SPACE OF CHORD DIAGRAMS ON SPHERICAL CURVES

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ABSTRACT. In this paper, we give a definition of $\mathbb{Z}$-valued functions from the ambient isotopy classes of spherical curves derived from chord diagrams, denoted by $\sum_i \alpha_i x_i$. Then, we introduce certain elements of the free $\mathbb{Z}$-module generated by the chord diagrams with at most l chords, called relators of Type (I) ((SII), (WII), (SIII), or (WIII), resp.), and introduce another function $\sum_i \alpha_i \tilde{x}_i$ derived from $\sum_i \alpha_i x_i$. The main result (Theorem 1) shows that if $\sum_i \alpha_i \tilde{x}_i$ vanishes for the relators of Type (I) ((SII), (WII), (SIII), or (WIII), resp.), then $\sum_i \alpha_i x_i$ is invariant under the Reidemeister move of type RI (strong RII, weak RII, strong RIII, or weak RIII, resp.) that is defined in [4].

1. INTRODUCTION

A spherical curve (plane curve, resp.) is the image of a generic immersion of a circle into a 2-sphere (plane, resp.). In this paper, we study certain equivalence classes of spherical curves. Any two spherical/plane curves can be transformed into each other by a finite sequence of Reidemeister moves, each of which is either one of types RI, RII, or RIII that is a replacement of a part of the curve as Figure 1. These moves are obtained from Reidemeister moves of types Ω_1, Ω_2, and Ω_3 on
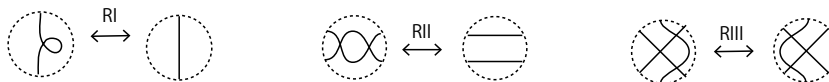

FIGURE 1. Reidemeister moves: types RI, RII, and RIII.

knot diagrams by ignoring over/under information. It is well-known that two knot diagrams represent the same knot if and only if one knot diagram is transformed into the other knot diagram by a finite sequence of Reidemeister moves of types Ω_1, Ω_2, and Ω_3.

In 1990, Vassiliev [9] defined an infinite sequence of knot invariants that are at least as powerful as all of the quantum group invariants of knots. After that some researchers gave alternating definitions of Vassiliev invariants (see [1]). In 1994, Polyak and Viro [8] introduced a formulation of Vassiliev invariants using chord diagram, and presented explicit formulas for some Vassiliev invariants. Goussarov [2] proved that the formulation of Polyak-Viro provides all Vassiliev invariants of knots. In 2001, Östlund further developed the theory of Polyak and Viro by using singularity theory of plane curves, and obtained the following result [7, Theorem 6]: if v is a function on the set of knot diagrams that is invariant under plane isotopy, Ω_1, and Ω_3 and is so-called Vassiliev-type, then v is invariant under Ω_2. Motivated

1

by this result, Östlund [7] posed the following question: is every plane curve transformed into the simple closed curve by a finite sequence of Reidemeister moves of types RI and RⅢ?

In 2008, Hagge and Yazinski [3] show that the answer to this question is negative, i.e., they showed that the equivalence classes under RI and RⅢ are nontrivial. In fact, any finite sequence of Reidemeister moves that transforms the plane curve depicted in Figure 2 into the simple closed curve contains a Reidemeister move of type RⅡ. However, to the best of the author's knowledge, the equivalence class
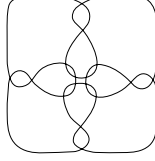

FIGURE 2. Example by Hagge and Yazinski [3].

containing the simple closed curve under RI and RⅢ has not to be determined yet. In general, none of the equivalence classes under RI and RⅢ have been determined yet.

Viro [10] suggested the idea that type RⅢ is decomposed into the following two types: suppose that P_1 is transformed into P_2 by a single RⅢ. Note that in RⅢ of Figure 1, a triangle is observed in each of the disks. We say that P_1 and P_2 are *related by a strong RⅢ* if the orientations on the edges of the triangle induced by an orientation of the spherical curve are coherent. If P_1 and P_2 are not related by a strong RⅢ, then we say that P_1 and P_2 are *related by a weak RⅢ*. Analogously, for type RⅡ, we define strong RⅡ and weak RⅡ as follows. Suppose that P_1 is transformed into P_2 by a single RⅡ. We say that P_1 and P_2 are *related by a strong RⅡ* if the orientations on the edges of the triangle induced by an orientation of the spherical curve are coherent. If P_1 and P_2 are not related by a strong RⅡ, then we say that P_1 and P_2 are *related by a weak RⅡ*. See Figure 3.
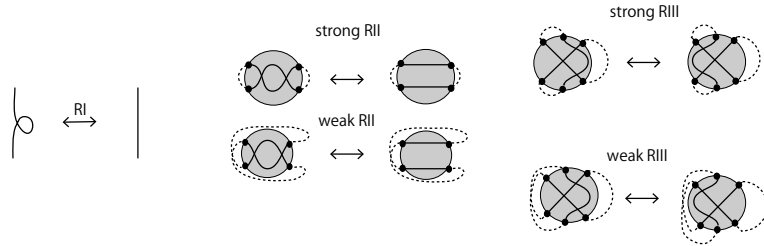

FIGURE 3. RI, strong RⅡ, weak RⅡ, strong RⅢ, and weak RⅢ.

From the viewpoint of Viro, the following problems arise from the above Östlund's question: which spherical curve is transformed into the simple closed curve by a finite sequence of Reidemeister moves of types RI and strong RⅢ (types RI and strong RⅢ, resp.)? First, in 2013, a necessary and sufficient condition that a spherical curve and the simple closed curve are related by a finite sequence of Reidemeister moves of types RI and weak RⅢ was given [4]. Second, in 2015, a necessary

and sufficient condition that a spherical curve and the simple closed curve are related by a finite sequence of Reidemeister moves of types RI and strong RIII was given [6]. After that, in 2015, different necessary and sufficient conditions using invariants were given. More concretely, for a chord diagram corresponding to a spherical curve, by counting the number of sub-chord diagrams of certain types, integer-valued functions λ, H, and $\widetilde{X}$ were defined, and the following was shown: a spherical curve P and the simple closed curve are related by a finite sequence of Reidemeister moves of types RI and strong RIII (types RI and weak RIII, resp.) if and only if $H(P) = 0$ and $\lambda(P) = 0$ ($\widetilde{X}(P) = 0$, resp.).

In this paper, we further develop the idea of counting sub-chord diagrams, and which leads to functions of all equivalence classes of spherical curves generated by all possible combinations of Reidemeister moves of types RI, strong RII, weak RII, strong RIII, and weak RIII. In [8], for a chord diagram A, and a chord diagram G obtained from a knot diagram, Polyak and Viro define an integer $\langle A, G \rangle$. They further consider a linear combination of $\langle A, \cdot \rangle$, that is denoted by $\langle \sum_i r_i A_i, \cdot \rangle$, which is a function on knot diagrams, and show that this type of the function induces a non-trivial Vassiliev knot invariant. We will mimic their idea to define a function, denoted by $\sum \alpha_i x_i$, where x_i denotes a chord diagram. Roughly speaking, $x_i(P)$ is obtained from $\langle A, G \rangle$ by ignoring "signs and arrows", and this means that we ignore the over/under informations of crossings and orientations of knot diagrams. In Section 2, we introduce certain elements of the free $\mathbb{Z}$-module $\mathbb{Z}[G_{\leq l}]$ generated by the chord diagrams with at most l chords, called relators of Type (I) ((SII), (WII), (SIII), or (WIII), resp.), and introduce another function $\sum_i \alpha_i \tilde{x}_i$ derived from $\sum_i \alpha_i x_i$. The main result (Theorem 1) shows that if $\sum_i \alpha_i \tilde{x}_i$ vanishes for the relators of Type (I) ((SII), (WII), (SIII), or (WIII), resp.), then $\sum_i \alpha_i x_i$ is invariant under RI (strong RII, weak RII, strong RIII, or weak RIII, resp.). In Section 4, we show that Theorem 1 produces a new function that is invariant under RI and strong RIII and is able to distinguish some pairs which the function λ of [5] cannot distinguish.

