Vacuum energy of the SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$

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2020/1/23 $\mathbb{C}P^N$ model: recent developments and future directions @Keio U.

- K. Ishikawa, O.M., K. Shibata, and H. Suzuki, arXiv:2001.07302 [hep-th].

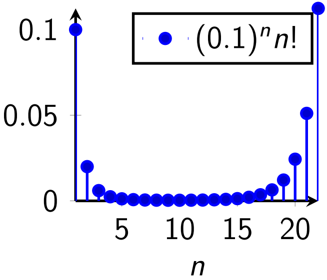
Contents

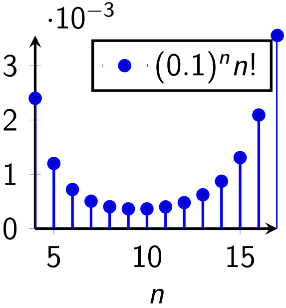
Contents

Factorial growth of perturbation series

- Perturbation theory (PT) of QM/QFT is a quite successful tool.
- But, perturbative series are typically divergent as

$$F(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n, \qquad c_n \sim n! \text{ at large } n.$$





- Accuracy of perturbative predictions is limited. . .

Factorial growth of perturbation series

- E.g., ground state energy in QM (Rayleigh–Schrödinger PT):

	Perturbative coefficients
Zeeman effect	$\sim (-1)^n (2n)!$
Stark effect	$\sim (2n)!$
Anharmonic oscillator	
$V(\phi) \sim \phi^3$	$\sim \Gamma(n + 1/2)$
$V(\phi) \sim \phi^4$	$\sim (-1)^n \Gamma(n + 1/2)$
Double well	$\sim n!$
periodic cosine well	$\sim n!$
$\vdots$	$\vdots$

- These are due to the factorial growth of $\#$ of Feynman diagrams.

Borel resummation

- The Borel (re)summation is useful for summing divergent asymptotic series.
- For the perturbative series of a quantity $f(\lambda)$,

$$f(\lambda) \sim \sum_{n=0}^{\infty} f_n \left(\frac{\lambda}{4\pi}\right)^{n+1},$$

we define the Borel transform by

$$B(u) \equiv \sum_{n=0}^{\infty} \frac{f_n}{n!} u^n.$$

- The Borel sum is given by

$$f(\lambda) \equiv \int_0^{\infty} du\, B(u) e^{-4\pi u/\lambda}.$$

Borel resummation

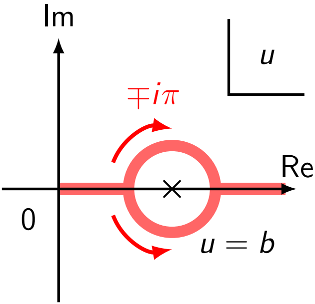
- If $f_n \sim b^{-n} n!$ as $n \to \infty$,

$$B(u) = \frac{1}{1 - u/b}.$$

$\to$ Pole singularity at $u = b$ (Finite radius of convergence!)

- The integral is convergent for $b < 0$ (alternating series).

- If $b > 0$, the Borel sum becomes ill-defined. $\Rightarrow$ non-Borel summable
- Then, one should avoid the pole by contour deformation.
- This induces the imaginary ambiguity proportional to $\sim e^{-4\pi b/\lambda}$.



Resurgence theory and semi-classical picture

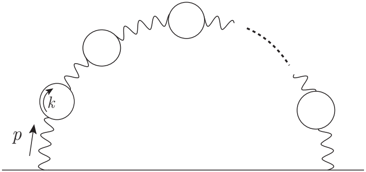
- $f(\lambda)$ is not analytic at $\lambda = 0$, but an asymptotic series.
- Asymptotic nature of perturbative series is related to
  1. possible instability of quantum theories,
     [Dyson '52, Hurst '52, Thirring '53, ...]
  2. nonperturbative effects such as quantum tunneling.
     [Vainshtein '64, Bender–Wu '73, Lipatov '77, ...]
- Ambiguity $\propto$ a nonperturbative factor $e^{-\text{const.}/\lambda}$

