POSITIVE KNOTS AND WEAK (1, 3) HOMOTOPY

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ABSTRACT. It is known that there exists a surjective map from the set of weak (1, 3) homotopy classes of knot projections to the set of positive knots. An interesting question whether this map is also injective, which question was formulated independently by S. Kamada and Y. Nakanishi in 2013. This paper obtains an answer to this question.

1. INTRODUCTION

A positive knot is ambient isotopic to the unknot if and only if its positive knot diagram on a 2-sphere is related to the simple closed curve by a sequence of the first Reidemeister moves (Figure 2 (a)), each of which decreases crossings up to ambient isotopy on S^2 [4, Theorem 1.5]. This statement gives rise to a problem (Question 1) that is solved by this paper.

For a knot projection on a 2-sphere (i.e., a generic immersed spherical curve up to ambient isotopy on S^2), each double point is replaced by a positive crossing as in Figure 1. This replacement induces a map

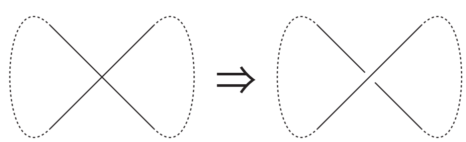


FIGURE 1. A local replacement of a double point with a positive crossing. Dotted curves indicate connections of a knot projection.

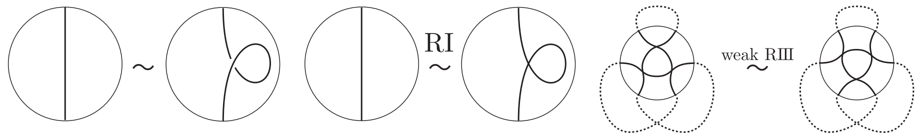
$$f : \{ \text{ knot projections } \} \to \{ \text{ knot diagrams } \}.$$

Then, letting RI (*weak* RIII, resp.) be a local deformation of a knot projection as in Figure 2 (b) ((c), resp.), we notice that if two knot projections P and P' are related by a deformations of type RI or type weak RIII, $f(P)$ represents the same knot type as that of $f(P')$ as in Figure 3. Thus, the following map F is well-defined [2].

$$F : \{\text{knot projections}\}/\,\mathrm{RI}, \text{weak RIII} \to \{\text{positive knots}\}/\text{isotopy} \tag{1}$$

where a *positive knot* is a knot having a knot diagram satisfying the condition that every crossing is a positive crossing. By [4, Theorem 1.5], we have Fact 1 [2, Corollary 4.2]. Here, we recall that knot projections P and P' are *weakly (1, 3) homotopic* if P and P' are related by a sequence of deformations of type RI or type weak RIII. For a knot projection P, let $[P]$ be the *weak (1, 3) homotopy class* including P.

Date: April 21, 2019.
Key words and phrases. positive knots, knot projection, Reidemeister moves.
MSC 2010: 57M25, 57R42, 57M99.



(a) (b) (c)

FIGURE 2. (a) : the first Reidemeister move, (b): a deformation of type RI, and (c) : a deformation of type weak RIII (dotted curves indicate connections of a knot projection). A deformation (b) or (c) is a local replacement of a disk with the other disk (each disk is used to represent x_n in the proof in Section 3).

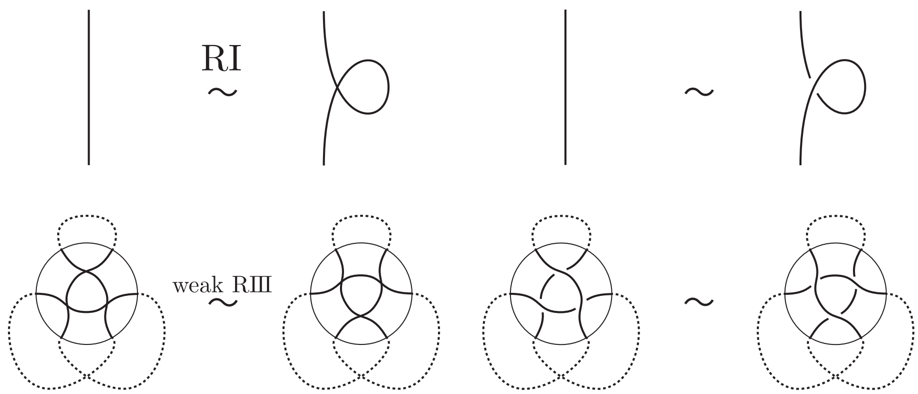


FIGURE 3. If two knot projections P and P' are related by a deformations of type RI or type weak RIII, $f(P)$ represents the same knot type as that of $f(P')$.