2. PRELIMINARIES AND MAIN RESULTS

2.1. Definitions and notations.

Definition 1 (Gauss word). A *word* w of length n is a map from $\hat{n} = \{1, 2, 3, \ldots, n\}$ to $\mathbb{N}$. This word is represented by $w(1)w(2)w(3)\cdots w(n)$. For a word $w : \hat{n} \to \mathbb{N}$, we call each element of $w(\hat{n})$ a *letter*. A word u of length q is a *sub-word* of w if there is an integer p ($q \leq p \leq n$) such that $u(j) = w(n - p + j)$ ($1 \leq j \leq q$). A *Gauss word* of length $2n$ is a word w of length $2n$ where each letter in $w(\hat{2n})$ appears exactly twice in $w(1)w(2)w(3)\cdots w(2n)$. Let cyc and rev be maps $\hat{2n} \to \hat{2n}$ such that $\mathrm{cyc}(p) \equiv p + 1 \pmod{2n}$ and $\mathrm{rev}(p) \equiv -p + 1 \pmod{2n}$. Two Gauss words, v and w, of length $2n$ are *isomorphic* if there exists a bijection $f : v(\hat{2n}) \to w(\hat{2n})$ satisfying the following: there exists $t \in \mathbb{Z}$ such that $w \circ (\mathrm{cyc})^t \circ (\mathrm{rev})^\epsilon = f \circ v$ ($\epsilon = 0$ or 1). The isomorphisms give an equivalence relation on the Gauss words. For a Gauss word v of length $2n$, $[v]$ denotes the equivalence class containing v. A Gauss word v' is a *sub-Gauss word* of the Gauss word v if v' is obtained from v by ignoring some letters of v. Then $\mathrm{Sub}(v)$ denotes the set of sub-Gauss words of v.

Definition 2 (chord diagram). Each configuration of n pair(s) of points on a circle up to ambient isotopy and reflection of the circle is called a *chord diagram*. The

integer n is called the *length* of the chord diagram. Traditionally, two points of each pair are connected by a straight arc, called a *chord*.

We note that the equivalence classes of the Gauss words of length $2n$ have one to one correspondence with the chord diagrams, each of which has n chords as in Figure 4. In the rest of this paper, we identify these four expressions in Figure 4,
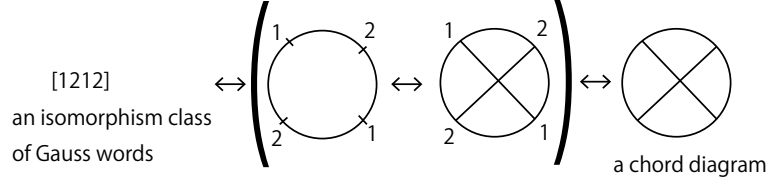

FIGURE 4. Four expressions.

and freely use either one of them depending on situations.

Notation 1 ($G_{\leq d}$, n_d, $G_{b,d}$). Let $G_{<\infty}$ be the set of chord diagrams, that is, the set of isomorphism classes of the Gauss words. It is clear that $G_{<\infty}$ consists of countably many elements. Hence, there exists a bijection between $G_{<\infty}$ and $\{x_i\}_{i\in\mathbb{N}}$, where x_i is a variable. Take and fix a bijection $f : G_{<\infty} \to \{x_i\}_{i\in\mathbb{N}}$ satisfying: the number of chords of $f^{-1}(x_i)$ is less than or equal to that of $f^{-1}(x_j)$ if and only if $i \leq j$ ($i, j \in \mathbb{N}$). For each positive integer d, let $G_{\leq d}$ be the set of chord diagrams consisting of at most d chords and let $n_d = |G_{\leq d}|$. Then, it is clear that $f|_{G_{\leq d}}$ is a bijection from $G_{\leq d}$ to $\{x_1, x_2, \ldots, x_{n_d}\}$. Further, for each pair of integers b and d ($2 \leq b \leq d$), let $G_{b,d} = G_{\leq d} \setminus G_{\leq b-1}$. Then, $f|_{G_{b,d}}$ is a bijection $G_{b,d} \to \{x_{n_{b-1}+1}, x_{n_{b-1}+2}, \ldots, x_{n_d}\}$.

In the rest of this paper, we use the notations in Notation 1 unless otherwise denoted, and we freely use this identification between $G_{<\infty}$ and $\{x_i\}_{i\in\mathbb{N}}$.

Definition 3 (a chord diagram CD_P of a spherical curve P). Let P be a spherical curve, i.e. there is a generic immersion $g : S^1 \to S^2$ such that $g(S^1) = P$. We define a chord diagram of P (e.g., Figure 5) as follows: let k be the number of the double points of P, and $m_1, m_2, \ldots, m_k$ mutually distinct positive integers. Fix a base point, which is not a double point on P, and choose an orientation of
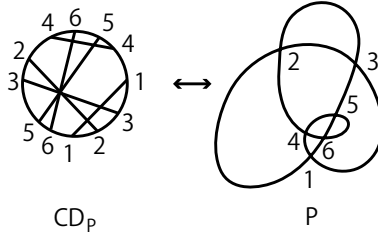

FIGURE 5. A chord diagram CD_P of a spherical curve P.

P. Starting from the base point, proceed along P according to the orientation of

P. We assign m_1 to the first double point that we encounter. Then we assign m_2 to the next double point that we encounter provided it is not the first double point. Suppose that we have already assigned $m_1, m_2, \ldots, m_p$. Then we assign m_{p+1} to the next double point that we encounter if it has not been assigned yet. Following the same procedure, we finally label all the double points. Note that g^{-1}(double point assigned m_i) consists of two points on S^1 and we shall assign m_i to them. The chord diagram represented by g^{-1}(double point assigned m_1), g^{-1}(double point assigned m_2), $\ldots$, g^{-1}(double point assigned m_k) on S^1 is denoted by CD_P and is called *a chord diagram of the spherical curve P*.

Recall that CD_P gives an equivalence class of Gauss words, say $[v_P]$. Then, by the definition of the equivalence relation, it is easy to see that the map $P \mapsto [v_P]$ is well-defined.

Notation 2 $(x(CD))$**.** Let $x \in \{x_i\}_{i \in \mathbb{N}}$ (hence, x_i corresponds to a chord diagram). For a given chord diagram CD, fix a Gauss word G representing CD. Let $\mathrm{Sub}_x(G) = \{H \mid H \in \mathrm{Sub}(G), [H] = x\}$. The cardinality of this subset is denoted by $x(G)$, i.e., $x(G) = |\mathrm{Sub}_x(G)|$. Let G' be another Gauss word representing CD. By the definition of the isomorphism of the Gauss words, it is easy to see $x(G') = x(G)$. Hence, we shall denote this number by $x(CD)$. If CD is a chord diagram of a spherical curve P, then $x(CD)$ can be denoted by $x(P)$.

Since each equivalence class of the Gauss words is identified with a chord diagram, we can calculate the number $x(CD)$ by using geometric observations. We explain this philosophy in the next example.

Example 1. We consider the chord diagram CD in Figure 6 (Note that $CD = CD_P$ in Figure 5). Then we label the chords of CD by α_i ($1 \le i \le 6$) as in Figure 6. Consider the subset of the power set of $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$, each element of which represents a chord diagram isomorphic to $\otimes$. It is elementary to see that this subset consists of ten elements, those are, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_2, \alpha_3\}$, $\{\alpha_2, \alpha_4\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_1, \alpha_5\}$, $\{\alpha_2, \alpha_5\}$, $\{\alpha_3, \alpha_6\}$, and $\{\alpha_4, \alpha_6\}$, and that fact shows that $\bigotimes(CD) = 10$. Similarly, we have $\bigotimes(CD) = 6$ and $\bigoplus(CD) = 8$.
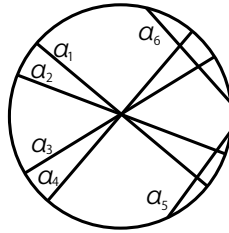

FIGURE 6. CD.

Let CD be a chord diagram consisting of chords $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then the chord diagram consisting of a subset of $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is called a *sub-chord diagram* of CD.

Definition 4 ($\tilde{x}(z), \tilde{x}([z])$)**.** Let x be a chord diagram. We define the function $\tilde{x}$ from the set of Gauss words to $\{0, 1\}$ by

$$\tilde{x}(F) = \begin{cases} 1 & [F] = x \\ 0 & [F] \neq x. \end{cases}$$

By definition, it is easy to see $\tilde{x}(F_1) = \tilde{x}(F_2)$ for each pair F_1 and F_2 with $[F_1] = [F_2]$. Hence, we shall denote this number by $\tilde{x}([F_1])$. If $[F]$ corresponds to a chord diagram of a spherical curve P, then $\tilde{x}([F])$ is denoted by $\tilde{x}(P)$. Further, let $\mathbb{Z}[G_{\leq l}]$ be the free $\mathbb{Z}$-module generated by the elements of $G_{\leq l}$, where l is sufficiently large. We linearly extend $\tilde{x}$ to the function from $\mathbb{Z}[G_{\leq l}]$ to $\mathbb{Z}$. It is clear that for any Gauss word with $[G] = CD$,

$$(2.1) \qquad x(CD) = \sum_{z \in \mathrm{Sub}(G)} \tilde{x}(z).$$

2.2. **Relators and Reidemeister moves.** Let $\mathbb{Z}[G_{\leq l}]$ be the free $\mathbb{Z}$-module defined in Subsection 2.1. In this subsection, first we define the elements of $\mathbb{Z}[G_{\leq l}]$ called relators of types (I), (SII), (WII), (SIII), and (WIII).

