Numerical study of ADE-type $\mathcal{N} = 2$ Landau–Ginzburg models

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- O.M. and Hiroshi Suzuki, PTEP **2018** (2018) 083B05 [arXiv:1806.10735 [hep-lat]].
- O.M., JHEP **1812** (2018) 045 [arXiv:1810.02519 [hep-lat]].
- O.M., arXiv:1906.00653 [hep-lat].

Motivation

- Superstring theory:
  4D spacetime + 6D space (Calabi–Yau (CY) manifold)
- On the 2D string world sheet,
  2D $\mathcal{N} = 2$ Superconformal field theory (SCFT)
- Solvable minimal models of SCFT (or the Gepner model)
  $\rightarrow$ points in moduli space of CY
- SCFT corresponding to general CY??

- 2D Landau–Ginzburg (LG) model $\xrightarrow{\text{IR limit}}$ SCFT
- LG/CY correspondence [Green-Vafa-Warner, Witten]

  LG model $\leftrightarrow$ *(SCFT?)* $\leftrightarrow$ CY

▶ Numerical simulation of LG model:
new approach to investigate superstring theory via LG/CY

LG model: 2D $\mathcal{N}=2$ Wess–Zumino model

- 2D $\mathcal{N}=2$ Wess–Zumino (WZ) model (Φ_I, $I=1,2,\ldots$)

$$S=\int d^2x\sum_I\Bigg[4\partial_z A_I^*\partial_{\bar z}A_I+\frac{\partial W(\{A\})^*}{\partial A_I^*}\frac{\partial W(\{A\})}{\partial A_I}$$
$$+(\bar\psi_{\dot 1},\psi_2)_I\sum_J\begin{pmatrix}2\delta_{IJ}\partial_z & \frac{\partial^2 W(\{A\})^*}{\partial A_I^*\partial A_J^*}\\ \frac{\partial^2 W(\{A\})}{\partial A_I\partial A_J} & 2\delta_{IJ}\partial_{\bar z}\end{pmatrix}\begin{pmatrix}\psi_1\\ \bar\psi_{\dot 2}\end{pmatrix}_J\Bigg]$$

$\xrightarrow{\text{IR limit}}$ $\mathcal{N}=2$ minimal model

(ADE classification [Vafa–Warner '89])

Algebra	Superpotential W	Central charge c
A_n	Φ_1^{n+1}, $n\geqq 1$	$3-6/(n+1)$
D_n	$\Phi_1^{n-1}+\Phi_1\Phi_2^2$, $n\geqq 3$	$3-6/2(n-1)$
E_7	$\Phi_1^3+\Phi_1\Phi_2^3$	$3-6/18$

($E_6\cong A_2\otimes A_3$, $E_8\cong A_2\otimes A_4$)

Preceding numerical studies

- Strong coupling at IR and IR divergences
  $\rightarrow$ No complete proof of the correspondence
- An alternative approach: Non-perturbative numerical study

- Preceding numerical studies of the A_2 model ($W = \Phi^3$)
  1. Scaling dimension $h + \bar{h}$ [Kawai–Kikukawa '09]
     based on a lattice formulation [Kikukawa–Nakayama '02]
     - ⋆ $1 - h - \bar{h} = 0.660(11)$ [expected value: $0.666\ldots$ in SCFT]
  2. $h + \bar{h}$ and central charge c [Kamata–Suzuki '10]
     based on a SUSY-preserving numerical algorithm
     [Kadoh–Suzuki '09]
     - ⋆ $1 - h - \bar{h} = 0.616(25)(13)$ [$0.666\ldots$]
     - ⋆ $c = 1.09(14)(31)$ [1]

  $\rightarrow$ Non-perturbative evidence of the A_2-type correspondence

Today's talk...

1. Applying the SUSY-preserving formulation to $A_{2,3}$, $D_{3,4}$, $E_{6,7}$ we directly measure c (and $h+\bar{h}$). [O.M.–Suzuki, O.M. '18]
   - For A_2, A_3, D_3, D_4, E_6 ($\cong A_2 \otimes A_3$), E_7 models, non-perturbative evidences of the conjectured correspondence
2. Precision measurement of $h+\bar{h}$ through the continuum limit and finite-size scaling [O.M. '19]
   - More reliable result

- Supporting the validity of the formulation
- We hope that this numerical approach will be useful to investigate superstring theory.

