Infrared renormalon in the supersymmetric $\mathbb{C}P^{N-1}$ model on $\mathbb{R}\times S^1$

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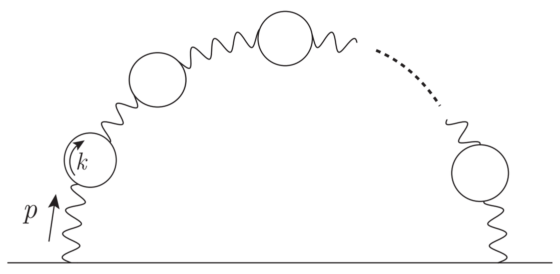
1. Introduction

- Perturbative series in quantum field theory are typically divergent.
  1. The number of the Feynman diagrams grows factorially $k!$.
  2. Amplitude of a single Feynman diagram grows factorially $\beta_0^k k!$. This is known as the renormalon ['t Hooft 1979].
- It is claimed that such ambiguities disappear thanks to semi-classical objects.
  1. $\leftarrow$ Instanton [Bogomolny 1980, Zinn-Justin 1981]
  2. $\leftarrow$ Bion [Argyres–Ünsal 2012, ...]?



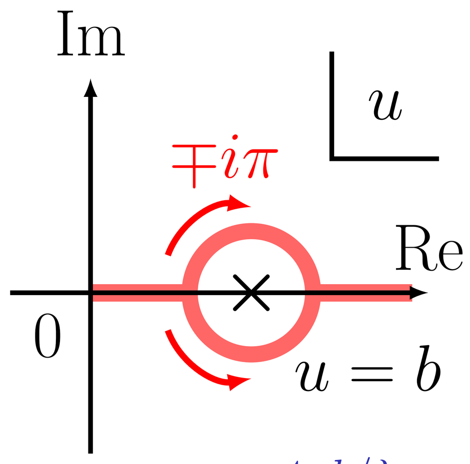
(a pair of fractional instanton and fractional anti-instanton)

- Semi-classical argument on $\mathbb{R}^{d-1}\times S^1$ [Radius $R \ll (\Lambda\text{ scale})^{-1}$]
- It is important to clarify the renormalon structure on $\mathbb{R}^{d-1}\times S^1$.
- We study the 2D $\mathcal{N}=(2,2)$ $\mathbb{C}P^{N-1}$ model in the large N limit.

2. Borel prescription

- For the perturbative series of a quantity $f(\lambda)$,
$$f(\lambda)\sim\sum_{k=0}^{\infty}f_k\,(\lambda/4\pi)^{k+1},$$
we define the Borel transform by
$$B(u)\equiv\sum_{k=0}^{\infty}(f_k/k!)\,u^k.$$
- The Borel sum is given by
$$f(\lambda)\equiv\int_0^\infty du\,B(u)e^{-4\pi u/\lambda}.$$

- If f_k grows factorially $f_k\sim b^{-k}k!$ as $k\to\infty$,
$$B(u)=1/(1-u/b).$$
This possesses a pole singularity at $u=b$.
- If $b>0$, the Borel sum becomes ill-defined; one should avoid the pole by contour deformation.



- This induces the imaginary ambiguity proportional to $\sim e^{-4\pi b/\lambda}$.
- *Bion* action $=\frac{4\pi}{\lambda}$ in the present system $\to u=$ positive integers.