Definition 5 (Relators, cf. Figure 7)**.**
- Type (I). An element r of $\mathbb{Z}[G_{\leq l}]$ is called a *Type* (I) *relator* if there exist a Gauss word S and a letter i not in S such that $r = [Sii]$.
- Type (SII). An element r of $\mathbb{Z}[G_{\leq l}]$ is called a *Type* (SII) *relator* if there exist a Gauss word ST and letters i and j not in ST such that $r = [SijTji] + [SiTi] + [SjTj]$.
- Type (WII). An element r of $\mathbb{Z}[G_{\leq l}]$ is called a *Type* (WII) *relator* if there exists a Gauss word ST and letters i and j not in ST such that $r = [SijTij] + [SiTi] + [SjTj]$.
- Type (SIII). An element r of $\mathbb{Z}[G_{\leq l}]$ is called a *Type* (SIII) *relator* if there exists a Gauss word STU and letters i, j, k not in STU such that $r = ([SijTkiUjk] + [SijTiUj] + [SiTkiUk] + [SjTkUjk]) - ([SjiTikUkj] + [SjiTiUj] + [SiTikUk] + [SjTkUkj])$.
- Type (WIII). An element r of $\mathbb{Z}[G_{\leq l}]$ is called a *Type* (WIII) *relator* if there exists a Gauss word STU and letters i, j, k not in STU such that $r = ([SijTikUjk] + [SijTiUj] + [SiTikUk] + [SjTkUjk]) - ([SjiTkiUkj] + [SjiTiUj] + [SiTkiUk] + [SjTkUkj])$.

We note that in Definition 5, each relator is induced by the corresponding Reidemeister move: Type (I) relator corresponding to RI, Type (SII) relator corresponding to strong RII, Type (WII) relator corresponding to weak RII, Type (SIII) relator corresponding to strong RIII, and Type (WIII) relator corresponding to weak RIII. For the precise statement of this note, we introduce the following setting.

We first introduce the following notations. Let P and P' be two spherical curves. If P and P' are related by a single RI (strong RII, weak RII, strong RIII, or weak RIII resp.), then there are Gauss words G and G' such that (by exchanging P and P', if necessary), $G = Sii$ ($SijTji$, $SijTij$, $SijTkiUjk$, or $SijTikUjk$ resp.) and $G' = S$ (ST, ST, $SjiTikUkj$, or $SjiTkiUkj$ resp.) such that $[G] = CD_P$ and $[G'] = CD_{P'}$ (Figure 8). The subset of $\mathrm{Sub}(G)$ such that each element has exactly m letters of i, j, and k is denoted by $\mathrm{Sub}^{(m)}(G)$. By definition,

$$(2.2) \qquad \mathrm{Sub}(G) = \mathrm{Sub}^{(0)}(G) \amalg \mathrm{Sub}^{(1)}(G) \amalg \mathrm{Sub}^{(2)}(G) \amalg \mathrm{Sub}^{(3)}(G).$$

(I)

(SII)

(WII)

(SIII)

(WIII)

FIGURE 7. Relators.

(I) $[Sii] \longleftrightarrow [S]$

(SII) $[SijTji] \longleftrightarrow [ST]$

(WII) $[SijTij] \longleftrightarrow [ST]$

(SIII) $[SijTkiUjk] \longleftrightarrow [SjiTikUkj]$

(WIII) $[SijTikUjk] \longleftrightarrow [SjiTkiUkj]$

FIGURE 8. Reidemeister moves.

Similarly, for a chord diagram x, $\mathrm{Sub}_x^{(m)}(G)$ denotes the subset of $\mathrm{Sub}_x(G)$ consisting of elements, each of which has exactly m letters of i, j, and k. Then,

$$(2.3) \qquad \mathrm{Sub}_x(G) = \mathrm{Sub}_x^{(0)}(G) \amalg \mathrm{Sub}_x^{(1)}(G) \amalg \mathrm{Sub}_x^{(2)}(G) \amalg \mathrm{Sub}_x^{(3)}(G).$$

Let $\mathcal{C}$ be the set of ambient isotopy classes of the spherical curves. Next, for each element $\sum_i \alpha_i x_i \in \mathbb{Z}[G_{\leq l}]$, we define a function $\mathcal{C} \to \mathbb{Z}$, also denoted by $\sum_i \alpha_i x_i$, and another function denoted by $\sum_i \alpha_i \tilde{x}_i$.

Definition 6 ($\sum_i \alpha_i x_i$, $\sum_i \alpha_i \tilde{x}_i$). Let b and d ($2 \leq b \leq d$) be integers and $G_{\leq d}$, $G_{b,d}$, $\{x_i\}_{i \in \mathbb{N}}$, and $\mathbb{Z}[G_{\leq l}]$ be as in Subsection 2.1. Recall that $n_d = |G_{\leq d}|$ and $G_{b,d} = \{x_i\}_{n_{b-1}+1 \leq i \leq n_d}$ and each x_i represents the chord diagram $f^{-1}(x_i)$. For each element

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i \in \mathbb{Z}[G_{\leq l}],$$

we define an integer-valued function $\mathcal{C} \to \mathbb{Z}$, also denoted by $\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i$, by

$$P \mapsto \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P),$$

where $x_i(P)$ is the integer introduced in Notation 2.

Analogously, for each $\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i \in \mathbb{Z}[G_{\leq l}]$, we define the function

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i : \mathbb{Z}[G_{\leq l}] \to \mathbb{Z}$$

by

$$\left(\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i \right)(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i(P),$$

where $\tilde{x}_i(P)$ is the integer introduced in Definition 4.

Recall that a spherical curve P and another spherical curve P' are related by a single RI and that there exist a letter i and an oriented Gauss word S such that $CD_P = [Sii]$ and $CD_{P'} = [S]$. Note that in this case in the decomposition (2.2), $\mathrm{Sub}^{(2)}(G) = \emptyset$ and $\mathrm{Sub}^{(3)}(G) = \emptyset$, where $G = Sii$. Then, by (2.1) in Definition 4 and (2.2),

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z \in \mathrm{Sub}(G)} \tilde{x}_i(z) \right)$$

$$= \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \tilde{x}_i(z_0) + \sum_{z_1 \in \mathrm{Sub}^{(1)}(G)} \tilde{x}_i(z_1) \right).$$

Note that each element $z_0 \in \mathrm{Sub}^{(0)}(G)$ is a sub-Gauss word of S. Then it is clear that $\mathrm{Sub}^{(1)}(G) = \{z_0 ii \mid z_0 \in \mathrm{Sub}^{(0)}(G)\}$. Hence,

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \tilde{x}_i(z_0) + \sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \tilde{x}_i(z_0 ii) \right)$$

$$= \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \tilde{x}_i([z_0]) + \sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \tilde{x}_i([z_0 ii]) \right).$$

On the other hand, since $\mathrm{Sub}(G')$ is identified with $\mathrm{Sub}^{(0)}(G)$,

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P') = \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \left(\sum_{z'\in\mathrm{Sub}(G')} \tilde{x}_i(z') \right) (\because (2.1))$$

$$= \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \left(\sum_{z_0\in\mathrm{Sub}^{(0)}(G)} \tilde{x}_i(z_0) \right)$$

$$= \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \left(\sum_{z_0\in\mathrm{Sub}^{(0)}(G)} \tilde{x}_i([z_0]) \right).$$

As a conclusion, the difference of the values is calculated as follows.

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P) - \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P')$$

$$= \sum_{n_{b-1}+1\leq i\leq n_d} \sum_{z_0\in\mathrm{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i\left([z_0 ii]\right).$$

We note that this is a linear combination of the values of Type (I) relators via $\tilde{x}_i$.

For the cases of Type (SII), (WII), (SIII), or (WIII) relators, the arguments are slightly more complicated than that of Type (I) relator. We will explain them in the proof of Theorem 1 and thus, we omit them here.

