RII NUMBER OF KNOT PROJECTIONS

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ABSTRACT. Every knot projection is simplified to the trivial spherical curve not increasing double points by using deformations of types 1, 2, and 3 analogously to Reidemeister moves of types 1, 2, and 3 on knot diagrams. We introduce RII number of a knot projection that is the minimum number of deformations of negative type 2 among such sequences. By definition, it is invariant under deformations of types 1 and 3. This is motivated by Östlund conjecture: Deformations of type 1 and 3 are sufficient to describe a homotopy from any generic immersion of a circle in a two dimensional plane to an embedding of the circle (2001), which implies RII number always would be zero. However, Hagge and Yazinski disproved the conjecture by showing the first counterexample with 16 double points, which implies that RII number is nontrivial. This paper shows that RII number can be any nonnegative number.

1. INTRODUCTION

A *knot projection* is the image of a generic immersion of a circle into the 2-sphere. Two knot projections are identified by ambient isotopy. Any self-intersection of a knot projection is a transverse double point and is simply called a *double point*. The *trivial spherical curve* is a knot projection with no double points. Deformations of types 1, 2, and 3 are local replacements defined in Fig. 1. This deformations are
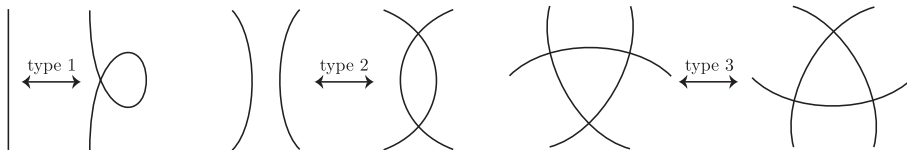

FIGURE 1. Deformations of types 1, 2, and 3

analogously to Reidemeistar moves of knot diagrams. Every knot projection is related to the trivial spherical curve by a finite sequence of deformations of types 1, 2, and 3.

In 2001 [7], Östlund conjectured the following:

Östlund Conjecture. *Deformations of types 1 and 3 are sufficient to obtain a homotopy from any generic immersion $S^1 \to \mathbb{R}^2$ to an embedding.*

In 2014, Hagge and Yazinski [2] disproved this conjecture as follows:

Hagge-Yazinski Theorem. *For P_{HY} that appears as Fig. 2, there is no finite deformations of types 1 and 3 from P_{HY} to the trivial spherical curve up to ambient isotopies.*
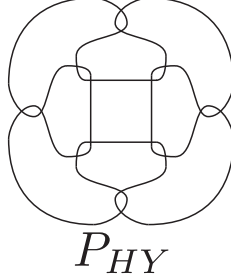

FIGURE 2. Hagge-Yazinski's example P_{HY} having 16 double points

In 2016, in [4], the authors obtain a generalization of the Hagge-Yazinski Theorem, that is, we show that there exists an infinite family of knot projections, which are counterexamples of the Östlund Conjecture for any n double points ($n \geq 15$, for 15 double points, see Fig. 3), including $P(l, m, n)$ [4, Remark 1, Page 28, Fig. 13]. Throughout of this paper, let $P(m, n) = P(1, m, n)$ (Fig. 4).
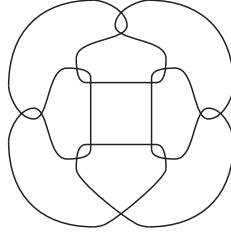

FIGURE 3. Our example having 15 double points

By experts, Fact 1 is known fact [6, Page 171].

Fact 1 ([6])**.** *Every knot projection is related to the trivial spherical curve by a finite sequence of deformations of types 1, 2, and 3 not increasing double points.*

We define that a deformation of type 2 decreasing double points is of *negative type 2*. In this paper, for a given knot projection P, we introduce RII *number* that is the minimum number of deformations of negative type 2 among sequences, each of which consists of deformations of type 1, negative type 2, and type 3. This number is denoted by RII(P). By definition, Östlund conjecture implies RII number always would be zero. Hagge-Yazinski Theorem implies that RII number is nontrivial. In contrast to them, we have Theorem 1. It is easy to see that there exists a knot projection P such that RII(P) = 0. This fact together with Theorem 1 implies that RII number can be any nonnegative number.