3. 2D $\mathcal{N}=(2,2)$ $\mathbb{C}P^{N-1}$ model

- The system in terms of the homogeneous coordinates of $\mathbb{C}P^{N-1}$:
$$S=\frac{N}{\lambda}\int d^2x\Big[\partial_\mu\bar z^A\partial_\mu z^A+\bar\chi^A\gamma_\mu(\partial_\mu-ij_\mu)\chi^A-j_\mu j_\mu-\left[(\bar\chi^A\chi^A)^2-(\bar\chi^A\gamma_5\chi^A)^2-(\bar\chi^A\gamma_\mu\chi^A)^2\right]/4\Big],$$
where $j_\mu=(1/2i)(\bar z^A\partial_\mu z^A-z^A\partial_\mu\bar z^A)$, $\gamma_5=-i\gamma_x\gamma_y$, A, B etc. run over $1,\dots,N$, and we impose $\bar z^A z^A=1$ and $\bar z^A\chi_\pm^A=0$.
  - ▸ (In)Homogeneous coordinates are given by [Cremmer–Scherk 1978]
$$\varphi^a\equiv z^a/z_N,\quad \psi_\pm^a\equiv(1/z^N)\chi_\pm^a-[z^a/(z^N)^2]\chi_\pm^N\quad(a=1,\dots,N-1).$$
The above action can be obtained by dimensional reduction of 4D $\mathcal{N}=1$ Wess–Zumino model with the Kähler potential $K=(N/\lambda)\ln(1+\bar\Phi^a\Phi^a)$.
- $\mathbb{Z}_N$ invariant twisted boundary condition ($m_a=a/NR$, $m_N=0$):
$$z^A(x,y+2\pi R)=e^{2\pi i m_A R}z^A(x,y),$$
$$\chi_\pm^A(x,y+2\pi R)=e^{2\pi i m_A R}\chi_\pm^A(x,y).$$

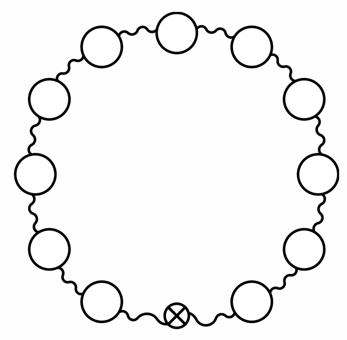
- We introduce auxiliary fields as [Lindström–Roček 1983],
$$S'=\frac{N}{\lambda}\int d^2x\Big[-f+\bar\sigma\sigma+\bar z^A(-D_\mu D_\mu+f)z^A+\bar\chi^A(\not{D}+\bar\sigma P_++\sigma P_-)\chi^A+2\bar\eta\bar z^A\chi^A+2\bar\chi^A z^A\eta\Big]$$
where $D_\mu=\partial_\mu+iA_\mu$, and $P_\pm=(1\pm\gamma_5)/2$.
  - ▸ $\sim$ 4D theory with $K=\bar Z^A Z^A e^{2V}-2V$ [$U(1)$ gauge field $V\ni(A_\mu,\sigma,\eta,f)$].
- We impose the periodic boundary conditions for all the auxiliary fields, and the Fourier transformation is defined by
$$\phi(x)=\int\frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y=n/R,n\in\mathbb{Z}}e^{ipx}\phi(p).$$

4. Effective action

- Fluctuations of the auxiliary fields around the large N saddle point:
$$A_\mu\equiv A_{\mu 0}+\delta A_\mu,\quad f\equiv f_0+\delta f,\quad \sigma\equiv\sigma_0+\delta\sigma,\quad \eta_0=0,$$
where $f_0=\bar\sigma_0\sigma_0=\mu^2e^{-4\pi/\lambda_R(\mu)}=\Lambda^2$ as $N\to\infty$.
  - ▸ We apply dimensional regularization as $2\to D=2-2\epsilon$, and introduce
$$\lambda=\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon}\lambda_R(\mu)\left[1+\frac{\lambda_R(\mu)}{4\pi}\frac{1}{\epsilon}\right]^{-1}\quad\to\quad\mu^2\frac{d}{d\mu^2}\lambda_R(\mu)=-\frac{\beta_0}{4\pi}\lambda_R^2(\mu)$$
- The effective action to the quadratic order in the fluctuations:
$$S_{\text{eff}}|_{\text{quadratic}}=\frac{N}{4\pi}\int\frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y}$$
$$\times\Big(\tfrac{1}{2}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\mathcal{L}_\infty(p)\delta A_\mu(p)\delta A_\nu(-p)+\tfrac{1}{2\Lambda^2}\mathcal{L}_\infty(p)\left[(p^2+4\Lambda^2)\delta R(p)\delta R(-p)+p^2\delta I(p)\delta I(-p)\right]-\tfrac{1}{2}\mathcal{L}_\infty(p)\delta f(p)\delta f(-p)-2\bar\eta(i\not{p}+2\sigma_0P_++2\bar\sigma_0P_-)\mathcal{L}_\infty(p)\eta(-p)+\epsilon_{\mu\nu}p_\mu\mathcal{L}_\infty(p)\left[\delta A_\nu(p)\delta I(-p)-\delta I(p)\delta A_\nu(-p)\right]+\mathcal{H}\Big),$$
- Here $\delta R\equiv(1/2)[\bar\sigma_0\delta\sigma+\sigma_0\delta\bar\sigma]$, $\delta I\equiv(1/2i)[\bar\sigma_0\delta\sigma-\sigma_0\delta\bar\sigma]$, $|\mathcal{H}|\lesssim e^{-\pi\Lambda RN}$, and
$$\mathcal{L}_\infty(p)\equiv\frac{2}{\sqrt{p^2(p^2+4\Lambda^2)}}\ln\left(\frac{\sqrt{p^2+4\Lambda^2}+\sqrt{p^2}}{\sqrt{p^2+4\Lambda^2}-\sqrt{p^2}}\right)=\frac{2}{p^2}\ln(p^2/\Lambda^2)+\mathcal{O}(\Lambda^2).$$
As $NR\Lambda\to\infty$, $S_{\text{eff}}|_{\mathbb{R}\times S^1}\to S_{\text{eff}}|_{\mathbb{R}^2}$ in [D'Adda–Di Vecchia–Lüscher 1979].
- A_μ propagator in the $NR\Lambda\to\infty$ limit is given by
$$\langle\delta A_\mu(p)\delta A_\nu(q)\rangle=\frac{4\pi}{N}\frac{\delta_{\mu\nu}+4\Lambda^2p_\mu p_\nu/(p^2)^2}{(p^2+4\Lambda^2)\mathcal{L}_\infty(p)}2\pi\delta(p_x+q_x)2\pi R\delta_{p_y+q_y,0}$$

