Numerical simulation of the $\mathcal{N} = 2$ Landau–Ginzburg model

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2017/9/28 SSI2017 @Yamaguchi

Introduction

- Superconformal symmetry
  - Superstring theory: a symmetry on world sheet
  - Critical phenomenon: conformal inv. $\sim$ scale inv.

    QFT on RG fixed point = Conformal field theory (CFT)
- CFT in terms of Lagrangian: Landau–Ginzburg (LG) description
  - RG fixed point of LG model $\rightarrow$ CFT

- IR limit of 2D $\mathcal{N} = (2,2)$ Wess–Zumino ($\mathcal{N} = 2$ WZ) model $\Rightarrow \mathcal{N} = 2$ SCFT [Vafa–Warner 1988, Howe–West 1989, Witten 1993,...]
- Application to spacetime compactification
  - $\mathcal{N} = 2$ SCFT on world sheet and Calabi–Yau (CY) manifold
  - σ-model on CY and LG model [Greene–Vafa–Warner 1989, Witten 1993]

Introduction

- (IR) Strong coupling/divergence $\rightarrow$ Non-perturbative approach
- SUSY regulator...
  - Lattice regularization breaks SUSY!
  - cf. Lattice simulation for superpotential Φ^3 [Kawai–Kikukawa 2009]. (But, Φ^4??)

- SUSY-preserving simulation method [Kadoh–Suzuki 2009]
- Calculation of scaling dimension and central charge for Φ^3 [Kamata–Suzuki 2010]

- <u>Scaling dimension and central charge for Φ^3 and Φ^4</u>

Supersymmetric formulation [Kadoh–Suzuki 2009]

- Continuum physical box $L \times L \longrightarrow$ Discretized momentum

$$\varphi(x) = \frac{1}{L^2} \sum_p e^{ipx} \varphi(p),$$

$$p_\mu = \frac{2\pi}{L} n_\mu. \qquad (n_\mu = 0, \pm 1, \pm 2, \dots)$$

  Formulation is defined in this momentum space.

- UV cutoff $\Lambda = \pi/a$ (a: "lattice spacing")

$$p^2 \leq \Lambda^2$$

  - L/a: even integer

- Not lattice space → SUSY, translational, R symmetries OK!
- Problem of locality (Perturbatively, locality restores as $\Lambda \to \infty$.)

WZ model and Nicolai map [Nicolai 1980]

- Action of complex scalar A and fermion ψ

$$S = S_B + \frac{1}{L^2}\sum_p \left[(\bar{\psi}_{\dot{1}}, \psi_2)\,(-p) \begin{pmatrix} 2ip_z & W''(A)^* * \\ W''(A)* & 2ip_{\bar{z}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \bar{\psi}_{\dot{2}} \end{pmatrix} (p) \right],$$

where we have integrated auxiliary field, $*$ is convolution, and

$$S_B = \frac{1}{L^2}\sum_p N(p)^* N(p), \quad N(p) \equiv 2ip_z A(p) + W'(A)^*(p).$$

- Integrate fermion, variable transf. $A(p), A^*(p) \to N(p), N^*(p)$.

$$\mathcal{Z} = \int \prod_{p^2 \le \Lambda^2} [dA(p) dA^*(p)]\, e^{-S_B} \det \frac{\partial(N, N^*)}{\partial(A, A^*)}$$

$$= \int \prod_{p^2 \le \Lambda^2} [dN(p) dN^*(p)]\, e^{-S_B} \sum_i \operatorname{sign} \det \frac{\partial(N, N^*)}{\partial(A, A^*)}\bigg|_{A=A_i}.$$

  - $A_i(p)$ $(i = 1, 2, \dots)$ are solutions with respect to $N(p)$.
  - Gaussian function (Nicolai map)

Algorithm

1. Generate Gaussian random numbers $(N(p), N^*(p))$
2. Solve numerically the equation

$$2ip_z A(p) + W'(A)^*(p) - N(p) = 0.$$

$\rightarrow$ all solutions $A(p)_i$ $(i = 1, 2, \dots)$

3. Calculate following sums

$$\sum_i \text{sign} \det \frac{\partial(N, N^*)}{\partial(A, A^*)} \mathcal{O}(A, A^*) \Big|_{A=A_i}, \quad \sum_i \text{sign} \det \frac{\partial(N, N^*)}{\partial(A, A^*)} \Big|_{A=A_i}$$

4. Repeat (1)$\sim$(3), and average $\longrightarrow \langle \mathcal{O} \rangle$

- Partition function (Witten index Δ)

$$\Delta \equiv \left\langle \sum_i \text{sign} \det \frac{\partial(N, N^*)}{\partial(A, A^*)} \Big|_{A=A_i} \right\rangle$$

Signs for $\{A_i(p)\}_N$: $(+ \cdots + - \cdots -) \Rightarrow \Delta_N = n_+ - n_-$

Classification of configurations

- $W(\Phi) = \lambda\Phi^k/k$, $a\lambda = 0.3$ $(\Delta = k - 1)$
- Solve by Newton method; Convergent init. conf.×100

$$[2ip_z A(p) + W'(A)^*(p) - N(p)] / \sqrt{\sum |N(q)|^2} \text{ — } (*)$$

$*$ $k = 3$ $(\Delta = 2)$
$L/a = 8$–34: 640 confs.