  $\updownarrow$ Cancellation (Resurgence structure)

  Ambiguity associated with nonperturbative effects
  - Instanton calculus [Bogomolny 1980, Zinn-Justin 1981]
- Reading out nonperturbative effects from PT: Resurgence theory

Renormalon and bion

- Another source of n-factorial: renormalon ['t Hooft 1979]
- Amplitude of a single Feynman diagram $\sim \beta_0^n n!$.
  (β_0: one-loop coefficient of the beta function)



- It is conjectured that renormalon ambiguities disappear thanks to the so-called bion [Argyres–Ünsal '12, Dunne–Ünsal '12, ...].
- Bion: a pair of $\underbrace{\text{fractional instanton/anti-instanton}}_{\text{on } \mathbb{R}^{d-1} \times S^1 \text{ with twisted boundary conditions (BC)}}$

Studies on $\mathbb{C}P^{N-1}$ model

- Bion calculus in 2D SUSY $\mathbb{C}P^{N-1}$ model with SUSY breaking term [Fujimori–Kamata–Misumi–Nitta–Sakai '18]
- Vacuum energy:
  ambiguity from bion $\Leftrightarrow$ renormalon on $\mathbb{R}^2$ at $u = 1$
  - $\mathbb{C}P^{N-1}$ QM [Fujimori–Kamata–Misumi–Nitta–Sakai '16, '17]
- Renormalon in 2D (SUSY) $\mathbb{C}P^{N-1}$ model in the large N limit [Ishikawa–O.M.–Nakayama–Shibata–Suzuki–Takaura '19]
- $\langle FF \rangle$: Borel singularity at $u = 2$ ($\mathbb{R}^2$) $\to$ $u = 3/2$ ($\mathbb{R} \times S^1$)
  - cf. 4D $SU(N)$ QCD(adj.) [Anber–Sulejmanpasic '14; Ashie–O.M.–Suzuki–Takaura–Takeuchi '19]
- No obvious semi-classical interpretation so far...

- Renormalon ambiguity for vacuum energy in the large N limit?
- Our answer:
  purely nonperturbative, no well-defined weak coupling expansion

Contents

2D SUSY $\mathbb{C}P^{N-1}$ model

- 2D $\mathcal{N} = (2,2)$ $\mathbb{C}P^{N-1}$ model in terms of homogeneous coordinates $(z^A,\ \chi^A)$ with $A = 1,\ 2,\ \ldots,\ N$

$$S = \frac{N}{\lambda}\int d^2x \left[\bar{z}^A\left(-D_\mu D_\mu + \bar{\sigma}\sigma\right)z^A + \bar{\chi}^A(\not{D} + \bar{\sigma}P_+ + \sigma P_-)\chi^A\right]$$

  where λ is the 't Hooft coupling, $D_\mu = \partial_\mu + iA_\mu$, $\gamma_5 = -i\gamma_x\gamma_y$, $P_\pm = (1 \pm \gamma_5)/2$, and we impose

$$\bar{z}^A z^A = 1 \qquad \text{and} \qquad \bar{z}^A\chi^A = 0.$$

- Lagrange multiplier fields f and $(\eta, \bar{\eta})$;

$$S' = S + \frac{N}{\lambda}\int d^2x \left[f(\bar{z}^A z^A - 1) + 2\bar{\eta}\bar{z}^A\chi^A + 2\bar{\chi}^A z^A\eta\right].$$

  - Auxiliary fields $(A_\mu, \sigma, \eta, f) \in U(1)$ supermultiplet
- Large N solution in $\mathbb{R}^2$ by [D'Adda–Di Vecchia–Lüscher '79]
  - Large N saddle point (subscript 0): $f_0 = 0$, $\bar{\sigma}_0\sigma_0 = \Lambda^2$, $\eta_0 = 0$.
  - $A_{\mu 0}$ undetermined $\rightarrow$ "vacuum moduli"