Definition 7 $(R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5})$. For each $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5)\in\{0,1\}^5$, let $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5} = \cup_{\epsilon_i=1}R_i$ $(\subset \cup_{l\geq 1}\mathbb{Z}[G_{\leq l}])$, where R_1 is the set of Type (I) relators, R_2 is the set of Type (SII) relators, R_3 is the set of Type (WII) relators, R_4 is the set of Type (SIII) relators, and R_5 is the set of Type (WIII) relators.

For integers b and d ($2\leq b\leq d$), let $O_{b,d}$ be the projection $\mathbb{Z}[G_{\leq l}] \to \mathbb{Z}[G_{b,d}]$. Here, note that $O_{b,d}$ is a linear map. By the definition, we immediately have:

Lemma 1. *If $n_{b-1}+1\leq i\leq n_d$, then for any $r\in\mathbb{Z}[G_{\leq l}]$,*

$$\tilde{x}_i(r) = \tilde{x}_i(O_{b,d}(r)).$$

Notation 3. Let $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b,d) = O_{b,d}(R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5})$.

By using Lemma 1, we have the next proposition.

Proposition 1. *For each pair of integers b and d ($2\leq b\leq d$), let $\sum_{n_{b-1}+1\leq i\leq n_d}\alpha_i\tilde{x}_i$ be a function as in Definition 6. For $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5)\in\{0,1\}^5$, let $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}$ be the set as in Definition 7. The following two statements are equivalent:*

(1) $\sum_{n_{b-1}+1\leq i\leq n_d}\alpha_i\tilde{x}_i(r) = 0$ $(\forall r\in R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5})$.
(2) $\sum_{n_{b-1}+1\leq i\leq n_d}\alpha_i\tilde{x}_i(r) = 0$ $(\forall r\in R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b,d))$.

Proof.
- $(1)\Rightarrow(2)$. This is obvious.
- $(1)\Leftarrow(2)$. Let $r\in R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}$. By Lemma 1,

$$\sum_{n_{b-1}+1\leq i\leq n_d}\alpha_i\tilde{x}_i(r) = \sum_{n_{b-1}+1\leq i\leq n_d}\alpha_i\tilde{x}_i(O_{b,d}(r)).$$

By the condition (2),

$$\sum_{n_{b-1}+1\leq i\leq n_d}\alpha_i\tilde{x}_i(O_{b,d}(r)) = 0.$$

Then,

$$\sum_{n_{b-1}+1 \le i \le n_d} \alpha_i \tilde{x}_i(r) = 0.$$

□

We note that it is simple task to give explicit presentation of $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b,d)$. In particular, if $d \le 4$, one can do it by hands. The next example shows the concrete such process for a certain case.

Example 2 ($R_{00010}(2,3)$)**.** $R_{00010}(2,3) = \{\bigotimes\!\!\!\bigtimes + 3\bigotimes - 3\bigcirc\!\!\!\!| - \bigcirc\!\!\!\!|, 3\bigotimes - 2\bigcirc\!\!\!| -$
$\bigcirc\!\!\!\!|, \bigotimes\!\!\!\bigtimes + \bigoplus - 2\bigotimes\}$. This fact is confirmed as follows.

Recall that a relator of Type (SⅢ) is of the form $r = ([SijTkiUjk] + [SijTiUj] + [SiTkiUk] + [SjTkUjk]) - ([SjiTikUkj] + [SjiTiUj] + [SiTikUk] + [SjTkUkj])$. We immediately see, from this expression, that each term of r consists of n or $n+1$ letters for some n. Note that $O_{2,3}(r)$ is obtained from r by removing the terms of length < 2, or > 3. Hence it is enough to suppose that each term of r consists of two or three letters (Case 1) or each term of r consists of three or four letters (Case 2).

• Case 1. Each term of r consists of two or three letters.

In this case, we have $STU = \emptyset$ (i.e., $S = \emptyset$, $T = \emptyset$, and $U = \emptyset$), and, hence,

$$O_{2,3}(r) = r = [ijkijk] + [ijij] + [ikik] + [jkjk] - [jiikkj] - [jiij] - [iikk] - [jkkj]$$
$$= [ijkijk] + 3[ijij] - 3[iijj] - [iijjkk].$$

• Case 2. Each term of r consists of three or four letters.

In this case, $STU = \sigma\sigma$ for some letter σ ($\ne i, j, k$).

Case 2.1. Both σ's are contained in one of the sub-words S, T, and U.

Suppose that both σ's are contained in S. Then,

$$r = [\sigma\sigma ijkijk] + [\sigma\sigma ijij] + [\sigma\sigma ikik] + [\sigma\sigma jkjk] - [\sigma\sigma jiikkj] - [\sigma\sigma jiij]$$
$$\quad - [\sigma\sigma iikk] - [\sigma\sigma jkkj]$$
$$= [\sigma\sigma ijkijk] - [\sigma\sigma jiikkj] + 3[ijijkk] - 2[ijkkji] - [iijjkk].$$

Hence,

$$O_{2,3}(r) = 3[ijijkk] - 2[ijkkji] - [iijjkk].$$

By using similar arguments, we can show that we obtain the same element as above in other cases. Details of the calculations are left to the reader.

Case 2.2. The two σ's are contained in mutually different sub-words S, T, U.

Suppose that one σ is contained in S and the other σ is contained in T. Then

$$r = [\sigma ij\sigma kijk] + [\sigma ij\sigma ij] + [\sigma i\sigma kik] + [\sigma j\sigma kjk] - [\sigma ji\sigma ikkj] - [\sigma ji\sigma ij]$$
$$\quad - [\sigma i\sigma ikk] - [\sigma j\sigma kkj].$$

Here, we note that

$$[\sigma i\sigma ikk] = [\sigma j\sigma kkj] = [ijijkk]$$

and

$$[\sigma i\sigma kik] = [\sigma j\sigma kjk] = [\sigma ji\sigma ij] = [ijkikj].$$

These show that

$$r = [\sigma ij\sigma kijk] - [\sigma ji\sigma ikkj] + [ijkijk] + [ijkikj] - 2[ijijkk].$$

Hence,

$$O_{2,3}(r) = [ijkijk] + [ijkikj] - 2[ijijkk].$$

By using similar arguments, we can show that we obtain the same element as above in other cases. Details of the calculations are left to the reader.

As a conclusion, $R_{00010}(2,3) = \{[ijkijk]+3[ijij]-3[iijj]-[iikkj],\ 3[ijijkk]-2[ijkkji]-[iijjkk],\ [ijkijk]+[ijkikj]-2[ijijkk]\} = \{⊗+3⊗-3◯-◐,\ 3⊗-2⦶-◐,\ ⊗+⊕-2⊗\}$.

Example 3 ($R_{00010}(2,4)$). By using the arguments in Example 2, we can show:
$R_{00010}(2,4) = \{⊗+3⊗-3◯-◐,\ ⊖+3⊗-⊖-2⦶-◐,\ ⊗+⊗+⊕-◐-2⊗,\ 2⊗-⊗-⊖,\ ⊗+⊕-⊖-2◐,\ ⊗+2◯-⊗-2⊖,\ 2⊗+⊕-2⊗-⊗,\ ⊗+2⊗-⊕-2◐,\ 3⊖-2⊖-◯,\ ⊖-2⊖-⊖,\ ◯-2⊖-◯,\ ⊖+⊗-⊖-2⊖\}$. Details of the proof are left to the reader.

Example 4 ($R_{00001}(2,3)$). By using the arguments in Example 2, we can show:

$R_{00001}(2,3) = \{⊕+⊗-◯-⊗,\ ⊗-⊖,\ 3⊕-⊗-2⊗,\ ⊗-2⦶+◐,\ ⊗-◐\}$.

Details of the proof are left to the reader.

2.3. **Main result and corollaries.**

Theorem 1. *Let b and d $(2 \le b \le d)$ be integers and $G_{\le d}$, $\{x_i\}_{i\in\mathbb{N}}$, $\mathbb{Z}[G_{\le l}]$, $n_d = |G_{\le d}|$, and $G_{b,d} = \{x_i\}_{n_{b-1}+1\le i\le n_d}$ be as in Subsection 2.1. Let $\sum_{n_{b-1}+1\le i\le n_d}\alpha_i x_i$ and $\sum_{n_{b-1}+1\le i\le n_d}\alpha_i \tilde{x}_i$ be functions as in Definition 6. For $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5) \in \{0,1\}^5$, let $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b,d)$ be as in Subsection 2.2. Suppose the following conditions are satisfied:*

- *If $\epsilon_1 = 1$, $\displaystyle\sum_{n_{b-1}+1\le i\le n_d}\alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{10000}(b,d)$.*

- *If $\epsilon_2 = 1$, $\displaystyle\sum_{n_{b-1}+1\le i\le n_d}\alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{01000}(b,d)$.*

- *If $\epsilon_3 = 1$, $\displaystyle\sum_{n_{b-1}+1\le i\le n_d}\alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{00100}(b,d)$.*

- *If $\epsilon_4 = 1$, $\displaystyle\sum_{n_{b-1}+1\le i\le n_d}\alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{00010}(b,d)$.*

- *If $\epsilon_5 = 1$, $\displaystyle\sum_{n_{b-1}+1\le i\le n_d}\alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{00001}(b,d)$.*

Then, the function $\displaystyle\sum_{n_{b-1}+1\le i\le n_d}\alpha_i x_i$ on the set of ambient isotopy classes of spherical curves is invariant under the Reidemeister moves corresponding to $\epsilon_j = 1$.

