POSITIVE KNOTS AND WEAK (1, 3) HOMOTOPY

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Abstract. It is known that there exists a surjective map from the set of weak (1, 3) homotopy classes of knot projections to the set of positive knots. An interesting question whether this map is also injective, which question was formulated independently by S. Kamada and Y. Nakanishi in 2013. This paper obtains an answer to this question.

1. Introduction

A positive knot is ambient isotopic to the unknot if and only if its positive knot diagram on a 2-sphere is related to the simple closed curve by a sequence of the first Reidemeister moves (Figure 2 (a)), each of which decreases crossings up to ambient isotopy on $S^2$ [4, Theorem 1.5]. This statement gives rise to a problem (Question 1) that is solved by this paper.

For a knot projection on a 2-sphere (i.e., a generic immersed spherical curve up to ambient isotopy on $S^2$), each double point is replaced by a positive crossing as in Figure 1. This replacement induces a map

$$f : \{\text{knot projections}\} \rightarrow \{\text{knot diagrams}\}.$$  

Then, letting RI (weak RI, resp.) be a local deformation of a knot projection as in Figure 2 (b) ((c), resp.), we notice that if two knot projections $P$ and $P'$ are related by a deformations of type RI or type weak RI, $f(P)$ represents the same knot type as that of $f(P')$ as in Figure 3. Thus, the following map $F$ is well-defined [2].

$$F : \{\text{knot projections}\}/\text{RI, weak RI} \rightarrow \{\text{positive knots}\}/\text{isotopy}$$

where a positive knot is a knot having a knot diagram satisfying the condition that every crossing is a positive crossing. By [4, Theorem 1.5], we have Fact 1 [2, Corollary 4.2]. Here, we recall that knot projections $P$ and $P'$ are weakly (1, 3) homotopic if $P$ and $P'$ are related by a sequence of deformations of type RI or type weak RI. For a knot projection $P$, let $[P]$ be the weak (1, 3) homotopy class including $P$.

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Figure 2. (a) : the first Reidemeister move, (b): a deformation of type RI, and (c) : a deformation of type weak R_{III} (dotted curves indicate connections of a knot projection). A deformation (b) or (c) is a local replacement of a disk with the other disk (each disk is used to represent $x_n$ in the proof in Section 3).

Figure 3. If two knot projections $P$ and $P'$ are related by a deformations of type RI or type weak R_{III}, $f(P)$ represents the same knot type as that of $f(P')$.

**Fact 1.** Let $U$ be the unknot and $[T]$ the weak (1, 3) homotopy class including the simple closed curve $T$. Then,

$$F([T]) = U, \text{ and } F^{-1}(U) = [T].$$

If we consider a generalization of Fact 1, Question 1 arises (in 2013, this question was formulated independently by S. Kamada and by Y. Nakanishi [1, Section 5.5]).

**Question 1.** Is $F$ is injective?

If it was true, positive knots would be caught by a plane curve theory. However, we have:

**Theorem 1.** Let $F$ be the map (1) as above. Let $\mathcal{P}$ and $\mathcal{R}$ be the two sets of knot projections as in Figure 4 $(i, j \geq 2)$. There exist infinitely many pairs $(P, R) \in \mathcal{P} \times \mathcal{R}$ such that $P$ and $R$ are not weakly (1, 3) homotopic and $F(P)$ and $F(R)$ are ambient isotopic.

In order to show Theorem 1, we show Theorem 2.

**Theorem 2.** Let $P_0$ be a knot projection. Suppose that $P_0$ has no 1-gons, no incoherent 2-gons, and no incoherent 3-gons as in Figure 5. Two knot projections $P$ and $P_0$ are weakly (1, 3) homotopic if and only if $P$ is obtained from $P_0$ by a sequence of deformations, each of which is of type RI increasing double points.
Remark 1. For strong (1, 3) homotopy classes, the counterpart of Theorem 2 appears in [3].

2. Proof of Theorem 1 from Theorem 2

Every knot projection \( R \) has no 1-gon, no incoherent 2-gon, and no incoherent 3-gon. This fact together with Theorem 2 implies that for any pair \((P, R)\), \( P \) and \( R \) are not weakly (1, 3) homotopic whereas it is easy to see that \( F(P) \) and \( F(R) \) are ambient isotopic by using Figure 4. It completes the proof of Theorem 1. \( \square \)
3. Proof of Theorem 2

Before starting the proof, we recall Fact 2, which is implied by [2, Corollary 4.1] with [2, Lemma 2.3].

Fact 2. Let (1a) be a deformation of type RI increasing double points. A knot projection $P$ and the simple closed curve are weakly $(1, 3)$ homotopic if and only if $P$ is obtained from the simple closed curve by a sequence of deformations, each of which is (1a).

The proof is given by induction on the length $n$ of a sequence:

$$P_0 \xrightarrow{O_{p_1}} P_1 \xrightarrow{O_{p_2}} P_2 \xrightarrow{O_{p_3}} \ldots \xrightarrow{O_{p_n}} P_n,$$

where each $O_{p_i}$ $(1 \leq i \leq n)$ is a deformation of type RI or weak RIII to obtain a knot projection $P_i$ (an example of $P_0$ is shown in Figure 6). Suppose that $P_n$

![Figure 6. An example of $P_0$](image)

satisfies the statement under the setting $P = P_n$, and we will prove $P_{n+1}$ also satisfies the statement of Theorem 2. Let (1a) ((1b) resp.) be a deformation of type RI increasing (decreasing, resp.) double points. Let $(w3a)$ ($(w3b)$, resp.) be a deformation of type weak RIII as shown in Figure 7. Recall that $O_{p_{n+1}}$ is a local replacement of a disk (cf. Figure 2), and let $x_n$ be the closed disk at which we will apply $O_{p_{n+1}}$. Note that $x_n$ is a subset of $S^2$ and intersects $P_n$. Let $B$ be a regular

