Numerical study of the $\mathcal{N}=2$ Landau–Ginzburg model

Okuto Morikawa and Hiroshi Suzuki (Kyushu Univ.)

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1. Introduction

- 2D massless $\mathcal{N}=2$ Wess–Zumino (WZ) model is believed to provide the Landau–Ginzburg (LG) description of SCFT.

ADE-type theories

- 2D $\mathcal{N}=2$ WZ model ($\{\Phi_I\}_{I=1,\ldots,N_\Phi}$; $\partial_{z,\bar{z}}=\frac{1}{2}(\partial_0 \mp i\partial_1)$)

$$S=\int d^2x \sum_I \left[4\partial_z A_I^* \partial_{\bar{z}} A_I + \frac{\partial W(A)^*}{\partial A_I^*}\frac{\partial W(A)}{\partial A_I} + (\bar{\psi}_1,\psi_2)_I \sum_J \begin{pmatrix} 2\delta_{IJ}\partial_z & \frac{\partial^2 W(A)^*}{\partial A_I^* \partial A_J^*} \\ \frac{\partial^2 W(A)}{\partial A_I \partial A_J} & 2\delta_{IJ}\partial_{\bar{z}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}_J \right]$$

$\xrightarrow{\text{IR limit}}$ $\mathcal{N}=2$ minimal model (ADE classification [Vafa–Warner '89])

Algebra	Superpotential W	Central charge c
A_n	$\frac{\lambda_1}{n+1}\Phi_1^{n+1}$, $n \geq 1$	$3-6/(n+1)$
D_n	$\frac{\lambda_1}{n-1}\Phi_1^{n-1}+\frac{\lambda_2}{2}\Phi_1\Phi_2^2$, $n\geq 3$	$3-6/2(n-1)$
E_7	$\frac{\lambda_1}{3}\Phi_1^3+\frac{\lambda_2}{3}\Phi_1\Phi_2^3$	$3-6/18$

($E_6 \cong A_2 \otimes A_3$, $E_8 \cong A_2 \otimes A_4$)

- This model is *strongly coupled* at IR, thus, it is difficult to prove this conjecture; LG description is a non-perturbative phenomenon.
- A non-perturbative calculational method on the basis of the lattice field theory may provide an alternative approach to this issue.
  - Numerical simulation of A_2 model [Kawai-Kikukawa '10, Kamata-Suzuki '11]

▶ Applying a SUSY-preserving numerical algorithm to ADE-type theories, we obtain the scaling dimension and the central charge, which are consistent with the conjecture.

2. Lattice formulation [Kadoh-Suzuki '09]

- System: continuum space with L^2. we work in the p-space,

$$p_\mu = 2\pi n_\mu/L, \quad n_\mu = 0, \pm 1, \ldots, \pm L/2a \ \text{(Momentum cutoff)}.$$

($a \sim$ lattice spacing.) Then, the action is given by

$$S=S_B+\frac{1}{L^2}\sum_{p,I,J}(\bar{\psi}_1,\psi_2)_I(-p)\begin{pmatrix} 2i\delta_{IJ}p_z & \frac{\partial^2 W(A)^*}{\partial A_I^* \partial A_J^*}* \\ \frac{\partial^2 W(A)}{\partial A_I \partial A_J} & 2i\delta_{IJ}p_{\bar{z}} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}_J (p),$$

$$S_B=\sum_{p,I}\frac{N_I^*(-p)N_I(p)}{L^2}, \quad \underbrace{N_I(p)\equiv 2ip_z A_I(p)+\frac{\partial W(A)^*}{\partial A_I^*}(p)}_{\text{Nicolai map } (A,A^*)\to(N,N^*) \text{ ['80]}}. \quad (*)$$

- Partition function ($\{A\}_k$: solutions of the algebraic eq. $(*)$):

$$\mathcal{Z}=\int \prod_{|p_\mu|\leq\pi/a}|dN(p)|^2 \underbrace{e^{-S_B}}_{\text{Gaussian}} \sum_k \operatorname{sign}\det\left.\frac{\partial(N,N^*)}{\partial(A,A^*)}\right|_{\{A\}=\{A\}_k}$$