Definition 8 (irreducible chord diagram). Let x be a chord diagram. A chord α in x is said to be an *isolated chord* if α does not intersect any other chord. If x has an isolated chord, x is called *reducible* and otherwise, x is called *irreducible*. The set of the irreducible chord diagrams is denoted by Irr. Let $I_{b,d}^{(\mathrm{Irr})} = \{i \mid n_{b-1}+1 \le i \le n_d, x_i \in \mathrm{Irr}\}$.

If we consider the function of the form $\sum_{i\in I_{b,d}^{(\mathrm{Irr})}}\alpha_i x_i$ for $\sum_{n_{b-1}+1\le i\le n_d}\alpha_i x_i$ in Theorem 1, we obtain:

Corollary 1. *Let b and d $(2 \le b \le d)$ be integers and $G_{\le d}$, $\{x_i\}_{i\in\mathbb{N}}$, $\mathbb{Z}[G_{\le l}]$, $n_d = |G_{\le d}|$, and $G_{b,d} = \{x_i\}_{n_{b-1}+1\le i\le n_d}$ be as in Subsection 2.1. Let $\sum_{i\in I_{b,d}^{(\mathrm{Irr})}}\alpha_i x_i$ and $\sum_{i\in I_{b,d}^{(\mathrm{Irr})}}\alpha_i\tilde{x}_i$ be functions as in Definition 6. For $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5) \in \{0,1\}^5$,*

let $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b,d)$ be as in Subsection 2.2. Suppose the following conditions are satisfied:

- If $\epsilon_2 = 1$, $\displaystyle\sum_{i \in I_{b,d}^{(\mathrm{Irr})}} \alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{01000}(b,d)$.

- If $\epsilon_3 = 1$, $\displaystyle\sum_{i \in I_{b,d}^{(\mathrm{Irr})}} \alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{00100}(b,d)$.

- If $\epsilon_4 = 1$, $\displaystyle\sum_{i \in I_{b,d}^{(\mathrm{Irr})}} \alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{00010}(b,d)$.

- If $\epsilon_5 = 1$, $\displaystyle\sum_{i \in I_{b,d}^{(\mathrm{Irr})}} \alpha_i\tilde{x}_i(r) = 0$ for each $r \in R_{00001}(b,d)$.

Then, the function $\displaystyle\sum_{i \in I_{b,d}^{(\mathrm{Irr})}} \alpha_i x_i$ on the set of ambient isotopy classes of spherical curves is invariant under RI and the Reidemeister moves corresponding to $\epsilon_j = 1$.

Proof of Corollary 1 from Theorem 1. By Theorem 1, it is enough to show

$$(2.4) \qquad \sum_{i \in I_{b,d}^{(\mathrm{Irr})}} \alpha_i\tilde{x}_i(r) = 0 \quad (\forall r \in R_{10000}(b,d))$$

for a proof of Corollary 1. We first note that if $x_i \in \mathrm{Irr}$, then x_i has no isolated chords. On the other hand, let $r \in R_{10000}(b,d)$, that is, there exist a Gauss word S and a letter j such that $r = [Sjj]$. Then, the chord corresponding to j is isolated. These show that $\tilde{x}_i(r) = 0$. This shows that (2.4) holds. $\qquad\square$

Example 5. It is easy to see that $\{x_i\}_{i \in I_{2,3}^{(\mathrm{Irr})}}$ consists of three elements, $y_1 = \otimes$, $y_2 = \otimes$, and $y_3 = \ominus$. The fact together with Corollary 1 implies that if there exist α_1, α_2, and α_3 such that $(\alpha_1\tilde{y}_1 + \alpha_2\tilde{y}_2 + \alpha_3\tilde{y}_3)(r) = 0$ $(\forall r \in R_{00010}(2,3))$, then $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3$ is invariant under RI and strong RIII. Recall that $R_{00010}(2,3)$ consists of the following three elements r_1, r_2, and r_3 (Example 2):

$$r_1 = \otimes + 3\otimes - 3\oslash - \oslash,$$
$$r_2 = 3\otimes - 2\oslash - \oslash, \text{ and}$$
$$r_3 = \otimes + \ominus - 2\otimes.$$

We can find such α_1, α_2, and α_3 by solving the linear equation $\mathbf{x}M = \mathbf{0}$, where $M = (\tilde{y}_i(r_j))$ (Figure 9). It is elementary to show that the set of solutions is $\{\mathbf{x} =$

$$
\begin{array}{c@{\qquad}c@{\qquad}c@{\qquad}c}
 & \otimes + 3\otimes - 3\oslash - \oslash & 3\otimes - 2\oslash - \oslash & \otimes + \ominus - 2\otimes \\[4pt]
\tilde{\otimes} & 3 & 0 & 0 \\[4pt]
\tilde{\otimes} & 1 & 0 & 1 \\[4pt]
\tilde{\ominus} & 0 & 0 & 1
\end{array}
$$

FIGURE 9. Matrix $(\tilde{y}_i(r_j))_{1 \le i,j \le 3}$.

$\gamma(1, -3, 3) \mid \gamma \in \mathbb{Z}\}$. Particularly, $y_1 - 3y_2 + 3y_3$ is invariant under RI and strong RIII. We note that $\frac{1}{4}(y_1 - 3y_2 + 3y_3)$ is the function λ introduced in [5] from a different view point.

Example 6. In contrast to Example 5, we will note that there does not exist a nonzero $(\alpha_1, \alpha_2, \alpha_3)$ such that $(\alpha_1 \tilde{y}_1 + \alpha_2 \tilde{y}_2 + \alpha_3 \tilde{y}_3)(r) = 0$ for each $r \in R_{00001}(2,3)$, i.e., there does not exist $\alpha_1 \tilde{y}_1 + \alpha_2 \tilde{y}_2 + \alpha_3 \tilde{y}_3$ satisfying the condition of Corollary 1 with $\epsilon_5 = 1$. In fact, we take $y_i(i = 1, 2, 3)$ as in Example 5, and name the elements of $R_{00001}(2,3)$ (see Example 4) as:

$$r_1' = \oplus + \otimes - \bigcirc - \otimes,$$
$$r_2' = \otimes - \oplus,$$
$$r_3' = 3\oplus - \otimes - 2\otimes,$$
$$r_4' = \otimes - 2\oplus + \bigcirc, \text{ and}$$
$$r_5' = \otimes - \bigcirc.$$

Let $M = (\tilde{y}_i(r_j'))_{1 \le i \le 3, 1 \le j \le 5}$ (Figure 10). Then it is easy to see that the linear equation $\mathbf{x}M = \mathbf{0}$ admits the trivial solution $\mathbf{x} = (0, 0, 0)$ only.

	$\oplus + \otimes - \bigcirc - \otimes$	$\otimes - \oplus$	$3\oplus - \otimes - 2\otimes$	$\otimes - 2\oplus + \bigcirc$	$\otimes - \bigcirc$
$\tilde{\otimes}$	1	0	0	0	0
$\tilde{\otimes}$	0	1	-1	0	0
$\tilde{\oplus}$	1	-1	3	0	0

FIGURE 10. Matrix $(\tilde{y}_i(r_j'))_{1 \le i \le 3, 1 \le j \le 5}$.

Definition 9 (connected chord diagram). Let v be a Gauss word of length $2m$ and w a Gauss word of length $2n$ where $v(\hat{2m}) \cap w(\hat{2n}) = \emptyset$. Then we define the Gauss word of length $2(m + n)$, denoted by vw, by $vw(i) = v(i)$ ($1 \le i \le 2m$) and $vw(2m+i) = w(i)$ ($1 \le i \le 2n$). The chord diagram $[vw]$ is called a *product* of chord diagrams $[v]$ and $[w]$. If a chord diagram is not a product of two non-empty chord diagrams, then the chord diagram is called a *connected* chord diagram. The set of the connected chord diagrams is denoted by Conn. Let $I_{b,d}^{(\text{Conn})} = \{i \mid n_{b-1} + 1 \le i \le n_d, x_i \in \text{Conn}\}$.