Two knot projections are *(1, 3) homotopic* if they are related by finite deformations of types 1 and 3 and ambient isotopies. The relation becomes an equivalence relation and is called *(1, 3) homotopy.*

Theorem 1. *Let P be a knot projection. The number $\mathrm{RII}(P)$ is an invariant under (1, 3) homotopy. For positive integers $m \geq 1$, and $n \geq 4$, let $P(m,n)$ be a knot projection, as in Fig. 4. Then, $\mathrm{RII}(P(m,n)) = m$ for any (m,n).*

$P(m,4)$ $\qquad$ $P(m,5)$ $\qquad$ $P(m,6)$ $\qquad$ $\cdots$

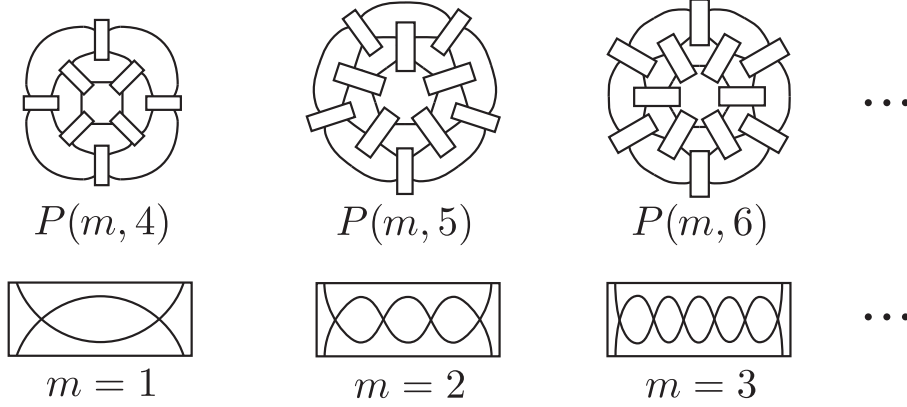$m = 1$ $\qquad\qquad$ $m = 2$ $\qquad\qquad$ $m = 3$ $\qquad$ $\cdots$

FIGURE 4. $P(m,n)$. Note that $P(1,4) = P_{HY}$.

For a proof of Theorem 1, see Section 3.

2. PRELIMINARIES

Definition 1 (RII number $\mathrm{RII}(P)$). Let P be a knot projection. The RII *number* is the minimum number of deformations of negative type 2 among sequences, each of which consists of deformations of type 1, negative type 2, and type 3. The number is denoted by $\mathrm{RII}(P)$.

Definition 2 ($(3,3)$-tangle). An unoriented $(3,3)$-tangle is the image of a generic immersion of 3 arcs into $[0,1] \times [0,1]$ such that:

- The boundary points of the arcs map bijectively to 6 points
$$\left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\} \times \{1\}, \left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\} \times \{0\}.$$
- Near the endpoints, the arcs are perpendicular to the boundary $[0,1]$.

In a $(3, 3)$-tangle, we call an image of the map of a single arc a *strand*.

Definition 3. Let P be a knot projection and F the closure of a connected component in $S^2 \setminus P$. Let n be a positive integer. Suppose that the double points of P that lie on ∂F are removed, the reminder consists of n connected components, each of which is homeomorphic to an open interval. Then, ∂F is called an *n-gon*. When we do not specify n, an n-gon is called a *polygon*.

Notation 1 ([1]). Let P and P' be two knot projections that are equivalent under deformations of type 1 and type 3, i.e., there exists a finite sequence of knot projections $P = P_0, P_1, \ldots, P_m = P'$, where P_i is obtained from P_{i-1} by a deformation of type 1 or type 3. Then, Op_i denotes the deformation from P_{i-1} to P_i, and the setting are expressed by using the notation:

$$P = P_0 \overset{Op_1}{\to} P_1 \overset{Op_2}{\to} \cdots \overset{Op_m}{\to} P_m = P'.$$

3. Proof of Theorem 1

Before starting the proof, we explain our plan of the proof.

- We show that $\mathrm{R}\mathbb{I}(P)$ is invariant under $(1, 3)$ homotopy.
- We show that $P(m, n) \geq m$. Most of the part is obtained from a similar proof of Hagge-Yazinski Theorem [2] or [4]. Therefore, a reader who is familiar to [2] can skip this part except for Section 3.6. Since it is elementary to prove it, we prove it here. We use the terminology used in [2].
- We show that $P(m, n) \leq m$. We introduce new techniques and show this part.

• **Proof of the invariance of** $\mathrm{R}\mathbb{I}(P)$**.** Suppose that arbitrary two knot projections P and P' are $(1, 3)$ homotopic. Let $m = \mathrm{R}\mathbb{I}(P')$, hence, there exists a sequence of deformations consisting of deformations of type 1, negative type 2, or type 3, and it contains exactly m deformations of negative type 2. By combining the deformations from P and P', and P' to the trivial spherical curve, we obtain a sequence from P to the trivial spherical curve which contains exactly m deformities of negative type 2. This shows that $\mathrm{R}\mathbb{I}(P) \leq \mathrm{R}\mathbb{I}(P')$. Since the argument is symmetric, we see that $\mathrm{R}\mathbb{I}(P) \geq \mathrm{R}\mathbb{I}(P')$ holds, too. Hence, $\mathrm{R}\mathbb{I}(P) = \mathrm{R}\mathbb{I}(P')$.