$\mathbb{Z}_N$ invariant twisted BC

- Spacetime is $\mathbb{R} \times S^1$; $x \in \mathbb{R}$ and $y \in S^1$ with S^1 radius R.
- Kaluza–Klein momentum along S^1 is given by $p_y = n/R$.
- We assume the following BC

$$
\begin{aligned}
z^A(x, y + 2\pi R) &= e^{2\pi i m_A R} z^A(x, y), \\
\chi^A(x, y + 2\pi R) &= e^{2\pi i m_A R} \chi^A(x, y),
\end{aligned}
$$

where

$$
m_A = \begin{cases} A/NR & \text{for } A = 1,\, 2,\, \ldots,\, N-1 \\ 0 & \text{for } A = N. \end{cases}
$$

- We impose periodic BC for all auxiliary fields A_μ, f, σ, η.

Twisted BC, Large N and volume (in)dependence

- When a loop momentum is associated with the twisted BC,

$$\sum_A e^{2\pi i m_A R n} \times \cdots \qquad (n \in \mathbb{Z})$$

- If $\cdots = 1$, noting that

$$\sum_A e^{2\pi i m_A R n} \times 1 = \begin{cases} N, & \text{for } n = 0 \bmod N, \\ 0, & \text{for } n \neq 0 \bmod N, \end{cases}$$

  the effective radius becomes NR.
- Assume "large N limit" as $NR\Lambda \to \infty$ (Decompactification!)
- Large N volume independence
  - $S_{\text{eff}}|_{\mathbb{R}\times S^1} \to S_{\text{eff}}|_{\mathbb{R}^2}$ [D'Adda–Di Vecchia–Lüscher] (twisted for z)
  - $\langle FF \rangle$ in large-N $SU(N)$ QCD(adj.): $u = 2 \to 2$ (twisted for A_μ)
- Volume *dependent* cases
  - $\langle FF \rangle$ in large-N $\mathbb{C}P^{N-1}$: $u = 2 \to 3/2$ (periodic for A_μ)

Contents

SUSY breaking term

- Vacuum energy in the large N $\Leftrightarrow$ that by bion calculus
- Let's introduce the SUSY breaking term as [Fujimori *et al.*]

$$\delta S = \int d^2x \, \frac{\delta\epsilon}{\pi R} \sum_{A=1}^{N} m_A \left(\bar{z}^A z^A - \frac{1}{N} \right)$$

- We have the vacuum energy as a function of $\delta\epsilon$, and

$$E(\delta\epsilon) = \underbrace{E^{(0)}}_{=0} + E^{(1)}\delta\epsilon + E^{(2)}\delta\epsilon^2 + \dots$$

- Bion contribution [Fujimori *et al.*]

$$E_{\text{bion}} = 2\Lambda \sum_{b=1}^{N-1} (-1)^b \frac{b}{(b!)^2} (NR\Lambda)^{2b-1}$$
$$\times \left\{ \delta\epsilon + \left[-2\gamma_E - 2\ln\left(4\pi b/\lambda_R\right) \mp \pi i \right] \delta\epsilon^2 + \dots \right\}$$

  - $b = 1$ term $\propto \Lambda^2 \leftarrow$ Expected order of the renormalon on $\mathbb{R}^2$

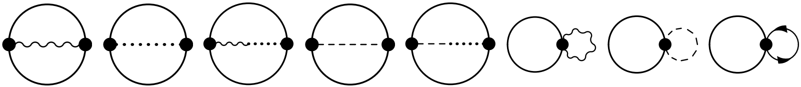
Vacuum bubble diagrams

- We need to compute the vacuum energy to NLO of $1/N$:

$$E^{(1)} = 2\sum_A m_A \left\langle \bar{z}^A z^A - 1/N \right\rangle_{\delta\epsilon=0}$$

$$E^{(2)} = -\frac{1}{\pi R}\int d^2x \sum_{A,B} m_A m_B \left\langle \bar{z}^A z^A(x) \bar{z}^B(0) z^B(0) \right\rangle_{\delta\epsilon=0}$$

  - 2-loop vacuum bubble diagrams (bubble chain!)