L/a	36
$(++)_2$	1248
$(+++-)_2$	32
Δ	2.0
Max norm$(*)$	1.3×10^{-15}

$*$ $k = 4$ $(\Delta = 3)$
$L/a = 8$–28: 320 confs.

L/a	24
$(+++)_3$	1253
$(++++-)_3$	24
$(++++++--)_3$	2
$(++++)_4$	1
Δ	3.001(1)
Max norm$(*)$	1.4×10^{-15}

- More difficult to solve equation for the case Φ^4. $\Delta \cong 3$.

Correlation function

- Simulation of scaling dimension and central charge
- Two-point function

$$\langle \varphi_1(p)\varphi_2(-p)\rangle = L^2 \int d^2x\, e^{-ipx} \underline{\langle \varphi_1(x)\varphi_2(0)\rangle}$$

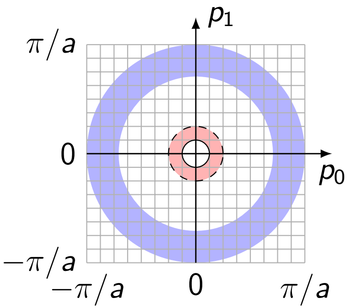
- SCFT: scalar field $A(x)$ and supercurrent $S_z^{\pm}(x)$

$$\langle A(x)A^*(0)\rangle \propto 1/z^{2h}\bar{z}^{2\bar{h}}, \qquad \left\langle S_z^+(x)S_z^-(0)\right\rangle = 2c/3z^3.$$

IR behavior of $\langle \varphi_1(p)\varphi_2(-p)\rangle$

$\Longrightarrow$ Scaling dimension $h+\bar{h}$

Central charge c



- Fitting $\langle \varphi_1(p)\varphi_2(-p)\rangle$
  - IR: $2\pi/L \leq |p| < 2\pi/L \times 2$

    (UV: free SCFT)