It is easy to see that if a connected chord diagram has at least two chords, it is irreducible. Note that the chord diagram consisting of exactly one chord is connected but not irreducible.

Recall that $\mathcal{C}$ is the set of ambient isotopy classes of the spherical curves.

Definition 10 (connected sum). Let P_1, $P_2 \in \mathcal{C}$. We suppose that the ambient 2-spheres are oriented. Let p_i be a point on P_i such that p_i is not a double point ($i = 1, 2$). Let d_i be a sufficiently small disk with center p_i ($i = 1, 2$) such that $d_i \cap P_i$ consists of an arc properly embedded in d_i. Let $\hat{d}_i = cl(S^2 \setminus d_i)$ and $\hat{P}_i = P_i \cap \hat{d}_i$; let $h : \partial \hat{d}_1 \to \partial \hat{d}_2$ be an orientation reversing homeomorphism such that $h(\partial \hat{P}_1) = \partial \hat{P}_2$. Then, $\hat{P}_1 \cup_h \hat{P}_2$ gives a spherical curve in the oriented 2-sphere $\hat{d}_1 \cup_h \hat{d}_2$. The spherical curve $\hat{P}_1 \cup_h \hat{P}_2$ in the oriented 2-sphere, denoted by $P_1 \sharp_{(p_1, p_2), h} P_2$,

is called a *connected sum* of the spherical curves P_1 and P_2 at the pair of points p_1 and p_2 (see Figure 11).
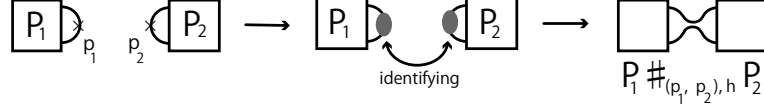

FIGURE 11. Connected sum $P_1\sharp_{(p_1,\ p_2),h}P_2$ of two spherical curves P_1 and P_2.

Definition 11 (additivity). Let I be a function on $\mathcal{C}$. We say that I is *additive* if $I(P_1\sharp_{(p_1,p_2),h}P_2) = I(P_1) + I(P_2)$ for any spherical curves P_1 and P_2 and for any $P_1\sharp_{(p_1,p_2),h}P_2$ in Definition 10.

Definition 12 (prime). A spherical curve P is *trivial* if P is a simple closed curve. A spherical curve P is *prime* if P cannot be a connected sum of two nontrivial spherical curves.

If we consider the function of the form $\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i x_i$ for $\sum_{n_{b-1}+1\le i\le n_d} \alpha_i x_i$ in Theorem 1, we obtain:

Corollary 2. *Let b and d ($2 \le b \le d$) be integers and $G_{\le d}$, $\{x_i\}_{i\in\mathbb{N}}$, $\mathbb{Z}[G_{\le l}]$, $n_d = |G_{\le d}|$, and $G_{b,d} = \{x_i\}_{n_{b-1}+1\le i\le n_d}$ be as in Subsection 2.1. Let $\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i x_i$ and $\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i \tilde{x}_i$ be functions as in Definition 6. For $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5) \in \{0,1\}^5$, let $R_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}(b,d)$ be as in Subsection 2.2. Suppose the following conditions are satisfied:*

- *If $\epsilon_2 = 1$, $\displaystyle\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i \tilde{x}_i(r) = 0$ for each $r \in R_{01000}(b,d)$.*

- *If $\epsilon_3 = 1$, $\displaystyle\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i \tilde{x}_i(r) = 0$ for each $r \in R_{00100}(b,d)$.*

- *If $\epsilon_4 = 1$, $\displaystyle\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i \tilde{x}_i(r) = 0$ for each $r \in R_{00010}(b,d)$.*

- *If $\epsilon_5 = 1$, $\displaystyle\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i \tilde{x}_i(r) = 0$ for each $r \in R_{00001}(b,d)$.*

Then, the function $\displaystyle\sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i x_i$ on the set of ambient isotopy classes of spherical curves is additive and invariant under RI and the Reidemeister moves corresponding to $\epsilon_j = 1$.

Proof of Corollary 2 from Theorem 1. Since $b \ge 2$ and $i \ge n_{b-1} + 1$, each x_i consists of more than one chords, i.e., $x_i \in \mathrm{Irr}$ (see the note preceding Definition 10).

By Theorem 1, it is enough to show

$$(2.5)\qquad \sum_{i\in I_{b,d}^{(\mathrm{Conn})}} \alpha_i \tilde{x}_i(r) = 0 \quad (\forall r \in R_{10000}(b,d))$$

for a proof of Corollary 2. We first note that if $x_i \in \mathrm{Irr}$, then x_i has no isolated chords. On the other hand, let $r \in R_{10000}(b,d)$, that is, there exist a Gauss word S

and a letter j such that $r = [Sjj]$. Then, the chord corresponding to j is isolated. These show that $\tilde{x}_i(r) = 0$. This shows that (2.5) holds.

Further, by using geometric observations as in Example 1 it is clear that if $x_i \in \mathrm{Conn}$, then

$$(2.6) \qquad x_i(P_1 \sharp_{(p_1,\ p_2),h} P_2) = x_i(P_1) + x_i(P_2)$$

for any $P_1 \sharp_{(p_1,p_2),h} P_2$. This fact implies $\sum_{i \in I_{b,d}^{(\mathrm{Conn})}} \alpha_i x_i$ is additive. $\qquad \square$

3. Proof of Theorem 1

$\bullet$ (Proof of the case $\epsilon_1 = 1$.) Let P and P' be two spherical curves where P and P' are related by a single RI, hence, there exist a letter i and two Gauss words $G = Sii$ and $G' = S$ corresponding to P and P', respectively, i.e., $CD_P = [Sii]$ and $CD_{P'} = [S]$. As we observed in a paragraph proceeding of Definition 7 of Subsection 2.2, we have

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) - \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P') = \sum_{n_{b-1}+1 \leq i \leq n_d} \sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i\left([z_0 ii]\right).$$

By the assumption of this case, for each $z_0 \in \mathrm{Sub}^{(0)}(G)$,

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \tilde{x}_i\left([z_0 ii]\right) = 0$$

and this shows that

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P').$$

Hence, $\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i$ is invariant under RI.

$\bullet$ (Proof of the case $\epsilon_2 = 1$.) Let P and P' be two spherical curves where P and P' are related by a single strong RII, hence, there exist two Gauss words $G = SijTji$ and $G' = ST$ corresponding to P and P', respectively, i.e., $CD_P = [SijTji]$ and $CD_{P'} = [ST]$.

By (2.1) in Subsection 2.1 and (2.2) in Subsection 2.2, we obtain (note that $\mathrm{Sub}^{(3)}(G) = \emptyset$):

$$\sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z \in \mathrm{Sub}(G)} \tilde{x}_i(z) \right)$$

$$= \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \tilde{x}_i(z_0) \right) + \sum_{n_{b-1}+1 \leq i \leq n_d} \sum_{z_{12} \in \mathrm{Sub}^{(1)}(G) \cup \mathrm{Sub}^{(2)}(G)} \alpha_i \tilde{x}_i(z_{12})$$

$$= \sum_{n_{b-1}+1 \leq i \leq n_d} \alpha_i \left(\sum_{z' \in \mathrm{Sub}(G')} \tilde{x}_i(z') \right) + \sum_{n_{b-1}+1 \leq i \leq n_d} \sum_{z_{12} \in \mathrm{Sub}^{(1)}(G) \cup \mathrm{Sub}^{(2)}(G)} \alpha_i \tilde{x}_i(z_{12})$$

$(\because \mathrm{Sub}^{(0)}(G)$ is identified with $\mathrm{Sub}(G'))$.

