_m} P_m = P'
\]
of the original sequence, where \( P_2^{(2)} \) is RI-minimal, \( n(P_2^{(2)}) = q - 2, f_c(P_2^{(2)}) = 1 \).

By repeating the arguments, we obtain a sequence

\[
P = P_0 \xrightarrow{Op^{(1)}_1} P_1^{(1)} \xrightarrow{Op^{(1)}_2} P_1^{(1)} \xrightarrow{Op^{(2)}_3} P_2^{(2)} \xrightarrow{Op^{(2)}_2} P_2^{(2)} \xrightarrow{Op^{(2)}_1} P_2^{(2)} \xrightarrow{Op^{(2)}_1} \cdots 
\]

\[
\cdots \xrightarrow{P_j^{(j+1)}} P_{j+1}^{(j+1)} \xrightarrow{Op^{(j+1)}_k} P_{j+1}^{(j+1)} \xrightarrow{Op^{(j+1)}_{k+1}} P_{j+1}^{(j+1)} \xrightarrow{Op^{(j+1)}_{k+2}} \cdots \xrightarrow{Op^{(q)}_k} P_q \xrightarrow{Op^{(q)}_m} \cdots \xrightarrow{P_m} P_m,
\]

where all of the deformations in the subsequence \( P_{q} \rightarrow \cdots \rightarrow P_m \) are of type RI. 
Here we note that \( P_q^{(q)} \) is RI minimal. Hence by Fact 3.2 we have \( P_m \) is ambient 
isotopic to \( P_q^{(q)} \).
Claim 3.16. Each $P_j^{(j)} \xrightarrow{\delta_j} P_{j+1}^{(j+1)}$ represents a deformation of type weak $\alpha$.

Proof. The above arguments imply that $P_j^{(j)}$ is RI-minimal, and that $CD_{P_j^{(j)}}$ contains exactly one isolated chord, say $\beta_j$, and $CD_{P_{j+1}^{(j+1)}}$ is obtained from $CD_{P_j^{(j)}}$ by removing $\beta_j$. Since $f_c(P_j^{(j)}) = 1$, this deformation is realized by a deformation of type negative RI (see Figure 18). It is directly observed from Figure 18, that these represent a deformation of type weak $\alpha$. □

Claim 3.16 shows that by putting $\tilde{P}' = P_q^{(q)}$ we obtain the conclusion of Theorem 3. □

4. Distance between elements of the equivalence class (6).

In this section, we prove:

**Theorem 4.** The graph of the equivalence class (6) in Figure 3 is tautly embedded in $K_{w,3}$, i.e., for each pair $P, P'$ of elements in the equivalence class (6) in Figure 3, $d_{w,3}([P], [P'])$ is realized by the minimal number of arrows in the paths joining $P$ and $P'$ in the diagram of the equivalence class (6).

Note that Example 2.17 shows that Theorem 4 holds for each pair $P, P'$ of elements in the equivalence class (6) in Figure 3, such that the minimal number of arrows in the paths joining $P$ and $P'$ is 2. Further we note that the diameter of the graph of equivalence class (6) is 3. Hence it is enough to show the next assertion, for a proof of Theorem 4.

**Assertion 4.1.** For any pair of elements $P$ and $P'$ in equivalence class (6) in Figure 3 such that the minimal number of arrows in the paths in Figure 3 joining $P$ and $P'$ is 3, we have $d_{w,3}([P],[P']) = 3$.

Proof. We first consider the pair $(5_1, 7_6)$. By Figure 3 we have $d_{w,3}(5_1, 7_6) \leq 3$. Since $\bar{\otimes}(5_1) = 10$, $\bar{\otimes}(7_6) = 11$, we obtain, by Proposition 2.15 that $d_{w,3}(5_1, 7_6) \equiv 1 \pmod{2}$. On the other hand, since $f_c(5_1) = f_c(7_6) = 1$, we have $d_{w,3}(5_1, 7_6) \geq 3$. Theorem 4 follows.

**Figure 18**
$7 - 5 = 2$ by Theorem 1. These show that $d_{w3}(5_1, 7_6) = 3$. Similar arguments work for the pairs $(5_1, 7_6), (5_1, 7_A)$. (Details are left to the reader.)

Then we consider the pair $(7_6, 7_B)$. By Figure 3 we have $d_{w3}(7_6, 7_B) \leq 3$. Since $\otimes(7_6) = 11, \otimes(7_B) = 12$, we obtain $d_{w3}(7_6, 7_B) \equiv 1 \pmod{2}$. Hence, it is enough to show $d_{w3}(7_6, 7_B) \neq 1$. Suppose that $d_{w3}(7_6, 7_B) = 1$. Since $n(7_6) = n(7_B) = 7$, and $f_c(7_6) = f_c(7_B) = 1$, we see that $7_6$ and $7_B$ are related by a deformation of type weak $R\overline{II}$ only by Theorem 2. For $7_B$, we have four regions where we can apply deformations of type weak $R\overline{II}$, that are depicted in Figure 19. It is directly observed that $7_6$ is not produced by applying the deformation of type weak $R\overline{II}$ there. This contradicts Theorem 2. These show that $d_{w3}(7_6, 7_B) = 3$. Similar arguments work for the pairs $(7_6, 7^*_B), (7_6, 7^*_B), (7^*_6, 7_B), (7_A, 7_7)$.