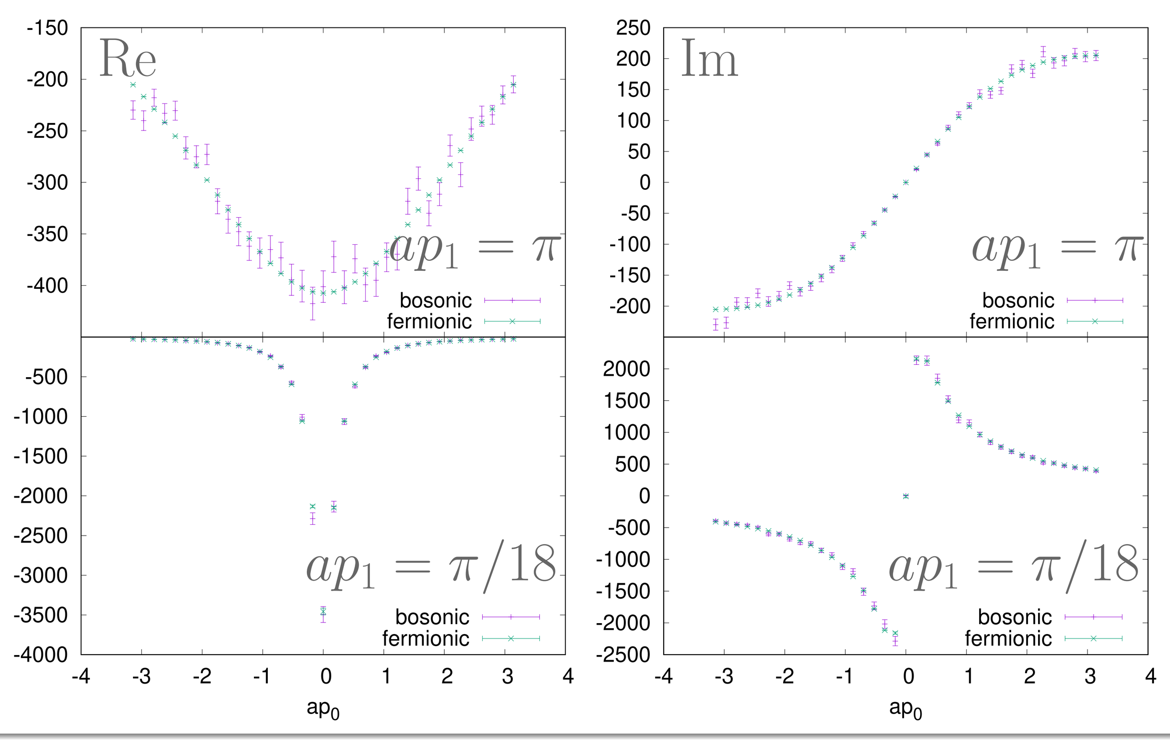
- This formulation manifestly preserves *translational inv. and SUSY*, and is free from autocorrelation among configurations.

3. Numerical setup

- Consider ADE-type superpotentials with the couplings $a\lambda_{1,2}=0.3$
  - A_n $(n=2,3)$, D_n $(n=3,4)$, E_7
- L/a: various even integers (typically ~ 30)
- Generate ~ 640 confs. of $N(p)$ using *the Gaussian random number*
- To solve Eq. $(*)$ numerically with respect to A, employ the Newton method

4. SUSY WT relation [1805.10735]

- In our formulation, SUSY WT relations should *hold exactly*.
- e.g.,
  $2ip_{\bar{z}}\langle A(p)A^*(-p)\rangle$
  $=-\langle \psi_1(p)\bar{\psi}_1(-p)\rangle$
  for A_2, $L/a=36$.