Fact 1. *Let U be the unknot and $[T]$ the weak (1, 3) homotopy class including the simple closed curve T. Then,*

$$F([T]) = U, \text{ and } F^{-1}(U) = [T].$$

If we consider a generalization of Fact 1, Question 1 arises (in 2013, this question was formulated independently by S. Kamada and by Y. Nakanishi [1, Section 5.5]).

Question 1. *Is F is injective?*

If it was true, positive knots would be caught by a plane curve theory. However, we have:

Theorem 1. *Let F be the map (1) as above. Let $\mathcal{P}$ and $\mathcal{R}$ be the two sets of knot projections as in Figure 4 ($i, j \geq 2$). There exist infinitely many pairs (P, R) $\in \mathcal{P} \times \mathcal{R}$ such that P and R are not weakly (1, 3) homotopic and $F(P)$ and $F(R)$ are ambient isotopic.*

In order to show Theorem 1, we show Theorem 2.

Theorem 2. *Let P_0 be a knot projection. Suppose that P_0 has no 1-gons, no incoherent 2-gons, and no incoherent 3-gons as in Figure 5. Two knot projections P and P_0 are weakly (1, 3) homotopic if and only if P is obtained from P_0 by a sequence of deformations, each of which is of type* RI *increasing double points.*

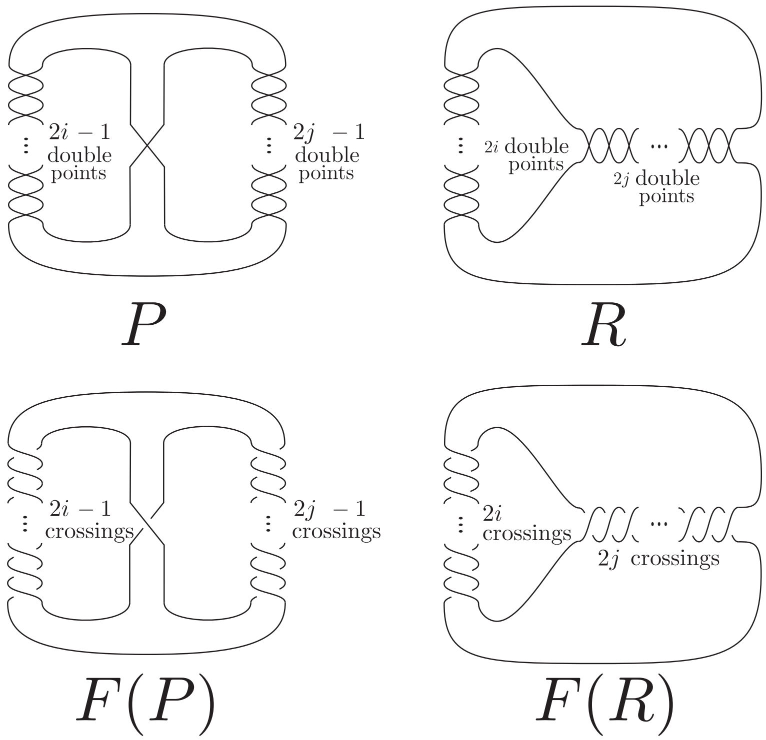


FIGURE 4. $i, j \geq 2$. Elements in $\mathcal{P}$ (upper left) and in $\mathcal{R}$ (upper right). Elements in $\{F(P) \mid P \in \mathcal{P}\}$ (lower left) and in $\{F(R) \mid R \in \mathcal{R}\}$ (lower right).