Supersymmetric formulation [Kadoh–Suzuki 2009]

- Continuum space $L \times L \longrightarrow$ we work in the momentum space:

$$\varphi(x) = \frac{1}{L^2} \sum_p e^{ipx} \varphi(p),$$

$$p_\mu = \frac{2\pi}{L} n_\mu. \qquad (n_\mu = 0, \pm 1, \pm 2, \dots)$$

- Let us introduce a UV cutoff Λ as

$$-\Lambda \le p_\mu \le \Lambda$$

  - $\Lambda = \pi/a$ ("lattice spacing" a; "continuum limit $a \to 0$")
- Manifestly preserving SUSY, translational inv., ...

  $\rightarrow$ Straightforward construction of supercurrent, EMT, ...
- Non-locality because of the cutoff
  - For massive 2D WZ, it is restored in continuum lim; massless??

WZ model and Nicolai map [Nicolai 1980]

- WZ action (complex scalar A, fermion ψ)

$$S = S_B + \frac{1}{L^2}\sum_p \left[(\bar{\psi}_{\dot{1}}, \psi_2)(-p) \begin{pmatrix} 2ip_z & W''(A)^{*}* \\ W''(A)* & 2ip_{\bar{z}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \bar{\psi}_{\dot{2}} \end{pmatrix} (p) \right],$$

where $p_z = \frac{1}{2}(p_0 - ip_1)$, $p_{\bar{z}} = \frac{1}{2}(p_0 + ip_1)$, $*$: convolution,

$$S_B = \frac{1}{L^2}\sum_p N(p)^* N(p), \quad N(p) \equiv 2ip_z A(p) + W'(A)^*(p). \quad (\star)$$

- Integrating over ψ, changing variables $A(p) \to N(p)$,

$$\mathcal{Z} = \int \prod_{|p_\mu| \leq \Lambda} [dN(p)dN^*(p)]\, e^{-S_B} \sum_i \text{sign det} \left. \frac{\partial(N, N^*)}{\partial(A, A^*)} \right|_{A=A_i}.$$

  - Jacobian cancellation $\Rightarrow$ weight is Gaussian (Nicolai map)
  - $A_i(p)$ $(i = 1, 2, \ldots)$: solutions of $(\star)$

Algorithm

1. Generate Gaussian random numbers $(N(p), N^*(p))$
2. Numerically solve the algebraic equation

$$2ip_z A(p) + W'(A)^*(p) - N(p) = 0,$$

   with respect to $A(p)$; find all solutions A_i $(i = 1, 2, \dots)$
3. Calculate sums

$$\sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)} \mathcal{O}(A, A^*)\Big|_{A=A_i}, \quad \sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)}\Big|_{A=A_i}$$

4. Repeat steps (1)–(3), and average

$$\langle \mathcal{O} \rangle = \frac{1}{\Delta} \left\langle \sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)} \mathcal{O}(A, A^*)\Big|_{A=A_i} \right\rangle$$

$$\Delta \equiv \left\langle \sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)}\Big|_{A=A_i} \right\rangle \qquad : \text{ Witten index}$$

IR behavior of correlation functions

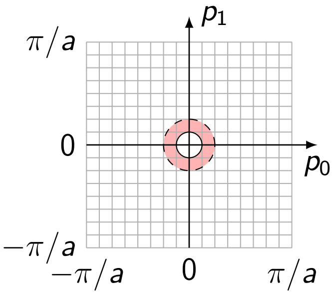
- Computation of scaling dimension and central charge [Kamata–Suzuki '10; O.M.-Suzuki, O.M. '18]
- 2-point function

$$\langle \varphi_1(p)\varphi_2(-p)\rangle = L^2 \int d^2x\, e^{-ipx} \underline{\langle \varphi_1(x)\varphi_2(0)\rangle}$$

- r.h.s $\to$ 2-point function in SCFT

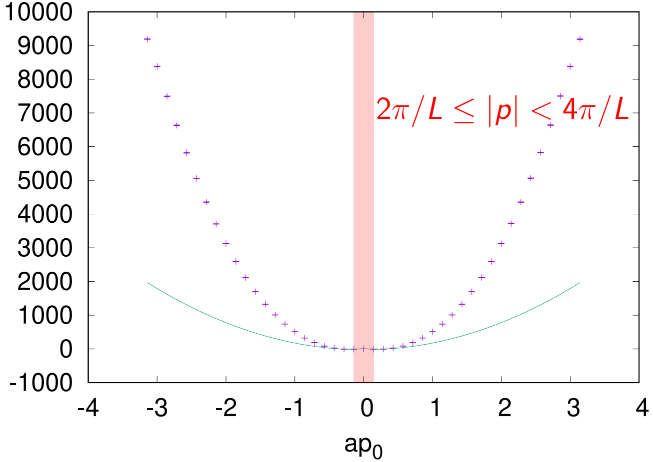
$$\langle A(x)A^*(0)\rangle \propto 1/z^{2h}\bar{z}^{2\bar{h}},$$
$$\langle T_{zz}(x)T_{zz}(0)\rangle = c/2z^4, \ldots$$