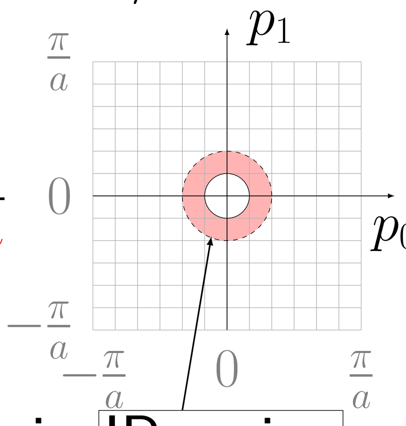
5. Scaling dimension [1805.10735]

- Two-point correlation function

$$\underbrace{\langle\varphi_1(p)\varphi_2(-p)\rangle}_{\text{Numerical simulation}} = L^2\int_{L^2} d^2x\, e^{-ipx}\underbrace{\langle\varphi_1(x)\varphi_2(0)\rangle}_{\text{Relations in SCFT}}$$

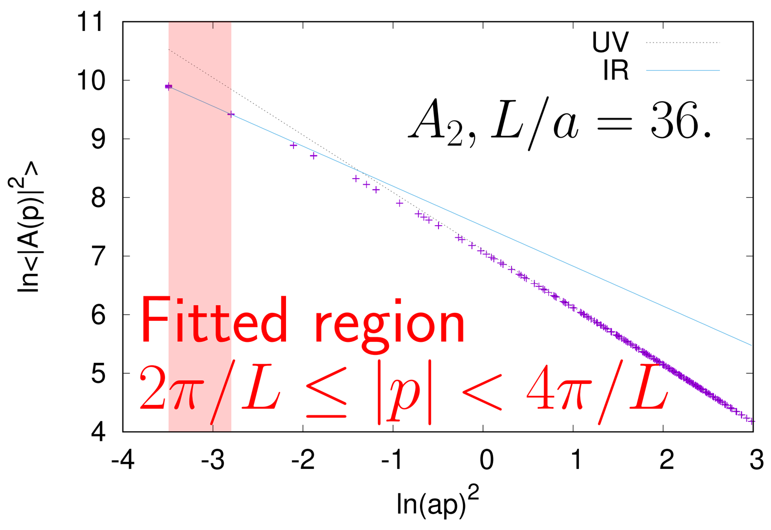
- e.g., $\langle A(x)A^*(0)\rangle \propto 1/z^{2h}\bar{z}^{2\bar{h}}$

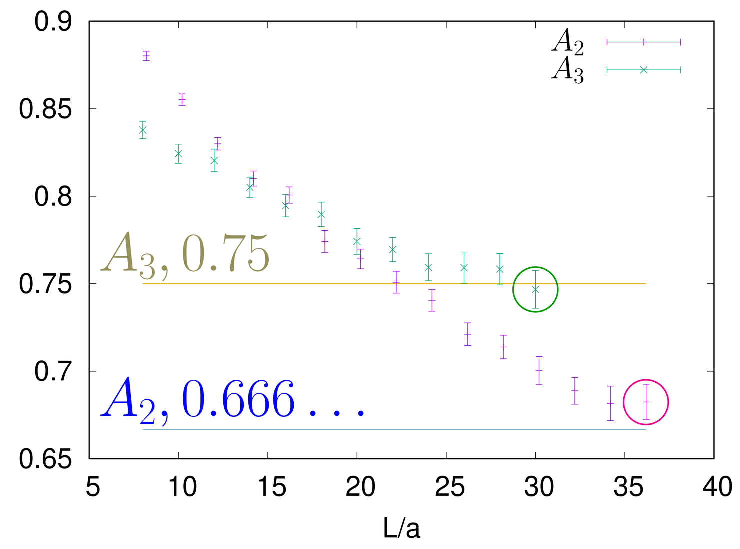
$\Rightarrow \langle A(p)A^*(-p)\rangle \propto \dfrac{1}{(p^2)^{1-h-\bar{h}}}$



- $h+\bar{h}$ obtained from the fit in IR region

Algebra	$1-h-\bar{h}$
A_2	0.682(10)(7)
A_3	0.747(11)(12)