![Figure 7. $(w3a)$ and $(w3b)$](image)

neighborhood of $P_0$ (Figure 8 (the right figure)). For every knot projection $P$, each connected component of $P \setminus \{\text{all the double points}\}$ is called an edge. For each edge of $P_0$, we put exactly one sufficiently small closed disk on the edge where the closed disk $\cap P_0$ is a simple arc, and this closed disk is called a swelling (Figure 9). Figure 10 obtains an example of $B$ and the swellings of $P_0$. 
Let $d(P_0)$ be the shared label of each double point of $P_0$. By using Fact 2 and the assumption of the induction, a double point of $P_n$ inherits the label $d(P_0)$ since double points of $P_n$ consist of those of $P_0$ with the added double points to $P_0$. Suppose that $x_n$ contains $m$ double points, which are inherited by $P_0$ and are labeled by $d(P_0)$’s. By the definition of $x_n$, $0 \leq m \leq 3$.

Here, assume for a contradiction, that $m \geq 1$.

\begin{itemize}
  \item Case $m = 1$ ($m = 2$, resp.).
  \begin{itemize}
    \item If $Op_n$ is a deformation of type RI and $x_n \cap P_n$ has exactly one double point with $d(P_0)$, then the one double point of the focused 1-gon ($\subset x_n$) is included in a swelling. However, by the definition of swellings and the assumption of the induction, there is no swelling such that a double point having the label $d(P_0)$ is included in the swelling, which implies a contradiction since $P_0$ has no 1-gon.

    If $Op_n$ is a deformation of type weak RI I I and $x_n \cap P_n$ has exactly one (two, resp.) double point(s) with $d(P_0)$ ($d(P_0)$’s, resp.), then the two (one, resp.) double point(s) of the focused incoherent 3-gon ($\subset x_n$) should be included in a swelling. By the assumption of the induction, if we take a closure of the swelling $\cap P_n$ as in Figure 11, then the closure is obtained from the simple closed curve by a sequence of deformations, each of which is (1a). However, it is easy to see that we cannot take a swelling satisfying the above condition that the closure is obtained from the
simple closed curve by a sequence of deformations, each of which is (1a) (it is easy to see it, an explanation is given by the next paragraph).

Recall that since $P_n$ is a knot projection, there exists an immersion $g$ such that $g(S^1) = P_n$. Since $P_n$ includes an incoherent 3-gon with the three double points $d_1, d_2,$ and $d_3$, a configuration $g^{-1}({d_1, d_2, d_3})$ of three paired points on $S^1$, where each pairing is represented a chord is as shown in Figure 12. In contrast,

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure12.png}
\caption{A configuration $g^{-1}({d_1, d_2, d_3})$ of three paired points on $S^1$ (the lower line) where each pairing is represented a chord (two types appearing in a deformation of type weak R\textsc{III}) for an immersion $g$.}
\end{figure}

every deformation of type RI cannot produce any intersection of chords (e.g., see Figure 13). Thus, Fact 2 implies that the chord corresponding to a double point in a swelling must be an isolated chord, and the only possibility is as shown in Fig. 14. By using the lower line of Figure 12, it is easy to see that the other choice of a single chord relates to an intersection of chords and any choice of two chords also relates an intersection of chords. The details are left to the reader. The only possibility as in Fig. 14 also implies that $P_0$ has a 1-gon, which contradicts to the assumption of the statement (recall that every double point with the label $d(P_0)$ of $P_n$ inherits a double point $P_0$).
Figure 13. Every deformation of type RI on a knot projection (left) cannot produce any intersection of chords (right). The solid chords, in the right figure, correspond to the double points in the left figure.

Figure 14. The only possibility (left) of a swelling that includes a double point of a triangle appearing in a deformation of type weak RI at the disk $x_n$, which implies that $P_0$ has a 1-gon (right).

Case $m = 3$. If $x_n \cap P_n$ has exactly three double points with labels $d(P_0)$'s, then the incoherent 3-gon appears in $P_0$, which contradicts to the assumption of the statement (recall that each double point with the label $d(P_0)$ of $P_n$ inherits a double point of $P_0$).

In conclusion, $m = 0$. Here, note that there exists a double point with the label $d(P_0)$ between two swellings, which implies that $x_n$ does not intersect both of two double points where one belongs to a swelling and another belongs to the other swelling (see Figure 15).

Figure 15. The disk $x_n$ cannot include two swellings (left) because there is a double point between two swelling (right).

In summary,

- The disk $x_n$ intersects at most one swelling (by using the argument of the above paragraph).
- Every double point in $x_n$ is also included in a swelling ($\therefore$ the assumption of the induction).
- By $m = 0$, $x_n$ contains no double point with the label $d(P_0)$ (by retaking $x_n$ if necessary).
Therefore, by retaking $x_n$ keeping its definition if necessary,

$$x_n \subset a \text{ swelling.}$$

Thus, if we take the closure of a sub-curve that is the swelling $\cap P_n$, as shown in Figure 11 where the swelling includes $x_n$, then the closure and the simple closed curve are weakly $(1, 3)$ homotopic. It is easy to see that this fact and Fact 2 imply that $Op_{n+1}$ is a deformation of type RI, which also implies the statement holds in the $n + 1$ case of the induction. \qed

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