3.1. **Proof of** $P(m, n) \geq 1$**.** For any $P(m, n)$, there exist $2n$ boxes such that the intersections of $P(m, n)$ and $2n$ boxes are $(3, 3)$-tangles, and arcs with no double points outside the boxes (Figs. 4 and 5). Each box is equivalent to $[0, 1] \times [0, 1]$ corresponding to a $(3, 3)$-tangle. For example, for the case $m = 1$ and $n = 4$, 8 boxes are shown in Fig. 5. Each $P(m, n)$ satisfies the following two conditions:
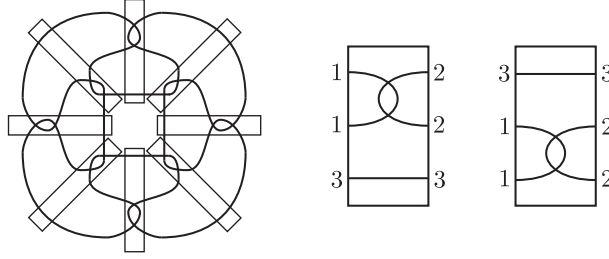

FIGURE 5. $P(1, 4)$ with boxes (left), strands 1, 2, and 3 (right)

(1) No double points are placed outside the boxes. Three arcs connect two adjacent boxes concentrically. There exist exactly two polygons, each of which has at least n sides partially outside boxes.

(2) If we fix our gazing direction from infinity, which is selected as shown Figs. 4 and 5, we define the *left-side* and *right-side* of each box. In each box, strand 1 (2, resp.) is a strand that begins and ends on the left-side (right-side, resp.). In each box, strand 3 has one endpoint on the left-side and another endpoint the right-side. There exists a pair of strand 1 and strand 2; for each pair, strand 1 and strand 2 intersect at exactly $2m$ double points. Strand 1 of a pair cannot intersect strand 2 belonging to a different pair.

Let m and n be positive integers ($m \geq 1, n \geq 4$). In general, for a knot projection Q, we say that Q satisfies $(2m, 2n)$ *box property* if there exist $2n$ boxes B_i ($1 \leq i \leq 2n$) in S^2, as shown in Fig. 6, satisfying the following (A) and (B).

(A) There exist no double points of Q in $S^2 \setminus (B_1 \cup B_2 \cup \dots B_{2n})$, and $Q \cap (S^2 \setminus (B_1 \cup B_2 \cup \dots B_{2n}))$ consists of $3 \times 2n$ simple arcs as in Fig. 6.

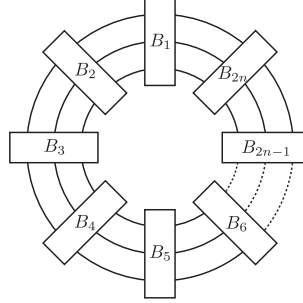FIGURE 6. Boxes B_1, B_2, ..., B_{2n}

(B) For each i $(1 \le i \le 2n)$, $Q \cap B_i$ is a $(3, 3)$-tangle that is the union of three immersed arcs. Then it satisfies the following condition.
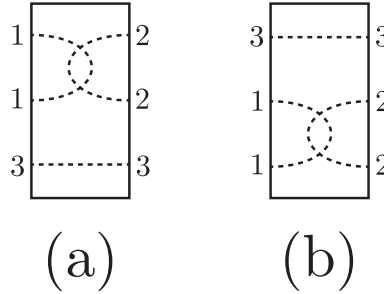

FIGURE 7. Types (a) and (b)

- If i is an odd (even, resp.), the endpoints of the three arcs are located on ∂B_i as in Fig. 7 (a) ((b), resp.), where we name the immersed arcs 1, 2, 3 as in Fig. 7 (a) ((b), resp.). Further, there exist at least $2m$ double points formed by a subarc of 1, and a subarc of 2.

• *Proof of* RII$(P(m,n)) \ge 1$. We consider an inductive proof with respect to the number of deformations, type 1 or 3, applied to $P(m,n)$. This induction proves Claim 1, which implies that $P(m,n)$ cannot be $(1, 3)$ homotopic to the trivial spherical curve; that is, RII$(P(m,n)) \ge 1$. Recall Notation 1.

Claim 1. *Let Q be a knot projection satisfying a $(2m, 2n)$ box property. Then, for each sequence of knot projections.*

$$Q = Q_0 \overset{Op_1}{\to} Q_1 \overset{Op_2}{\to} \dots \overset{Op_r}{\to} Q_r$$

such that each Op_i $(1 \le i \le r)$ is of type 1 or 3, we have:
*   by retaking the boxes if necessary, each Q_i $(0 \le i \le r - 1)$ satisfies $(2m, 2n)$ box property.*

First, for Q_0, it is clear that Claim 1 holds. Second, we suppose that Claim 1 holds for Q_{r-1} and prove it Q_r.