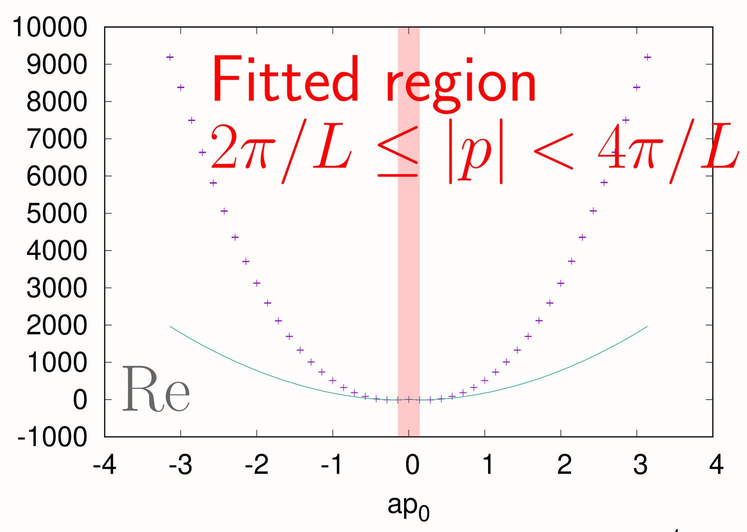


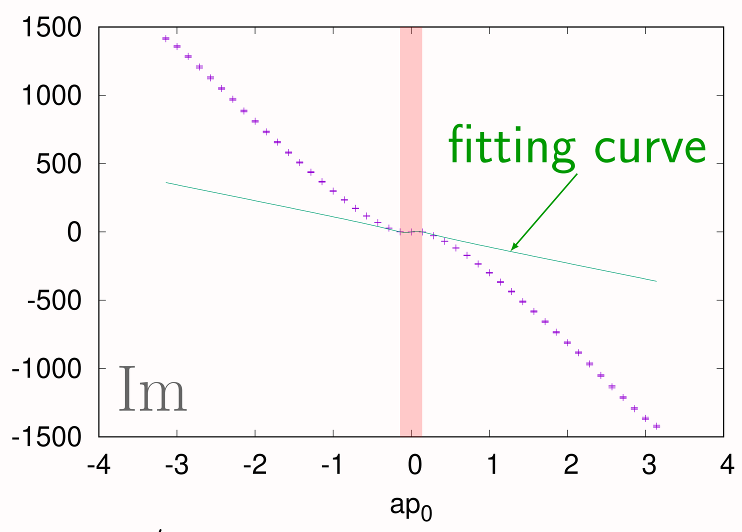


(cf. Kawai-Kikukawa (A_2): 0.660(11), Kamata-Suzuki (A_2): 0.616(25)(13))

6. Central charge [1805.10735, 1810.02519]

- Energy-momentum tensor correlator $\langle T_{zz}(p)T_{zz}(-p)\rangle = L^2\pi\, c\, \frac{p_z^3}{12p_{\bar{z}}}$

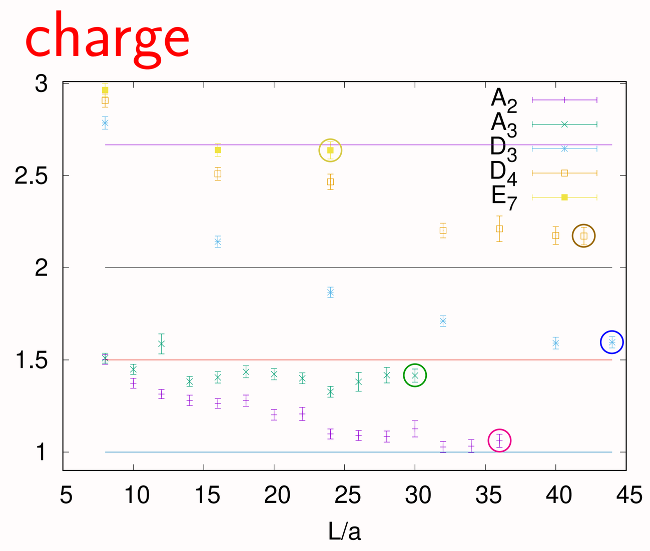




D_3, $L/a=44$, $ap_1=\pi/22$.