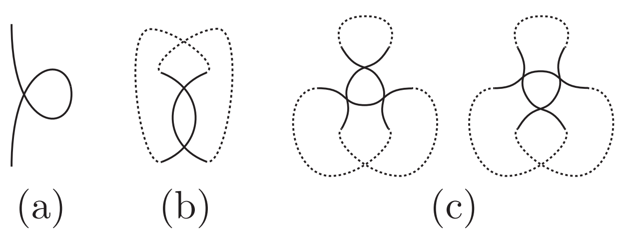


FIGURE 5. (a) : 1-gon, (b) : incoherent 2-gon, and (c) : incoherent 3-gons. Dotted curves indicate their connections of knot projections.

Remark 1. For strong (1, 3) homotopy classes, the counterpart of Theorem 2 appears in [3].

2. Proof of Theorem 1 from Theorem 2

Every knot projection R has no 1-gon, no incoherent 2-gon, and no incoherent 3-gon. This fact together with Theorem 2 implies that for any pair (P, R), P and R are not weakly (1, 3) homotopic whereas it is easy to see that $F(P)$ and $F(R)$ are ambient isotopic by using Figure 4. It completes the proof of Theorem 1. $\square$

3. Proof of Theorem 2

Before starting the proof, we recall Fact 2, which is implied by [2, Corollary 4.1] with [2, Lemma 2.3].

Fact 2. *Let* (1*a*) *be a deformation of type* RI *increasing double points. A knot projection P and the simple closed curve are weakly (1, 3) homotopic if and only if P is obtained from the simple closed curve by a sequence of deformations, each of which is* (1*a*).

The proof is given by induction on the length n of a sequence:

$$P_0 \overset{Op_1}{\to} P_1 \overset{Op_2}{\to} P_2 \overset{Op_3}{\to} \cdots \overset{Op_n}{\to} P_n,$$

where each Op_i $(1 \leq i \leq n)$ is a deformation of type RI or weak RⅢ to obtain a knot projection P_i (an example of P_0 is shown in Figure 6). Suppose that P_n

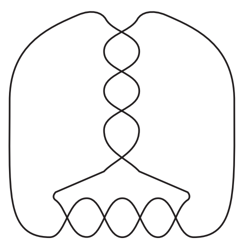


Figure 6. An example of P_0

satisfies the statement under the setting $P = P_n$, and we will prove P_{n+1} also satisfies the statement of Theorem 2. Let $(1a)$ ($(1b)$ resp.) be a deformation of type RI increasing (decreasing, resp.) double points. Let $(w3a)$ ($(w3b)$, resp.) be a deformation of type weak RⅢ as shown in Figure 7. Recall that Op_{n+1} is a local replacement of a disk (cf. Figure 2), and let x_n be the closed disk at which we will apply Op_{n+1}. Note that x_n is a subset of S^2 and intersects P_n. Let B be a regular

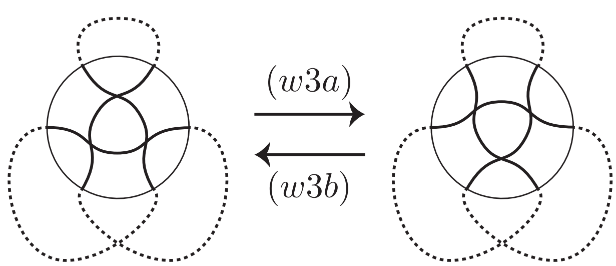


Figure 7. $(w3a)$ and $(w3b)$

neighborhood of P_0 (Figure 8 (the right figure)). For every knot projection P, each connected component of $P \setminus \{$all the double points$\}$ is called an *edge*. For each edge of P_0, we put exactly one sufficiently small closed disk on the edge where the closed disk $\cap$ P_0 is a simple arc, and this closed disk is called a *swelling* (Figure 9). Figure 10 obtains an example of B and the swellings of P_0.

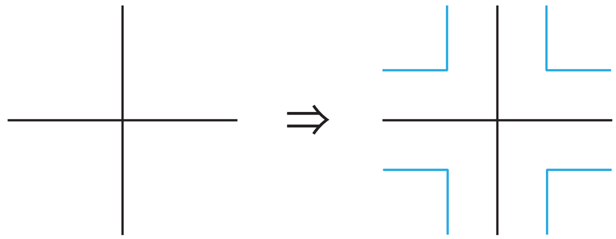


FIGURE 8. Taking a regular neighborhood B of P_0.