3.2. **On a deformation of type 1 or type 3 occurring within a box.** Suppose that the Op_r is a single deformation (type 1 or type 3) occurring entirely within a box. This deformation fixes the endpoints of the strands, and thus Claim 1 holds.

3.3. **On a deformation of type 1 not occurring within a box and increasing double points.** Suppose that Op_r is a single deformation of type 1 increasing double points. If the new 1-gon produced by Op_r is outside the boxes, by retaking the box by a sphere isotopy, as shown in Fig. 8, Op_r is entirely within a box. If the new 1-gon produced by Op_r is not completely outside the boxes, a similar modification works to retake the box.
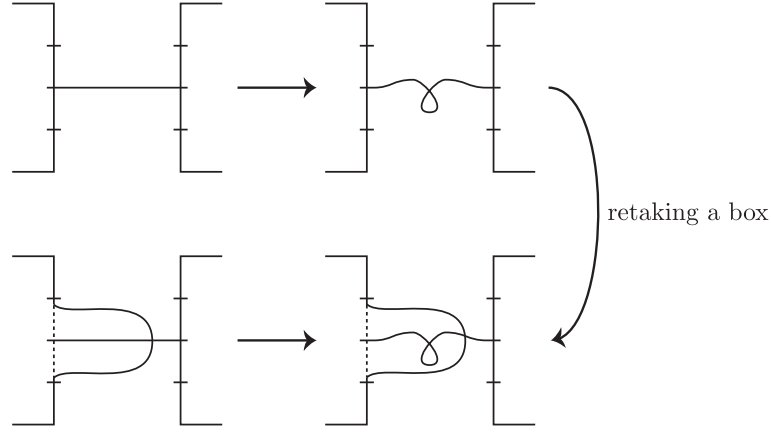
FIGURE 8. Retaking a box

3.4. **On a deformation of type 1 not occurring within a box and decreasing double points.** Suppose that the Op_r does not occur within a box. Since Op_r is a deformation of type 1 decreasing double points, there exists a 1-gon to be removed in Q_{r-1}. The two possibilities of appearing of a 1-gon are considered as follows.

- Suppose that the 1-gon contains a region having one side, as shown in Fig. 9. By the induction assumption of P_{r-1}, the region has at least four sides, which is a contradiction.
- Suppose that there exists the 1-gon containing a region having two sides outside the boxes, as shown in Fig. 10. If one of the two sides is directly
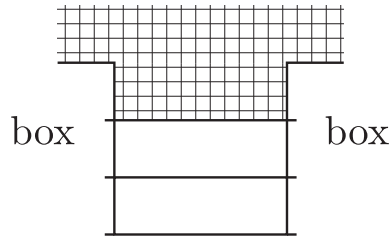

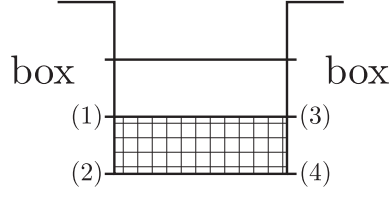FIGURE 9. A region having one side in $S^2 \setminus (B_1 \cup B_2 \cup \ldots B_{2n})$

FIGURE 10. Each of endpoints (1), (2), (3), and (4) connects to strand 1, 2, or 3.

connected to either strand 1 or 2, then the 1-gon has at least two double points, which is a contradiction (there is no 1-gon with two double points). If one of the two sides is directly connected to strand 3 that is connected strand 1 or 2 in the adjacent box, then this also implies a contradiction which is similar to the case above.

3.5. **On a deformation of type 3 not occurring within a box.** If Op_r is a single deformation of type 3, we focus on the 3-gon T_{r-1} in Q_{r-1} with respect to the single deformation of type 3, as shown in Fig. 11.