![Figure 19](image)

Then we consider the pair $(7_6, 6_3)$. In this case, since $\otimes(7_6) = 11, \otimes(6_3) = 10$ it is enough to show $d_{w3}(7_6, 6_3) \neq 1$. Suppose that $d_{w3}(7_6, 6_3) = 1$. Then we can apply Theorem 3 to show that $7_6$ is related to $6_3$ by exactly one deformation of type weak $\alpha$. However we can show that this is not the case, by using the analysis as in Figure 20. These show that $d_{w3}(7_6, 6_3) = 3$. Similar argument works for the pair $(7^*_6, 6^*_3)$. These complete the proof of Assertion 4.1.

![Figure 20](image)
Acknowledgements

The authors thank Ms. Misaki Kataoka for pointing out some typographical errors in the first version of this paper. M. H. was supported by JSPS KAKENHI Grant-in-Aid for JSPS Fellows (No. 26-3667). N. I. was partially supported by Sumitomo Foundation (Grant for Basic Science Research Projects, Project number: 160556). N. I. was a project researcher of Grant-in-Aid for Scientific Research (S) (No. 24224002) (2016.4–2017.3). T. K. was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) (No. 254000915). H. M. was partially supported by Nara Women’s University Intramural Grant for Project Research (2016.4–2017.3).

References

[1] N. Ito and Y. Takimura, (1, 2) and weak (1, 3) homotopies on knot projections, J. Knot Theory Ramifications 22 (2013), 1350085, 14pp.

Appendix A. Proof of Proposition 1.1

Proof of Proposition 1.1. By the definition of \( \hat{d}_{w,3} \) and \( d_{w,3} \), it is easy to see that \( \hat{d}_{w,3}(\{P\},\{P'\}) \leq d_{w,3}(P,P') \). Hence we show that \( d_{w,3}(P,P') \leq \hat{d}_{w,3}(\{P\},\{P'\}) \). Let \( m = \hat{d}_{w,3}(\{P\},\{P'\}) \). This means that there is a sequence \( P = P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_k \) (\( k \geq m \)) and ambient isotopies \( \phi_i \) (\( i = 0, 1, \ldots, k \)) such that there is a subset \( \{i_1, i_2, \ldots, i_m\} \subset \{i = 0, 1, \ldots, k\} \) (\( i_1 < i_2 < \cdots < i_m \)) such that

- \( \phi_0 = \text{id}_{S^2} \)
- \( \phi_i(P_j) \) is transformed to \( P_{j+1} \) (\( j = 0, 1, \ldots, k - 1 \)) by a deformation of type RI (if \( j \notin \{i_1, i_2, \ldots, i_m\} \)) or by a deformation of type RIII (if \( j \in \{i_1, i_2, \ldots, i_m\} \))
- \( \phi_1^k(P_k) = P' \).

Let \( D_j \) be the disk in \( S^2 \) such that the deformation \( \phi_i(P_j) \rightarrow P_{j+1} \) of type RI or type RIII is performed within \( D_j \). Then let \( D_0 = (\phi_0^1)^{-1}(D_0) \), and \( P_1 = (\phi_0^1)^{-1}(P_1) \) (hence \( P_1 \) is ambient isotopic to \( P_1 \)). Since \( (\phi_0^1)^{-1}(\phi_1^k(P_1)) = P_1 \), we see that \( P_1 \) is obtained from \( P_0 \) by a deformation of type RI or type weak RIII that is performed within \( D_0 \). Then let \( D_1 = (\phi_1^1)^{-1} \circ (\phi_1^1)^{-1}(D_1) \), \( P_2 = (\phi_1^1)^{-1} \circ (\phi_1^1)^{-1}(P_2) \) (hence \( P_2 \) is ambient isotopic to \( P_2 \)). It is easy to see that \( P_2 \) is obtained from \( P_0 \) by applying a deformation of type RI or type weak RIII that is performed within...
Then \( D^{(0)}_0 \), then applying a deformation of type RI or type weak RIII that is performed within \( D_1^{(1)} \). For each \( j \in \{3,4,\ldots,m-1\} \), let

\[
D_j^{(j)} = (\phi_1^{-1} \circ (\phi_1^{-1})^{-1} \circ \cdots \circ (\phi_1^{-1})^{-1}(D_j)),
\]

and

\[
P_j^{(j+1)} = (\phi_1^{-1} \circ (\phi_1^{-1})^{-1} \circ \cdots \circ (\phi_1^{-1})^{-1}(P_{j+1}) \text{ (hence } P_j^{(j+1)} \text{ is ambient isotopic to } P_{j+1}).
\]

Then \( P_m^{(m)} \) is obtained from \( P_0 \) by successively applying deformation of type RI or type weak RIII that is performed within the disk \( D_j^{(j)} \). Since \( P_m^{(m)} \) is ambient isotopic to \( P' \), this shows that \( d_{w3}(P,P') \leq \tilde{d}_{w3}([P],[P']) \).

\[\square\]
(Current address: Organization for the Strategic Coordination of Research and Intellectual Properties, Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo 164-8525, Japan)

E-mail address: megumihashizume@meiji.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: noboru@ms.u-tokyo.ac.jp

Department of Mathematics, Nara Women’s University, Kitauoya Nishimachi, Nara 630-8506, Japan

E-mail address: tsuyoshi@cc.nara-wu.ac.jp

Department of Mathematics, Nara Women’s University, Kitauoya Nishimachi, Nara 630-8506, Japan

E-mail address: murai@cc.nara-wu.ac.jp