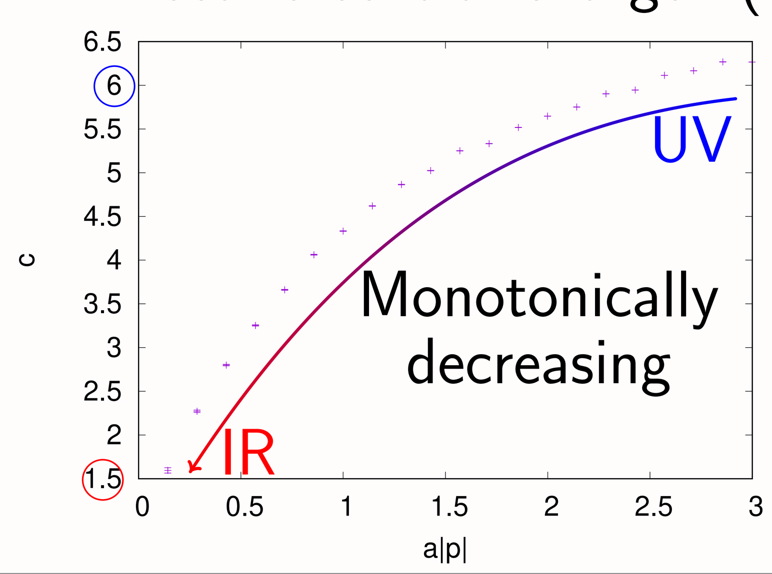
- Numerical determination of the central charge

Algebra	Central charge	Expected value
A_2	1.061(36)(34)	1
A_3	1.415(36)(36)	1.5
D_3	1.595(31)(41)	1.5
D_4	2.172(48)(39)	2
E_7	2.638(47)(59)	2.666…



(cf. Kamata-Suzuki (A_2): 1.09(14)(31))

- "Effective central charge" (D_3, $L/a=44$)



- Various fitted regions ($n=1,2,\ldots$): $\frac{2\pi}{L}n \leq |p| \leq \frac{2\pi}{L}(n+1)$
- $c_{\mathrm{IR}} \approx 1.5$
  $\Leftrightarrow c_{\mathrm{UV}} \approx 6 = 3N_\Phi$ (free theory)
- Analogous to *Zamolodchikov C-function*

7. Conclusion

- We numerically studied the IR behavior of 2D $\mathcal{N}=2$ WZ model, and determined $h+\bar{h}$ and c.
- This study supports the conjectured LG correspondence.
- *We hope that this numerical approach will be useful to investigate superstring theory via the LG/Calabi–Yau correspondence.*

Energy-momentum tensor and its correlator [1805.10735]

- Energy-momentum tensor (EMT)

$$T_{zz}=-4\pi\partial A^*\partial A - \pi\psi_2\partial\bar{\psi}_2 + \pi\partial\psi_2\bar{\psi}_2, \quad \ldots$$

Requirements: $T_{\mu\nu}=T_{\nu\mu}$, and $\sum_\mu T_{\mu\mu}=0$ in the UV limit (free $W\to 0$)

- EMT = SUSY transformation of the supercurrent

$$T_{zz}=\tfrac{1}{4}Q_2S_z^+ - \tfrac{1}{4}\bar{Q}_{\dot{2}}S_z^-, \qquad S_z^+=4\pi\bar{\psi}_{\dot{2}}\partial A,\ S_z^-=-4\pi\psi_2\partial A^*.$$
$$(Q_2\bar{\psi}_{\dot{2}}=-2\partial A^*, Q_2A=\psi_2, \bar{Q}_{\dot{2}}\psi_2=-2\partial A, \bar{Q}_{\dot{2}}A^*=\bar{\psi}_{\dot{2}}, \ldots)$$

- *SUSY WT relations* → a less noisy form of the EMT correlator:

$$\langle T_{zz}(p)T_{zz}(-p)\rangle = -\frac{2ip_z}{16}\left\langle S_z^+(p)S_z^-(-p)+S_z^-(p)S_z^+(-p)\right\rangle$$