- Suppose that T_{r-1} contains a region having one side, as shown in Fig. 9. By the induction assumption of Q_{r-1}, the region has at least four sides, which is a contradiction.
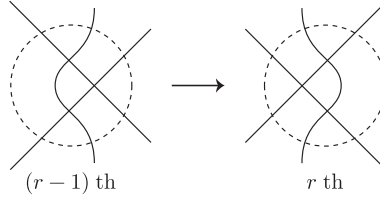

FIGURE 11. 3-gon T_{r-1} in Q_{r-1} with respect to a single deformation of type 3

- Suppose that T_{r-1} contains a region having two sides outside the boxes, as shown in Fig. 12.
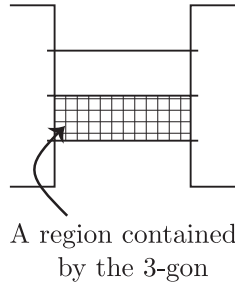

A region contained
by the 3-gon

FIGURE 12. A region surrounded by the 3-gon T_{r-1} appearing in Q_{r-1}

(1) Suppose that T_{r-1} does not have at least one double point inside a box. There is an arc inside a box, as shown in Fig. 13, which starts and ends on the same side. Then, by the induction assumption of
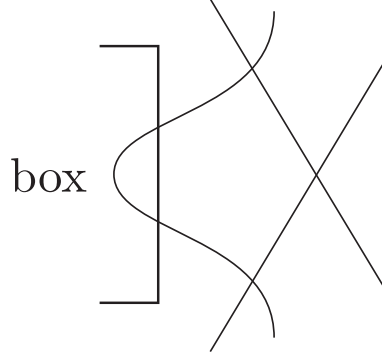

FIGURE 13. Impossible case

Q_{r-1}, the arc should be either subsrc 1 or 2, which is a contradiction because an arc has no intersection whereas both subarcs 1 and 2 have intersections.

(2) Suppose that T_{r-1} has at least one double point in a box. By the induction assumption, there is no double point outside the boxes for Q_{r-1}. Thus, if T_{r-1} is not inside a box, there are exactly two cases.
  (a) Case 1. A double point of T_{r-1} is inside a box and the other two double points of T_{r-1} are in another box, as shown in the left figure of Fig. 14.
  (b) Case 2. The three double points of T_{r-1} are in three different boxes, respectively, as shown in the right figure of Fig. 14.
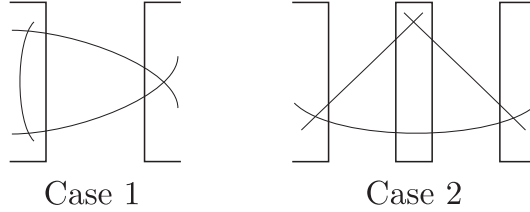

Case 1   Case 2

FIGURE 14. Case 1 (left) and Case 2 (right)

As a result, for Case 1 (Case 2, resp.), by a sphere isotopy, by retaking one box (two boxes, resp.), as shown in Fig. 15, T_{r-1} is contained entirely within a box.

This completes the proof of $\mathrm{R}\mathbb{II}(P(m,n)) \geq 1$.

3.6. **Proof of** $\mathrm{R}\mathbb{II}(P(m,n)) \geq m$. Recall Claim 1. By the argument of the above proof of Claim 1, we have Lemma 1.
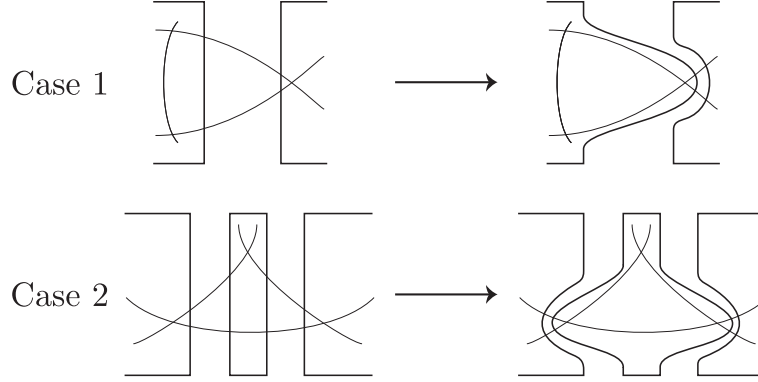
FIGURE 15. Retaking box(es) for Case 1 (upper) and Case 2 (lower)

Lemma 1. *Let k be an integer $(0 \leq k \leq r)$. Let $Q_0 = P(m,n)$ and Q_k be the knot projection obtained from $P(m,n)$ by Op_k using Notation 1:*

$$P(m,n) = Q_0 \stackrel{Op_1}{\to} Q_1 \stackrel{Op_2}{\to} \cdots \stackrel{Op_r}{\to} Q_r,$$

where the sequence consists of a single deformation of negative type 2 and deformations of type 1 and 3. Suppose that Op_{k+1} is a deformation of negative type 2 within a box.

Then, we have two following statements.