5. IR renormalon in gluon condensate

- Gluon condensate in the large N limit
$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle=\frac{4\pi}{N}\int\frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y}\frac{2p^2}{(p^2+4\Lambda^2)\mathcal{L}_\infty(p)}$$



- Positive powers of $\Lambda^2\sim e^{-4\pi/\lambda_R}$ are regarded as the non-perturbative part; $\langle FF\rangle$ in perturbation theory is given by
$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle_{\text{PT}}=\frac{4\pi}{N}\int\frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y}\frac{p^2}{\ln(p^2/\Lambda^2)}\bigg|_{\text{expansion in }\lambda_R}$$
$$=\frac{4\pi}{N}\sum_{k=0}^{\infty}\int\frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y}p^2\left[-\ln(p^2/\mu^2)\right]^k\left[\frac{\lambda_R(\mu)}{4\pi}\right]^{k+1}.$$
Note that $\ln(p^2/\Lambda^2)=\ln(p^2/\mu^2)+4\pi/\lambda_R(\mu)$.
- Only the $p_y=0$ term can be singular. Focusing on the IR region by introducing a UV cutoff q ($p^2\le q^2$),
$$B_{p_y=0}(u)=\frac{4\pi}{N}\frac{1}{2\pi R}\int_{|p_x|\le q}\frac{dp_x}{2\pi}p_x^2\left(\frac{\mu^2}{p_x^2}\right)^u=-\frac{\mu^{2u}}{\pi RN}\frac{q^{3-2u}}{u-3/2}$$
- The Borel singularity at $u=3/2$ gives rise to the renormalon:
$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle_{\text{renormalon}}=\pm\pi i\frac{\Lambda^3}{\pi RN}.$$
Peculiar to the compactified space $\mathbb{R}\times S^1$!
  - ▸ On the other hand, on $\mathbb{R}^2$ [$(1/R)\sum_{p_y}\to\int dp_y$], we have
$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle_{\text{renormalon on }\mathbb{R}^2\text{ (at }u=2)}=\pm\pi i\Lambda^4/N.$$

6. Conclusion

- We find an unfamiliar renormalon singularity at $u=3/2$.
- Despite large N volume independence of the integrand, that is shifted by effective reduction as $\int dp^d\to\int dp^{d-1}$ [1909.09579].
  - ▸ cf. $SU(N)$ QCD(adj.) [Anber–Sulejmanpasic 2014; 1909.05489]
- No obvious semi-classical interpretation so far...
  - ▸ Bion calculations with $NR\Lambda\ll 1$ [Fujimori *et al.*]