Scaling dimension

$$\langle A(p)A^*(-p)\rangle \sim 1/(p^2)^{1-h-\bar{h}}$$

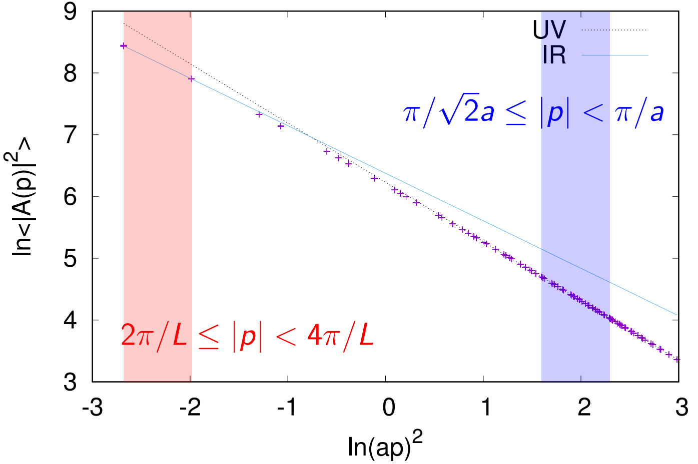


Figure: Φ^4, $L/a = 24$

Scaling dimension

$$\langle A(p)A^*(-p)\rangle \sim 1/(p^2)^{1-h-\bar{h}}$$

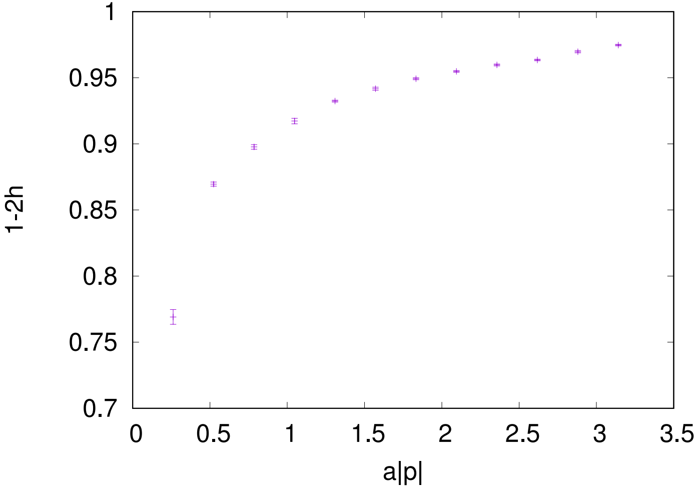


Figure: Φ^4, $L/a = 24$

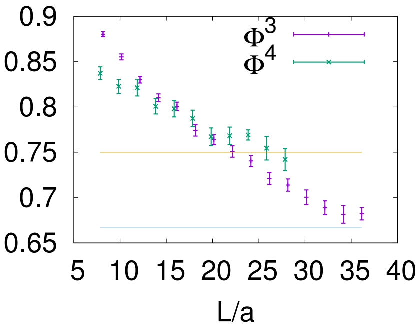
Scaling dimension $h + \bar{h}$

- Φ^3, $L/a = 36$

	$1 - h - \bar{h}$
IR	0.6822(68)
UV	0.979650(11)

- Φ^4, $L/a = 28$

	$1 - h - \bar{h}$
IR	0.742(12)
UV	0.95972(37)



- As $L \to$ large, $1 - h - \bar{h}$ "approaches" values for SCFT.
  - Φ^3: $2/3 = 0.666...$, Φ^4: $3/4 = 0.75$ (UV $\to 1$)

Central charge

$$\langle S_z^+(p) S_z^-(-p) \rangle = L^2 i\pi c p_z^2 / 3p_{\bar{z}}$$

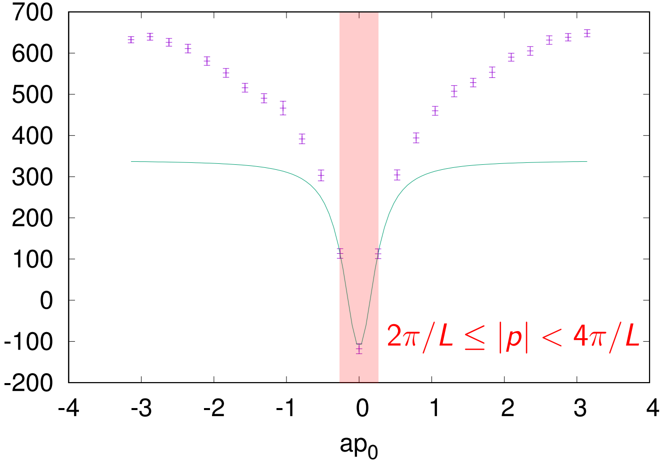


Figure: Φ^4, $L/a = 24$, $ap_1 = \pi/12$, Real part.

Central charge

$$\langle S_z^+(p) S_z^-(-p) \rangle = L^2 i\pi c p_z^2 / 3p_{\bar{z}}$$

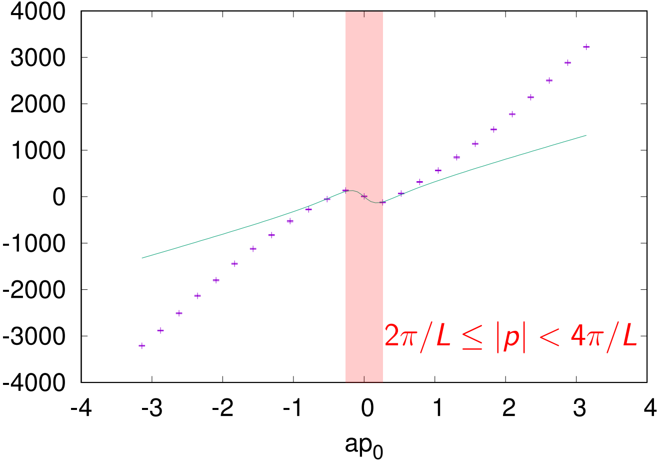


Figure: Φ^4, $L/a = 24$, $ap_1 = \pi/12$, Imaginary part.

Central charge

$$\langle S_z^+(p) S_z^-(-p) \rangle = L^2 i\pi c p_z^2 / 3 p_{\bar{z}}$$

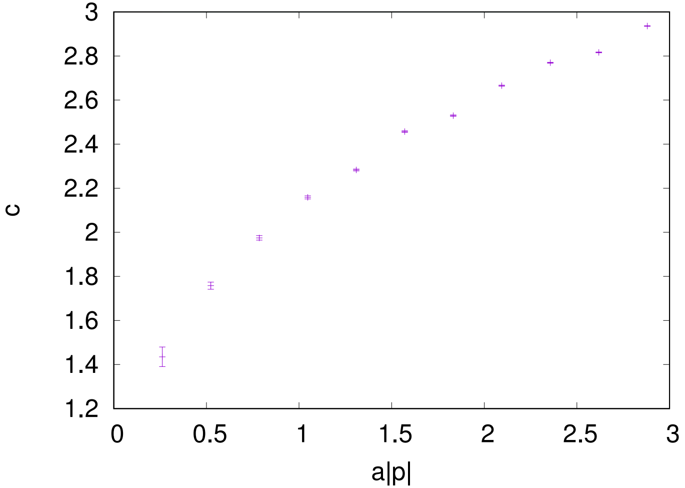


Figure: Φ^4, $L/a = 24$. Analogue of Zamolodchikov's c-theorem.