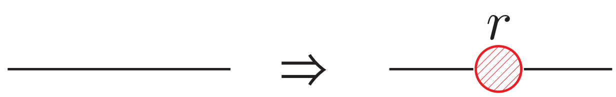


FIGURE 9. Putting a swelling r on an edge of P_0.

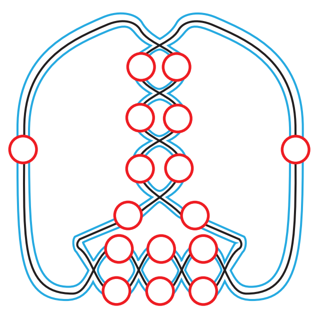


FIGURE 10. An example of P_0 with a regular neighborhood B and the swellings

Let $d(P_0)$ be the shared label of each double point of P_0. By using Fact 2 and the assumption of the induction, a double point of P_n inherits the label $d(P_0)$ since double points of P_n consist of those of P_0 with the added double points to P_0. Suppose that x_n contains m double points, which are inherited by P_0 and are labeled by $d(P_0)$'s. By the definition of x_n, $0 \leq m \leq 3$.

Here, assume for a contradiction, that $m \geq 1$.

• Case $m = 1$ ($m = 2$, resp.).

If Op_n is a deformation of type RI and $x_n \cap P_n$ has exactly one double point with $d(P_0)$, then the one double point of the focused 1-gon ($\subset x_n$) is included in a swelling. However, by the definition of swellings and the assumption of the induction, there is no swelling such that a double point having the label $d(P_0)$ is included in the swelling, which implies a contradiction since P_0 has no 1-gon.

If Op_n is a deformation of type weak RIII and $x_n \cap P_n$ has exactly one (two, resp.) double point(s) with $d(P_0)$ ($d(P_0)$'s, resp.), then the two (one, resp.) double point(s) of the focused incoherent 3-gon ($\subset x_n$) should be included in a swelling. By the assumption of the induction, if we take a closure of the swelling $\cap$ P_n as in Figure 11, then the closure is obtained from the simple closed curve by a sequence of deformations, each of which is ($1a$). However, it is easy to see that we cannot take a swelling satisfying the above condition that the closure is obtained from the

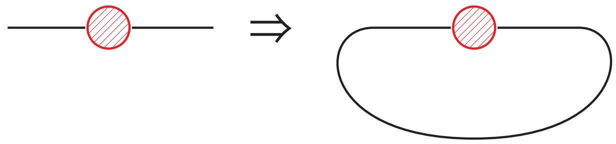


FIGURE 11. Taking the closure, which is a knot projection, of subcurves $g(A)$ (for an interval $A \subset S^1$) including exactly one swelling.

simple closed curve by a sequence of deformations, each of which is $(1a)$ (it is easy to see it, an explanation is given by the next paragraph).

Recall that since P_n is a knot projection, there exists an immersion g such that $g(S^1) = P_n$. Since P_n includes an incoherent 3-gon with the three double points d_1, d_2, and d_3, a configuration $g^{-1}(\{d_1, d_2, d_3\})$ of three paired points on S^1, where each pairing is represented a chord is as shown in Figure 12. In contrast,