Let $z_0 \in \mathrm{Sub}^{(0)}(G)$. We note that since G is an oriented Gauss word z_0 uniquely admits a decomposition into two sub-words, which are sub-words on S and T. Let

$\sigma(z_0)$ be the sub-word of S and $\tau(z_0)$ the sub-word of T satisfying $z_0 = \sigma(z_0)\tau(z_0)$. Under these notations, we define maps

$$z_2 : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(2)}(G); z_2(z_0) = \sigma(z_0)ij\tau(z_0)ji,$$

$$z_1 : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(1)}(G); z_1(z_0) = \sigma(z_0)i\tau(z_0)i, \text{ and}$$

$$z_1' : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(1)}(G); z_1'(z_0) = \sigma(z_0)j\tau(z_0)j.$$

Then, it is easy to see that $\mathrm{Sub}^{(1)}(G) \cup \mathrm{Sub}^{(2)}(G)$ admits a decomposition

$\mathrm{Sub}^{(1)}(G) \cup \mathrm{Sub}^{(2)}(G)$

$= \{z_1(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\} \amalg \{z_1'(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\} \amalg \{z_2(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\}.$

These notations together with the above give:

$$\sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i(P')$$
$$+ \sum_{n_{b-1}+1 \le i \le n_d} \sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i(z_1(z_0) + z_1'(z_0) + z_2(z_0))$$
$$= \sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i(P')$$
$$+ \sum_{n_{b-1}+1 \le i \le n_d} \sum_{z_0 \in \mathrm{Sub}^{(0)}(G)} \alpha_i \tilde{x}_i([z_1(z_0)] + [z_1'(z_0)] + [z_2(z_0)]).$$

By Proposition 1 and by the condition for the case $\epsilon_2 = 1$, for any $z_0 \in \mathrm{Sub}^{(0)}(G)$,

$$\sum_{n_{b-1}+1 \le i \le n_d} \alpha_i \tilde{x}_i([z_1(z_0)] + [z_1'(z_0)] + [z_2(z_0)]) = 0.$$

Thus,

$$\sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i(P').$$

Hence, $\sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i$ is invariant under strong RⅡ.

• (Proof of the case $\epsilon_3 = 1$.) Since the arguments are essentially the same as that of the case $\epsilon_2 = 1$, we omit this proof.

• (Proof of the case $\epsilon_4 = 1$.) Let P and P' be two spherical curves where P and P' are related by a single strong RⅢ, hence, there exist two Gauss words $G = SijTkiUjk$ and $G' = SjiTikUkj$ corresponding to P and P', respectively, i.e., $CD_P = [SijTkiUjk]$ and $CD_{P'} = [SjiTikUkj]$.

By (2.1) in Subsection 2.1 and (2.2) in Subsection 2.2, we obtain:

$$\sum_{n_{b-1}+1 \le i \le n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1 \le i \le n_d} \alpha_i \left(\sum_{z \in \mathrm{Sub}(G)} \tilde{x}_i(z) \right)$$
$$= \sum_{n_{b-1}+1 \le i \le n_d} \alpha_i \left(\sum_{z_{01} \in \mathrm{Sub}^{(0)}(G) \cup \mathrm{Sub}^{(1)}(G)} \tilde{x}_i(z_{01}) + \sum_{z_{23} \in \mathrm{Sub}^{(2)}(G) \cup \mathrm{Sub}^{(3)}(G)} \tilde{x}_i(z_{23}) \right).$$

and

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P') = \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \left(\sum_{z'\in\mathrm{Sub}(G')} \tilde{x}_i(z') \right)$$

$$= \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \left(\sum_{z'_{01}\in\mathrm{Sub}^{(0)}(G')\cup\mathrm{Sub}^{(1)}(G')} \tilde{x}_i(z'_{01}) + \sum_{z'_{23}\in\mathrm{Sub}^{(2)}(G')\cup\mathrm{Sub}^{(3)}(G')} \tilde{x}_i(z'_{23}) \right).$$

Since $\mathrm{Sub}^{(0)}(G)$ ($\mathrm{Sub}^{(1)}(G)$ resp.) is naturally identified with $\mathrm{Sub}^{(0)}(G')$ ($\mathrm{Sub}^{(1)}(G')$ resp.), the above equations show:

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P) - \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P')$$

$$= \sum_{n_{b-1}+1\leq i\leq n_d}\sum_{z_{23}\in\mathrm{Sub}^{(2)}(G)\cup\mathrm{Sub}^{(3)}(G)} \alpha_i \tilde{x}_i(z_{23})$$

$$- \sum_{n_{b-1}+1\leq i\leq n_d}\sum_{z'_{23}\in\mathrm{Sub}^{(2)}(G')\cup\mathrm{Sub}^{(3)}(G')} \alpha_i \tilde{x}_i(z'_{23}).$$

Let $z_0 \in \mathrm{Sub}^{(0)}(G)$, which is identified with $\mathrm{Sub}^{(0)}(G')$. We note that since G is an oriented Gauss word z_0 uniquely admits a decomposition into three sub-words, which are sub-words on S, T, and U. Let $\sigma(z_0)$ be the sub-word of S, $\tau(z_0)$ the sub-word of T, and $\mu(z_0)$ the sub-word of U satisfying $z_0 = \sigma(z_0)\tau(z_0)\mu(z_0)$. We define maps

$$z_3 : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(3)}(G); z_3(z_0) = \sigma(z_0)ij\tau(z_0)ki\mu(z_0)jk,$$

$$z_{2a} : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(2)}(G); z_{2a}(z_0) = \sigma(z_0)ij\tau(z_0)i\mu(z_0)j,$$

$$z_{2b} : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(2)}(G); z_{2b}(z_0) = \sigma(z_0)i\tau(z_0)ki\mu(z_0)k, \text{ and}$$

$$z_{2c} : \mathrm{Sub}^{(0)}(G) \to \mathrm{Sub}^{(2)}(G); z_{2c}(z_0) = \sigma(z_0)j\tau(z_0)k\mu(z_0)jk.$$

Similarly, let

$$z'_3 : \mathrm{Sub}^{(0)}(G') \to \mathrm{Sub}^{(3)}(G'); z'_3(z_0) = \sigma(z_0)ji\tau(z_0)ik\mu(z_0)kj,$$

$$z'_{2a} : \mathrm{Sub}^{(0)}(G') \to \mathrm{Sub}^{(2)}(G'); z'_{2a}(z_0) = \sigma(z_0)ji\tau(z_0)i\mu(z_0)j,$$

$$z'_{2b} : \mathrm{Sub}^{(0)}(G') \to \mathrm{Sub}^{(2)}(G'); z'_{2b}(z_0) = \sigma(z_0)i\tau(z_0)ik\mu(z_0)k, \text{ and}$$

$$z'_{2c} : \mathrm{Sub}^{(0)}(G') \to \mathrm{Sub}^{(2)}(G'); z'_{2c}(z_0) = \sigma(z_0)j\tau(z_0)k\mu(z_0)kj.$$

Then, it is easy to see that $\mathrm{Sub}^{(2)}(G) \cup \mathrm{Sub}^{(3)}(G)$ admits decompositions

$\mathrm{Sub}^{(2)}(G) \cup \mathrm{Sub}^{(3)}(G)$

$= \{z_3(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\} \amalg \{z_{2a}(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\} \amalg \{z_{2b}(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\}$

$\amalg \{z_{2c}(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\}$

and

$\mathrm{Sub}^{(2)}(G') \cup \mathrm{Sub}^{(3)}(G')$

$= \{z'_3(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\} \amalg \{z'_{2a}(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\} \amalg \{z'_{2b}(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\}$

$\amalg \{z'_{2c}(z_0) \mid z_0 \in \mathrm{Sub}^{(0)}(G)\}.$

By using these notations, we obtain:

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P) - \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P')$$

$$= \sum_{z_0\in\mathrm{Sub}^{(0)}(G)} \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \tilde{x}_i\Big((z_3(z_0) + z_{2a}(z_0) + z_{2b}(z_0) + z_{2c}(z_0))$$

$$- (z'_3(z_0) + z'_{2a}(z_0) + z'_{2b}(z_0) + z'_{2c}(z_0))\Big)$$

$$= \sum_{z_0\in\mathrm{Sub}^{(0)}(G)} \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \tilde{x}_i\Big(([z_3(z_0)] + [z_{2a}(z_0)] + [z_{2b}(z_0)] + [z_{2c}(z_0)])$$

$$- ([z'_3(z_0)] + [z'_{2a}(z_0)] + [z'_{2b}(z_0)] + [z'_{2c}(z_0)])\Big).$$

Here, we note that

$$([z_3(z_0)]+[z_{2a}(z_0)]+[z_{2b}(z_0)]+[z_{2c}(z_0)])-([z'_3(z_0)]+[z'_{2a}(z_0)]+[z'_{2b}(z_0)]+[z'_{2c}(z_0)]) \in R_{00010}.$$

Hence, by the assumption of Case $\epsilon_4 = 1$ and by Proposition 1, for each z_0,

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i \tilde{x}_i\Big(([z_3(z_0)] + [z_{2a}(z_0)] + [z_{2b}(z_0)] + [z_{2c}(z_0)])$$

$$- ([z'_3(z_0)] + [z'_{2a}(z_0)] + [z'_{2b}(z_0)] + [z'_{2c}(z_0)])\Big)$$

$$= 0.$$

These show that

$$\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P) = \sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i(P').$$

Hence, $\sum_{n_{b-1}+1\leq i\leq n_d} \alpha_i x_i$ is invariant under storng RⅢ.