Central charge c

- Φ^3, $L/a = 36$

fitting region	c
IR	1.152(48)
$\frac{17\pi}{18} \leq a\lvert p\rvert < \pi$	3.09901(73)

- Φ^4, $L/a = 26$

fitting region	c
IR	1.43(11)
$\frac{12\pi}{13} \leq a\lvert p\rvert < \pi$	2.9360(14)

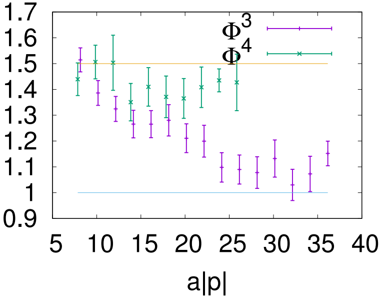


Figure: Φ^4, $L/a = 24$.

- As $L \to$ large, central charge c "approaches" values for SCFT.
  - Φ^3: 1, Φ^4: 1.5. (UV $\to$ 3)

Summary

- Numerical simulation of 2D $\mathcal{N}=2$ WZ model $\xrightarrow{\text{IR limit}}$ Non-perturbatively emerging SCFT
- For Φ^k $(k=3,4)$ theory, scaling dimension and central charge

k	$1-h-\bar{h}$		c	
3	0.6822(68)	[0.666...]	1.152(48)	[1]
4	0.742(12)	[0.75]	1.43(11)	[1.5]

$\Longrightarrow$ These results are "consistent" with SCFT.

- Future work
  - Precision: larger L, finite size scaling
  - Correlation functions of Energy-momentum tensor, $U(1)$ current
  - Multi-superfield models (ADE classification)
  - Application to Calabi–Yau space

Backup: Supercurrent

$$S_z^+(p) = \frac{4\pi}{L_0 L_1} \sum_q i(p-q)_z A(p-q) \bar{\psi}_{\dot{2}}(q),$$

$$S_{\bar{z}}^+(p) = \frac{2\pi}{L_0 L_1} \sum_q W'(A)(p-q) \psi_1(q),$$

$$S_z^-(p) = -\frac{4\pi}{L_0 L_1} \sum_q i(p-q)_z A^*(p-q) \psi_2(q),$$

$$S_{\bar{z}}^-(p) = \frac{2\pi}{L_0 L_1} \sum_q W'(A)^*(p-q) \bar{\psi}_{\dot{1}}(q).$$

Backup: Energy-momentum tensor

- Energy-momentum tensor is given by

$$\begin{aligned} T_{zz}(x) &= -4\pi\partial A^*(x)\partial A(x) \\ &\quad - \pi\psi_2(x)\bar{\psi}_{\dot{2}}(x) + \pi\partial\psi_2(x)\bar{\psi}_{\dot{2}}(x), \\ T_{\bar{z}\bar{z}}(x) &= -4\pi\bar{\partial}A^*(x)\bar{\partial}A(x) \\ &\quad - \pi\bar{\psi}_{\dot{1}}(x)\bar{\partial}\psi_1(x) + \pi\bar{\partial}\bar{\psi}_{\dot{1}}\psi_1(x), \\ T_{z\bar{z}}(x) &= T_{\bar{z}z} = \cdots . \end{aligned}$$

- Two-point function

$$\langle T_{zz}(x)T_{zz}(0)\rangle = \frac{c}{2z^4}, \text{ that is,}$$

$$\langle T_{zz}(p)T_{zz}(-p)\rangle = L^2\frac{\pi c}{12}\frac{p_z^3}{p_{\bar{z}}}.$$

Backup: C-function

- General forms ($\tau = \ln z\bar{z}$)

$$\langle T_{zz}(x) T_{zz}(0) \rangle = F(\tau)/z^4,$$
$$\langle T_{zz}(x) T_{z\bar{z}}(0) \rangle = G(\tau)/4z^3\bar{z},$$
$$\left\langle T_{z\bar{z}(x)} T_{z\bar{z}}(0) \right\rangle = H(\tau)/z^2\bar{z}^2.$$

- Conservation laws and reflection positivity imply

$$\frac{d}{d\tau}(2F - G - 3H/8) \leq 0.$$

- Zamolodchikov's C-function:

$$C = 2F - G - \frac{3}{8}H.$$

Monotonically decreasing function along RG flow $\rightarrow c$.