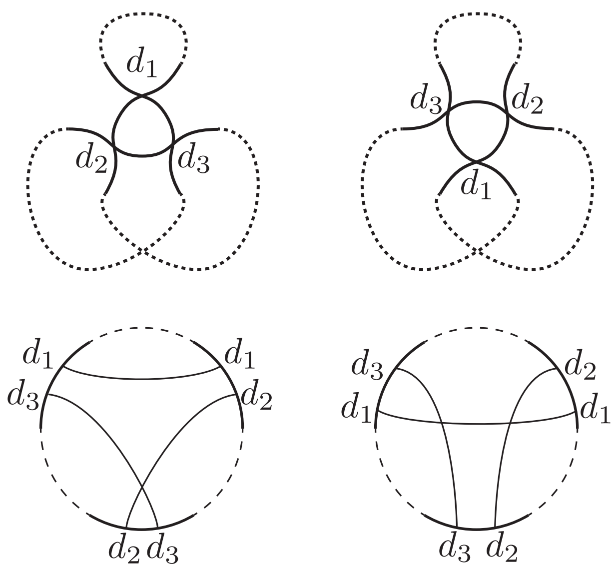


FIGURE 12. A configuration $g^{-1}(\{d_1, d_2, d_3\})$ of three paired points on S^1 (the lower line) where each pairing is represented a chord (two types appearing in a deformation of type weak RIII) for an immersion g.

every deformation of type RI cannot produce any intersection of chords (e.g., see Figure 13). Thus, Fact 2 implies that the chord corresponding to a double point in a swelling must be an isolated chord, and the only possibility is as shown in Fig. 14. By using the lower line of Figure 12, it is easy to see that the other choice of a single chord relates to an intersection of chords and any choice of two chords also relates an intersection of chords. The details are left to the reader. The only possibility as in Fig. 14 also implies that P_0 has a 1-gon, which contradicts to the assumption of the statement (recall that every double point with the label $d(P_0)$ of P_n inherits a double point P_0).

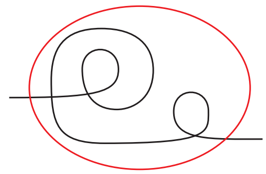


FIGURE 13. Every deformation of type RI on a knot projection (left) cannot produce any intersection of chords (right). The solid chords, in the right figure, correspond to the double points in the left figure.

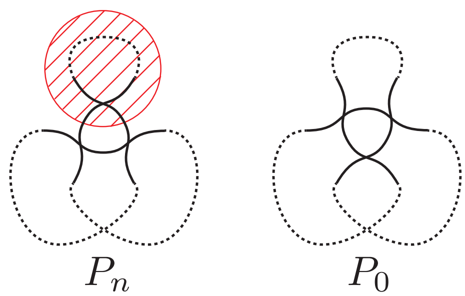


FIGURE 14. The only possibility (left) of a swelling that includes a double point of a triangle appearing in a deformation of type weak RIII at the disk x_n, which implies that P_0 has a 1-gon (right).

• Case $m = 3$. If $x_n \cap P_n$ has exactly three double points with labels $d(P_0)$'s, then the incoherent 3-gon appears in P_0, which contradicts to the assumption of the statement (recall that each double point with the label $d(P_0)$ of P_n inherits a double point of P_0).

In conclusion, $m = 0$. Here, note that there exists a double point with the label $d(P_0)$ between two swellings, which implies that x_n does not intersect both of two double points where one belongs to a swelling and another belongs to the other swelling (see Figure 15).

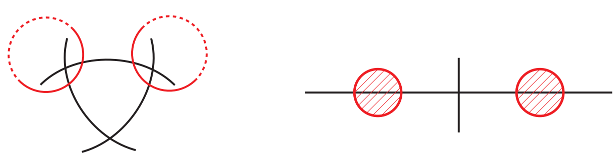


FIGURE 15. The disk x_n cannot include two swellings (left) because there is a double point between two swelling (right).

In summary,

- The disk x_n intersects at most one swelling (by using the argument of the above paragraph).
- Every double point in x_n is also included in a swelling ($\because$ the assumption of the induction).
- By $m = 0$, x_n contains no double point with the label $d(P_0)$ (by retaking x_n if necessary).

Therefore, by retaking x_n keeping its definition if necessary,

$$x_n \subset \text{a swelling}.$$

Thus, if we take the closure of a sub-curve that is the swelling $\cap\ P_n$, as shown in Figure 11 where the swelling includes x_n, then the closure and the simple closed curve are weakly (1, 3) homotopic. It is easy to see that this fact and Fact 2 imply that Op_{n+1} is a deformation of type RI, which also implies the statement holds in the $n+1$ case of the induction. $\square$

Acknowledgement

The authors would like to thank Professor Seiichi Kamada and Professor Yasutaka Nakanishi for giving us Question 1.

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