(1) *Each of $\{Q_0, Q_1, \ldots, Q_k\}$ preserves $(2m, 2n)$ box property.*
(2) *Q_{k+1} satisfies $(2m-2, 2n)$ box property.*
(3) *Each of $\{Q_{k+2}, Q_{k+3}, \ldots, Q_r\}$ preserves $(2m-2, 2n)$ box property.*

Proof. (1) ((3), resp.) By using the argument of the proof of Claim 1, we may suppose that every Op_j $(j \neq k+1)$ is a deformation of type 1 or 3 within a box by retaking boxes. Since Op_j of type 1 or 3 within a box preserve $(2m, 2n)$ $((2m-2, 2n)$, resp.) box property, the statement (1) ((3), resp.) holds.
(2) The deformation of negative type 2 decreases or preserve double points formed by a subarc of 1 and a subarc of 2. It implies the statement (2). □

Lemma 1 immediately implies $RII(P(m,n)) \geq m$.

3.7. **Proof of $P(m,n) \leq m$.** We prepare Lemma 2.

Lemma 2. *Let k be a positive integer. Each of replacements $T(2k-1)$ and $T(2k)$ as in Fig. 16 is always possible under $(1, 3)$ homotopy.*

Proof. The statement holds in the induction with respect to k for $T(2k)$ and $T(2k-1)$.

- Case $k = 1$: for $T(1)$, the statement is clear. $T(2)$ holds as in Fig. 17.
- Case $k = i$: Suppose that $T(2i-2)$ is a possible local replacement under $(1, 3)$ homotopy. In this case, $T(2i-1)$ holds, which also implies $T(2i)$. See Fig. 18.

It completes the induction step. □
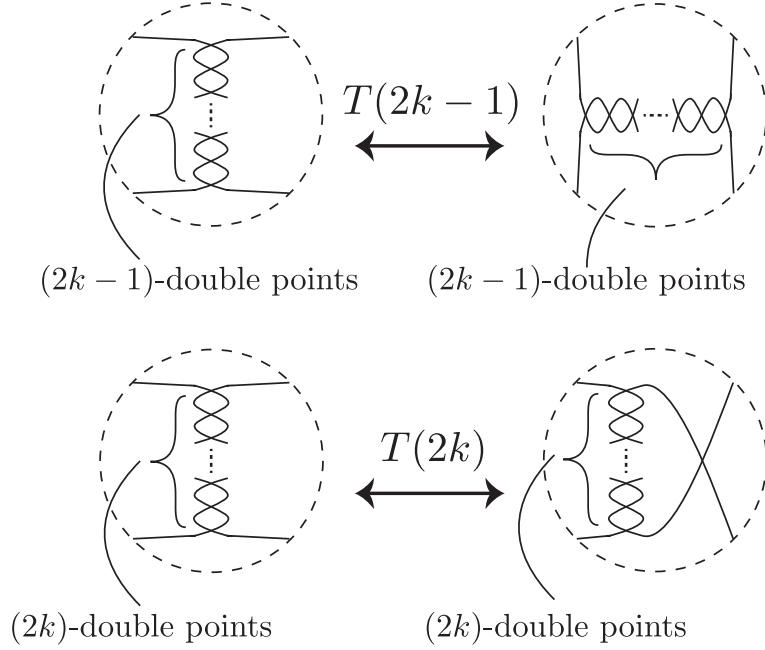
FIGURE 16. Definitions of $T(2k-1)$ and $T(2k)$
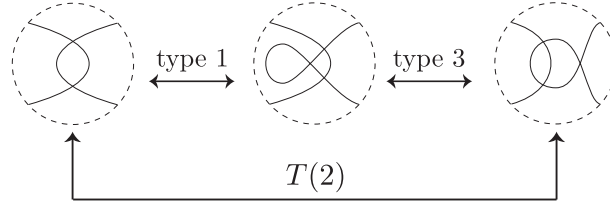
FIGURE 17. $T(2)$

• *Proof of* $\mathrm{R}\mathrm{II}(P(m,n)) \leq m$. By applying m deformations of negative type 2 to a $(3, 3)$-tangle corresponding to a box and a $(1, 3)$ homotopy, we may suppose that there is no double point within the box. Then, we apply $T(2m)$ $2m$ times decreasing double points to erase each single box as in Fig. 19.

Finally, we apply m deformations of type 1 as in Fig. 20.

$\square$

Remark 1. The operation $T(n)$ $(n \in \mathbb{Z}_{>0})$ of Lemma 2 corresponds to a generalization of a type of an edge of a complex [1] or the operation α [3].