- Behavior of $\langle \varphi_1(p)\varphi_2(-p)\rangle$ at IR
  $\Longrightarrow$ Scaling dimension $h + \bar{h}$
  Central charge c



EMT correlator [O.M.–Suzuki, O.M. '18]

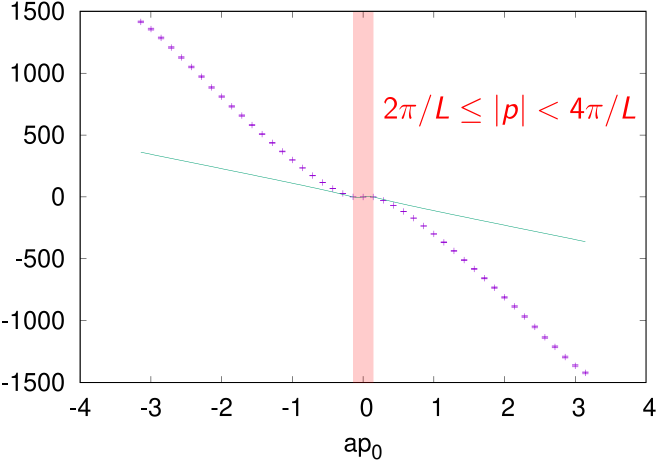
$$\mathrm{Re}\,\langle T_{zz}(p)\,T_{zz}(-p)\rangle = \mathrm{Re}[L^2\pi c\, p_z^3/12p_{\bar{z}}]$$



D_3: $W = \frac{\lambda_1}{2}\Phi_1^2 + \frac{\lambda_2}{2}\Phi_1\Phi_2^2$ with $a\lambda_1 = a\lambda_2 = 0.3$.
$L/a = 44$, $ap_1 = \pi/22$, 640 confs.

EMT correlator [O.M.–Suzuki, O.M. '18]

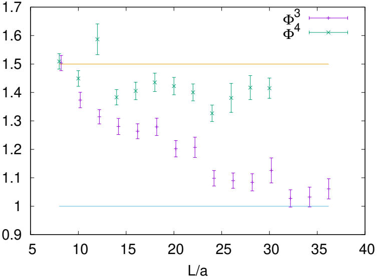
$$\mathrm{Im}\,\langle T_{zz}(p)\,T_{zz}(-p)\rangle = \mathrm{Im}[L^2\pi c p_z^3/12p_{\bar{z}}]$$



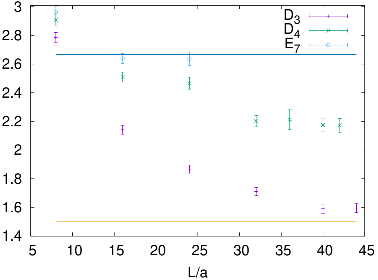
D_3: $W = \frac{\lambda_1}{2}\Phi_1^2 + \frac{\lambda_2}{2}\Phi_1\Phi_2^2$ with $a\lambda_1 = a\lambda_2 = 0.3$.
$L/a = 44$, $ap_1 = \pi/22$, 640 confs.

Central charge [O.M.–Suzuki, O.M. '18]

- Parameters
  - $a\lambda_{1,2} = 0.3$
    $A_n : \frac{\lambda_1}{n+1}\Phi_1^{n+1}$
    $D_n : \frac{\lambda_1}{n-1}\Phi_1^{n-1} + \frac{\lambda_2}{2}\Phi_1\Phi_2^2$
    $E_7 : \frac{\lambda_1}{3}\Phi_1^3 + \frac{\lambda_2}{3}\Phi_1\Phi_2^3$
  - L/a: various even integers
  - 640 confs of $N(p)$