• (Proof of the case $\epsilon_5 = 1$.) Since the arguments are essentially the same as that of the case $\epsilon_4 = 1$, we omit this proof.

$\square$

4. COMPLETE ANALYSIS OF INVARIANTS OBTAINED FROM $G_{2,4}$ AND $R_{00010}(2,4)$

In this section, by using Corollary 2, we completely analyze invariants obtained from $G_{2,4}\cap\mathrm{Conn}$ and $R_{00010}(2,4)$. First, it is easy to see that $\{x_i\}_{i\in I_{2,4}^{(\mathrm{Conn})}}$ consists of nine elements, $y_1 = \otimes$, $y_2 = \circledast$, $y_3 = \ominus$, $y_4 = \ominus$, $y_5 = \oplus$, $y_6 = \otimes$, $y_7 = \circledast$, $y_8 = \oslash$, and $y_9 = \circledast$. The fact together with Corollary 2 implies that if there exist α_i ($1 \leq i \leq 9$) such that $(\sum_{i=1}^{9} \alpha_i \tilde{y}_i)(r) = 0$ ($\forall r \in R_{00010}(2,4)$), then $\sum_{i=1}^{9} \alpha_i y_i$ is invariant under RI and strong RⅢ. Second, recall that $R_{00010}(2,4)$

consists of twelve elements r_j $(1 \leq j \leq 12)$ as follows (see Example 3):

$$r_1 = \otimes + 3\otimes - 3\bigcirc - \bigcirc,$$
$$r_2 = \ominus + 3\otimes - \ominus - 2\oplus - \bigcirc,$$
$$r_3 = \otimes + \otimes + \oplus - \bigcirc - 2\otimes,$$
$$r_4 = 2\otimes - \otimes - \ominus,$$
$$r_5 = \otimes + 2\oplus - \ominus - 2\oslash,$$
$$r_6 = \otimes + 2\otimes - \otimes - 2\ominus,$$
$$r_7 = 2\otimes + \oplus - 2\otimes - \otimes,$$
$$r_8 = \otimes + 2\otimes - \oplus - 2\bigcirc,$$
$$r_9 = 3\ominus - 2\ominus - \ominus,$$
$$r_{10} = 3\ominus - 2\ominus - \ominus,$$
$$r_{11} = 3\ominus - 2\ominus - \bigcirc, \text{ and}$$
$$r_{12} = \ominus + 2\otimes - \ominus - 2\ominus.$$

We can find α_i $(1 \leq i \leq 9)$ satisfying $(\sum_{i=1}^{9} \alpha_i \tilde{y}_i)(r) = 0$ $(\forall r \in R_{00010}(2,4))$ by solving the linear equation $\mathbf{x}M = \mathbf{0}$, where $M = (\tilde{y}_i(r_j))$ (Figure 12). It is elementary to show that the set of solutions is $\{\mathbf{x} = \gamma_1(1,-3,3,0,0,0,0,0,0), \gamma_2(0,0,2,8,5,1,-4,2,-2) \mid \gamma_1, \gamma_2 \in \mathbb{Z}\}$. Particularly, $y_1 - 3y_2 + 3y_3$ $(= \otimes -3\otimes +3\ominus)$ and $2y_3 + 8y_4 + 5y_5 + y_6 - 4y_7 + 2y_8 - 2y_9$ $(= 2\ominus + 8\ominus + 5\oplus + \otimes -4\otimes + 2\otimes -2\otimes)$ are invariant under RI and strong RIII.

Let $\lambda^{(3)} = y_1 - 3y_2 + 3y_3$ and let $\lambda^{(4)} = 2y_3 + 8y_4 + 5y_5 + y_6 - 4y_7 + 2y_8 - 2y_9$. Note that $\lambda^{(3)}$ is the function obtained in Example 5, hence $\lambda^{(3)} = 4\lambda$ where λ is the function that appears in Theorem 1 of [5]. We further note that in [5], it is shown that λ (hence, $\frac{1}{4}\lambda^{(3)}$) is a complete invariant of prime spherical curves up to seven double points (Table 1). We note that $\lambda^{(4)}$ is also a complete invariant

	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
$\tilde{\oplus}$	0	0	0	0	-1	0	1	0	0	0	0	0
$\tilde{\oplus}$	0	0	0	0	2	0	0	-1	0	0	0	0
$\tilde{\oplus}$	0	0	0	2	0	2	-2	1	0	0	0	0
$\tilde{\otimes}$	0	0	0	-1	0	0	-1	2	0	0	0	0
$\tilde{\otimes}$	0	0	0	0	0	1	0	0	0	0	0	0
$\tilde{\oplus}$	0	0	1	0	1	-1	2	0	0	0	0	0
$\tilde{\otimes}$	3	0	0	0	0	0	0	0	0	0	0	0
$\tilde{\otimes}$	1	0	1	0	0	0	0	0	0	0	0	0
$\tilde{\oplus}$	0	0	1	0	0	0	0	0	0	0	0	0

FIGURE 12. Matrix $(\tilde{y}_i(r_j))_{1 \leq i \leq 9, 1 \leq j \leq 12}$.

of prime spherical curves up to seven double points. We further note that $\lambda^{(4)}$ is stronger than $\lambda^{(3)}$ in some sense. In fact, $\lambda(4_1) = \lambda(8_9) = 4$ and $\lambda^{(4)}(4_1)$ $(= 16)$ $\neq \lambda^{(4)}(8_9)$ $(= 40)$. Similarly, we have $\lambda(5_1) = \lambda(8_2) = -5$ and $\lambda^{(4)}(5_1)$ $(= -20)$ $\neq \lambda^{(4)}(8_2)$ $(= -28)$, and $\lambda(8_5) = \lambda(8_{17}) = 12$ and $\lambda^{(4)}(8_5)$ $(= 128) \neq \lambda^{(4)}(8_{17})$ $(= 116)$.

Acknowledgements

References

[1] R. Bott and C. Taubes, On the self-linking of knots, Topology and physics, *J. Math. Phys.* **35** (1994), 5247–5287.

[2] M. Goussarov, M. Polyak, and O. Viro, Finite-type invariants of classical and virtual knots, *Topology* **39** (2000), 1-45–1068.

[3] T. Hagge and J. Yazinski, On the necessity of Reidemeister move 2 for simplifying immersed planar curves, *Banach Center Publ.* **103** (2014), 101–110.

[4] N. Ito and Y. Takimura, (1, 2) and weak (1, 3) homotopies on knot projections, *J. Knot Theory and Ramifications* **22** (2013), 1350085, 14pp.

[5] N. Ito and Y. Takimura, Sub-chord diagrams of knot projections, *Houston J. Math.* **41** (2015), 701–725.

[6] N. Ito, Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections, *Osaka J. Math.* **52** (2015), 617–646.

[7] O.-P. Östlund, A diagrammatic approach to link invariants of finite degree, *Math. Scand.* **94** (2004), 295–319.

[8] M. Polyak and O. Viro, Gauss diagram formulas for Vassiliev invariants, *Internat. Math. Res. Notices* **1994,** 445ff., approx. 8pp. (electronic).

[9] V. A. Vassiliev, Cohomology of knot spaces, *Theory of singularities and its applications,* 23–69, Adv. Soviet Math., 1, *Amer. Math. Soc., Providence, RI,* 1990.

[10] O. Viro, Generic immersions of the circle to surfaces and the complex topology of real algebraic curves. *Topology of real algebraic varieties and related topics,* 231–252, Amer. Math. Soc. Transl. Ser. 2, 173, Adv. Math. Sci., 29, *Amer. Math. Soc., Providence, RI,* 1996.

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TABLE 1. Values of $\frac{1}{4}\lambda^{(3)}$ $(=\lambda)$ and $\lambda^{(4)}$ for prime spherical curves at most seven double points. Any pair of prime spherical curves in the same box in the leftmost column are related by a finite sequence of Reidemeister moves of types RI and strong RIII.

P	$\frac{1}{4}\lambda^{(3)}(P)$	$\lambda^{(4)}(P)$
7_1	-21	-140
5_1	-5	-20
8_2	-5	-28
7_3	-3	-44
1_1 3_1	0	0
4_1 5_2 6_3	4	16
8_9	4	40
6_2 7_5 7_8	5	36
8_5	12	128
8_{17}	12	116
6_1 7_2 7_6 7_9	14	120
7_4 7_7 7_{10}	15	148