4. Applications

4.1. Every pretzel knot projection is (1, 3) homotopic to the trivial spherical curve.

Notation 2. A part consisting of m double points of a knot projection as in Fig. 4, it is called a *twist*.
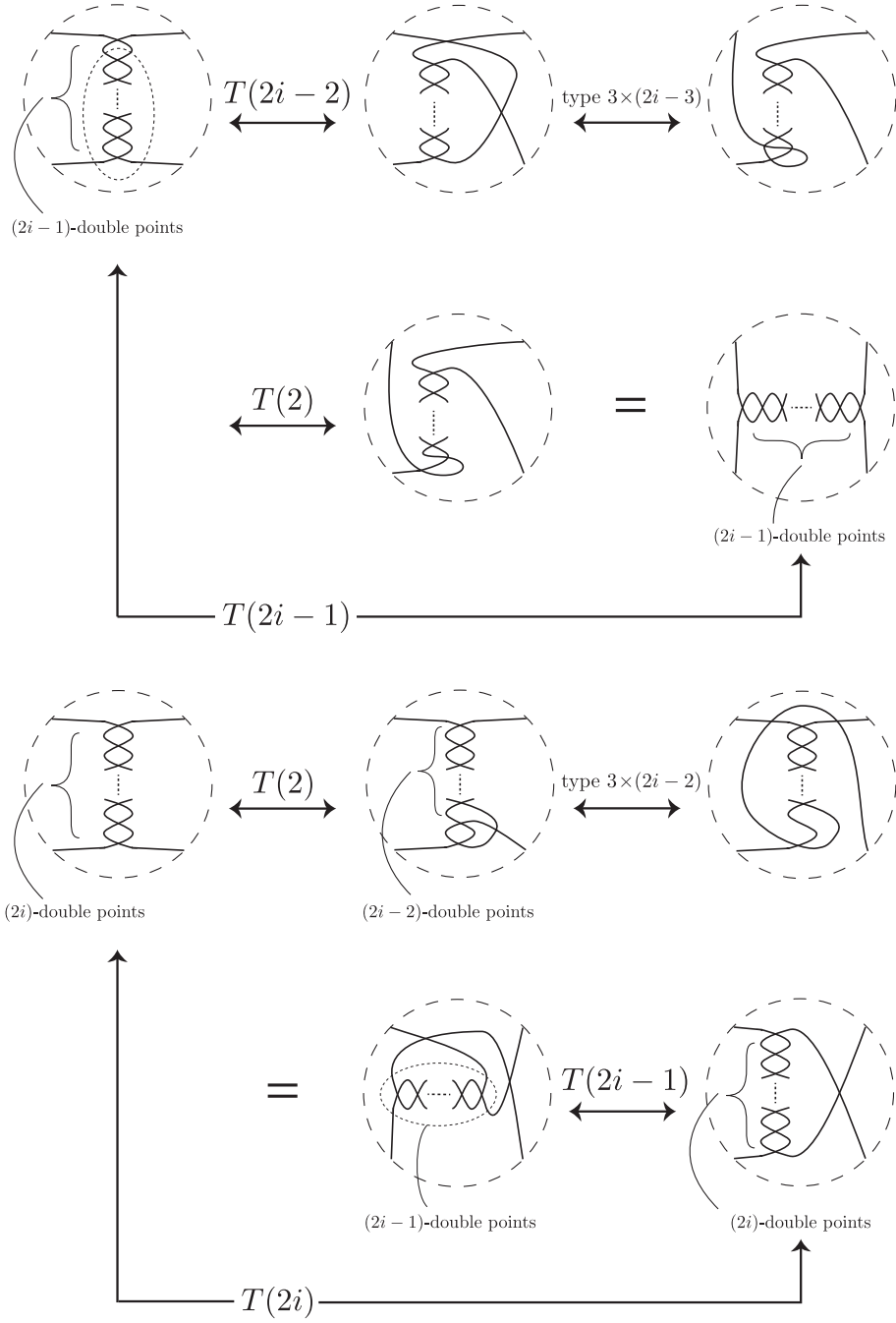
FIGURE 18. An inductive argument of $T(2i-1)$ and $T(2i)$

Proposition 1. *Every pretzel knot projection as in Fig. 21 is (1, 3) homotopic to the trivial spherical curve.*
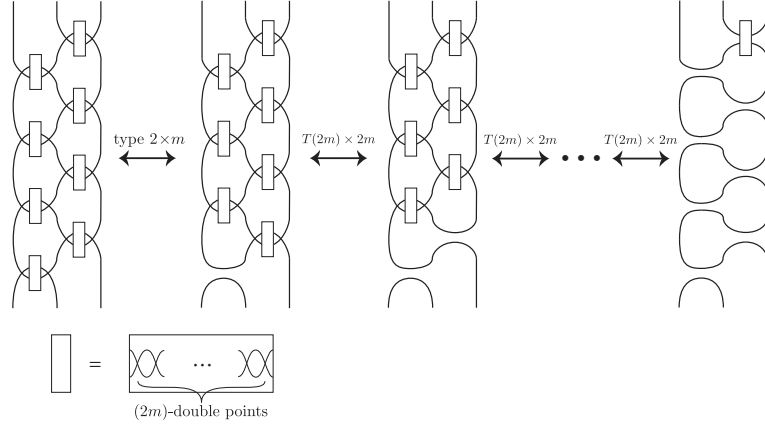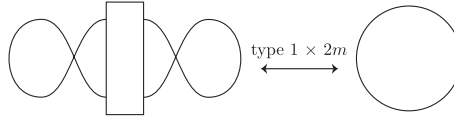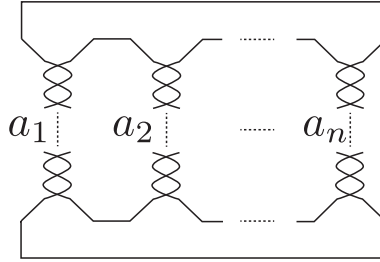
FIGURE 19. Applications of Lemma 2