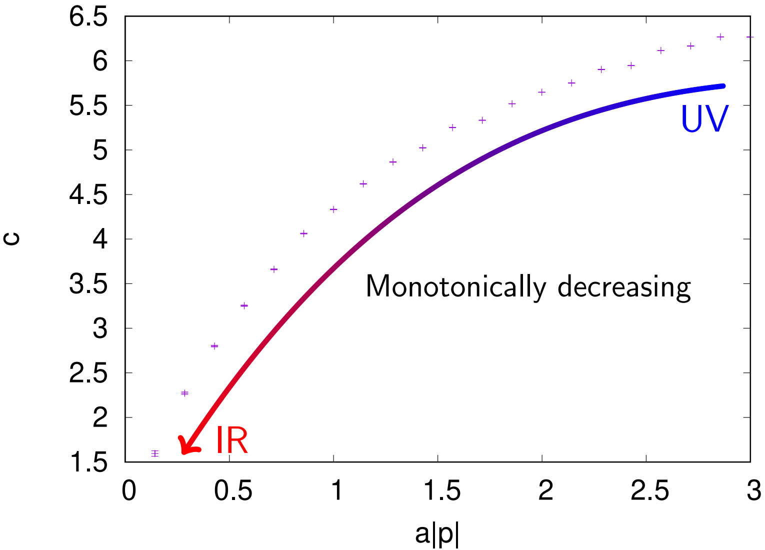


	L/a	Central charge	
A_2	36	1.061(36)(34)	1
A_3	30	1.415(36)(36)	1.5
D_3	44	1.595(31)(41)	1.5
D_4	42	2.172(48)(39)	2
E_7	24	2.638(47)(59)	2.66...



“Effective central charge” (D_3, $L/a = 44$)

(Analogous to Zamolodchikov C-function ['86])



Continuum limit and finite-size scaling

- We also obtained the scaling dimension

	L/a	$1-h-\bar{h}$	
A_2	36	0.682(10)(7)	0.666...
A_3	30	0.747(11)(12)	0.75

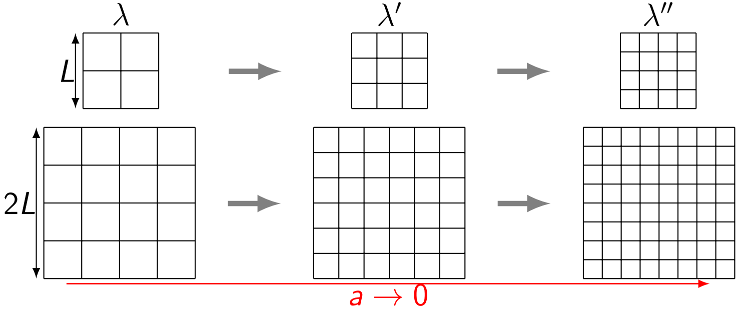
- In this analysis, L dependence is not quite clear $\overset{??}{\Rightarrow} L/a \to \infty$
- Susceptibility of the scalar field [Kawai–Kikukawa '09]

$$\chi(L) = \frac{1}{a^2} \int_{L^2} d^2x \, \langle A(x) A^*(0) \rangle \propto (L^2)^{1-h-\bar{h}}$$

- Measurement for various sizes of L
  $\to$ we can read $1-h-\bar{h}$ from L dependence (finite-size scaling)
- Sensitivity to UV details for finite L/a in Kamata–Suzuki
  - To obtain a precise and reliable result, take the continuum limit
- Let us try to extrapolate $\chi(L)$ to $a \to 0$

Continuum limit extrapolation [O.M. '19]

- Similar to the continuum limit of the step scaling function [Lüscher–Weisz–Wolff '91]

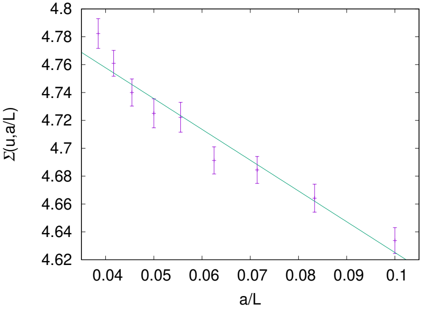


- $^{\forall}a$, λ is tuned such that $\chi(L) = u$ is fixed
  $\to \Sigma(u, a/L) = \ln \chi(2L)$ for (a, λ)
- From def. of the susceptibility, we have

$$1 - h - \bar{h} = \frac{1}{\ln 4}\left[\lim_{a \to 0} \Sigma(u, a/L) - u\right]$$

$\Sigma(u, a/L)$ with $u = 3.9175$ [O.M. '19]

- Simplest case: A_2 model ($W(A) = \frac{\lambda}{3} A^3$)
  - 2k to 8k confs of $N(p)$ with various λ (largest $2L/a = 52$)



- Scaling dimension from a linear fit

	$1 - h - \bar{h}$	
A_2	0.6699(77)(87)	0.6666...