FIGURE 20. $2m$ deformations of type 1

Let $P(a_1, a_2, \ldots, a_n)$ be a knot projection as in Fig. 21, where each a_i presents double points of a twist $(1 \le i \le n)$.


FIGURE 21. Pretzel knot projection $P(a_1, a_2, \ldots, a_n)$

It is sufficient to consider the two cases.

- Case 1: for each i, a_i are odd double points and n is an odd positive number.
- Case 2: a_1 are even double points and for $i \ne 1$, a_i are odd double points.

It is easy to see that every $(2, 2p + 1)$-torus knot projection $(p \in \mathbb{N})$, as shown in Fig. 22, is $(1, 3)$ homotopic to the trivial spherical curve by applying $T(2p + 1)$ and applying deformations of type 1 decreasing double points.

For Case 1, we apply $T(a_i)$ to each twist, and we have a $(2, 2p + 1)$-torus knot projection where $2p + 1 = \sum_{i=1}^{n} a_i$. For Case 2, since a_i $(i \ne 1)$ is an odd number, we apply $T(a_i)$ to each twist corresponding to a_i double points $(i \ne 1)$, and apply $T(a_1)$ $(\sum_{i=2}^{n} a_i$ times) to the resulting knot projection, which implies a knot projection
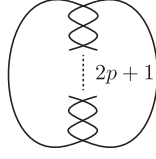
FIGURE 22. $(2, 2p + 1)$-torus knot projection

having exactly a_1 double points. After that, deformations of type 1 decreasing double points obtain the trivial spherical curve.

4.2. **Every two-bridge knot projection is (1, 3) homotopic to the trivial spherical curve.**

Proposition 2. *Every two-bridge knot projection as in Fig. 23 is (1, 3) homotopic to the trivial spherical curve.*
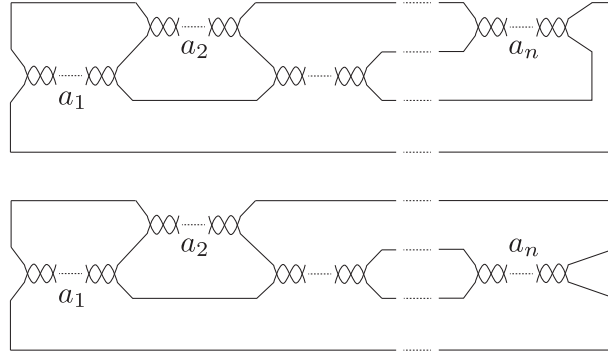
FIGURE 23. Two-bridge knot projection $P(a_1, a_2, \ldots, a_n)$

Let $P(a_1, a_2, \ldots, a_n)$ be a knot projection as in Fig. 23, where each a_i presents double points of a twist $(1 \le i \le n)$.

- Case 1: Suppose that a_1 is an odd number corresponding to a_1 double points. By applying a single $T(a_1)$, two twists a_1 and a_2 merge a twist consisting of $a_1 + a_2$ double points.
- Case 2: Suppose that a_1 is an even number corresponding to a_1 double points. By applying $T(a_1)$ a_2 times, a_2 double points are resolved, which implies a twist disappears.

By an induction of the number of twists, we resolve every twist.

Remark 2. By using Lemma 2, for a knot projection P consisting of two-bridge knot projections via tangle sums or connected sums, it is elementary to show that such P is (1, 3) homotopic to the trivial spherical curve by using Propositions 1 and 2.

Master thesis, on September 20, 2018. This work included $T(2k-1)$. Then, we sent a preprint (corresponding to an earlier version of this paper) with a slide of a talk in a Colloquium of Department of Mathematics, Faculty of Education, Waseda University on December 18, 2015, which includs Lemma 2. The motivations of the two works, [5] and this paper are different. The author would like to thank Professor Tsuyoshi Kobayashi for encouraging us to write this paper and giving us an information of [5]. Ms. Sumika Kobayashi gave another application of Lemma 2 by other motivation of hers [5].

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