- Kawai-Kikukawa 0.660(11)
- Kamata-Suzuki 0.616(25)(13)
- O.M.-Suzuki 0.682(10)(7)

Summary

- Numerical study of the IR behavior of the 2D $\mathcal{N} = 2$ WZ model

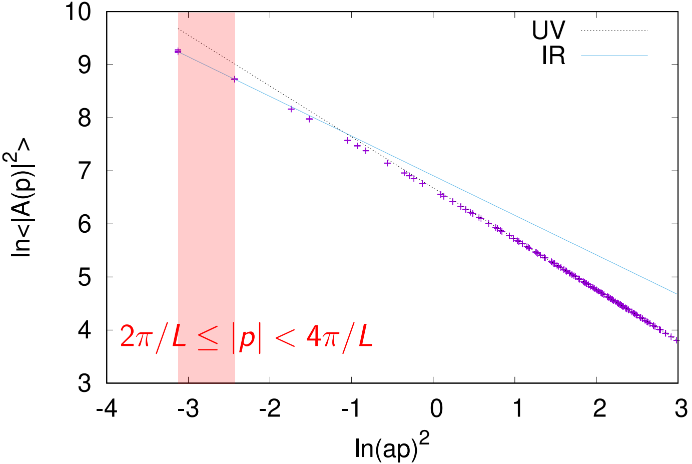
  $\xrightarrow{\text{IR}}$ Non-perturbative evidences of the WZ/SCFT correspondence
  - Central charge (& Scaling dimension) [O.M.–Suzuki, O.M. '18]
    $\Longrightarrow$ (First) Direct computation for typical ADE models
  - Finite-size effect & Continuum limit [O.M. '19]
    $\Longrightarrow$ Precise and reliable result of the scaling dimension
- Validity of the formulation (implication of locality restoration)

- E_8 ($\cong A_2 \otimes A_4$), A_4: $W = \Phi^5$?
- *Application to superstring theory*

  (Collaboration with Hiroshi Suzuki and Hisao Suzuki)
  - Not ADE-type (Gepner) models; Deformation of target space
  - Dynamics of Calabi–Yau?

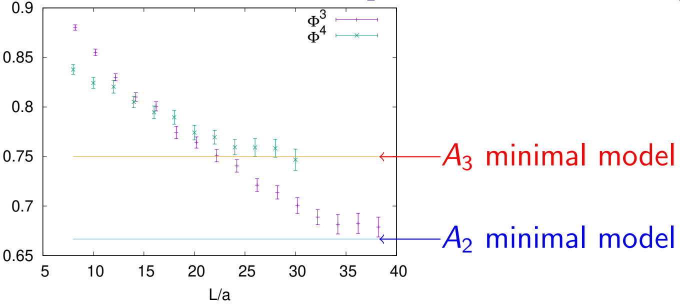
Backup: Scaling dimension from a scalar correlator

$$\langle A(p)A^*(-p)\rangle \sim 1/(p^2)^{1-h-\bar{h}}$$



A_3: $W = \frac{\lambda}{4}\Phi^4$ with $a\lambda = 0.3$.
$L/a = 30$. 640 confs.

Backup: Scaling dimension [O.M.–Suzuki '18]



- Scaling dimension (largest L)

	L/a	$1 - h - \bar{h}$	
A_2	36	0.682(10)(7)	0.666...
A_3	30	0.747(11)(12)	0.75

  - Kawai–Kikukawa (A_2): 0.660(11)
  - Kamata–Suzuki (A_2): 0.616(25)(13)

Backup: Energy-momentum tensor

- Translational inv. $\to$ Construction of EMT is straightforward
- EMT:

$$T_{zz}(x) = -4\pi\partial_z A^*(x)\partial_z A(x) - \pi\psi_2(x)\partial_z\bar{\psi}_{\dot{2}}(x) + \pi\partial_z\psi_2(x)\bar{\psi}_{\dot{2}}(x)$$

with requirements: $T_{\mu\nu} = T_{\nu\mu}$, $\sum_\mu T_{\mu\mu} \to 0$ in the UV limit (free SCFT)

- 2-point fn. of EMT

$$\langle T_{zz}(x)T_{zz}(0)\rangle = \frac{c}{2z^4} \quad \Rightarrow \quad \langle T_{zz}(p)T_{zz}(-p)\rangle = L^2\frac{\pi c}{12}\frac{p_z^3}{p_{\bar{z}}}$$

- $\to$ Linear combination of supercurrent correlators

$$\langle T_{zz}(p)T_{zz}(-p)\rangle = -\frac{2ip_z}{16}\langle S_z^+(p)S_z^-(-p) + S_z^-(p)S_z^+(-p)\rangle$$

$$S_z^+(p) \equiv 4\pi\partial_z A(x)\bar{\psi}_{\dot{2}}(x),$$
$$S_z^-(p) \equiv -4\pi\partial_z A^*(x)\psi_2(x)$$