- Now, in $E^{(1)}$ and $E^{(2)}$, we have

$$\sum_A \left\{ m_A \quad \text{or} \quad m_A^2 \right\} e^{2\pi i m_A R n} \neq 0 \qquad \text{for } n \neq 0 \bmod N.$$

We cannot expect large N volume independence of those.

Renormalizability

- For example,

$$
\begin{aligned}
&E^{(1)}\Big|_{\text{1-loop}} \\
&= \frac{\lambda}{4\pi R}\left(1-\frac{1}{N}\right)\sum_{m=1}^{\infty} 4\cos(A_{y0}2\pi RNm)K_0(2\pi RNm\Lambda) \\
&\quad + \frac{\lambda}{4\pi R}\frac{1}{N}\sum_{n\neq 0 \bmod N} 4\frac{e^{-iA_{y0}2\pi Rn}}{e^{-2\pi ni/N}-1}K_0(2\pi R|n|\Lambda) \xrightarrow{\int_0^1 (A_{y0}RN)} 0
\end{aligned}
$$

- We find that, to the 2-loop order,

$$E^{(1)}\delta\epsilon = \lambda\delta\epsilon \times (\text{finite}), \qquad E^{(2)}\delta\epsilon^2 = (\lambda\delta\epsilon)^2 \times (\text{finite}).$$

- Renormalized parameters as ($D = 2-2\epsilon$)

$$\lambda\delta\epsilon = \lambda_R\delta\epsilon_R \qquad \text{where } \lambda = \left(\frac{e^{\gamma_E}\mu}{4\pi}\right)^{\epsilon}\lambda_R\left(1+\frac{\lambda_R}{4\pi}\frac{1}{\epsilon}\right)^{-1}.$$

Then, the vacuum energy to the 2-loop order is UV finite.

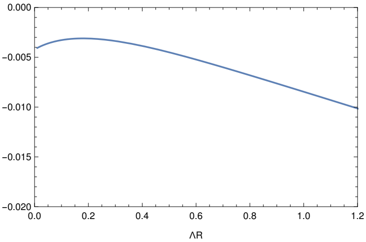
Results of $E^{(1)}$ and $E^{(2)}$

- Finally, we obtain

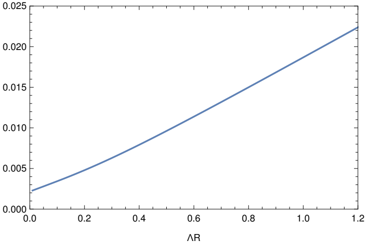
$$E^{(1)}\delta\epsilon = 0 \cdot N^0 + 0 \cdot N^{-1} + \mathcal{O}(N^{-2})$$

and

$$RE^{(2)}\delta\epsilon^2\Big|_{\mathcal{O}(N^{-1})} = N^{-1}(\lambda_R\delta\epsilon_R)^2(\Lambda R)^{-3}F(\Lambda R)$$
$$RE^{(2)}\delta\epsilon^2\Big|_{\mathcal{O}(N^{-2})} = N^{-2}(\lambda_R\delta\epsilon_R)^2(\Lambda R)^{-3}G(\Lambda R)$$



$F(\Lambda R)$



$G(\Lambda R)$

Discussion

- $E(\delta\epsilon)$ is a well-defined quantity to 2-loop order.
  - This still depends on the nonperturbative factor $\Lambda \sim e^{-2\pi/\lambda}$
- How to observe the renormalon?
  1. Extract the perturbative part of $E(\delta\epsilon)$ (weak coupling limit $R\Lambda \to 0$),
  2. Expand $E(\delta\epsilon)|_{\mathrm{PT}}$ in $\lambda_R(\mu)$,
  3. By the Borel prescription, obtain the renormalon ambiguity.
- Since F and G is finite at $R\Lambda = 0$,

$$RE^{(2)}(\delta\epsilon) \sim (R\Lambda)^{-3} \to \infty$$

  for $R\Lambda$ small!
- $E(\delta\epsilon)$ is purely nonperturbative, and has no well-defined weak coupling expansion.
- *We cannot discuss the renormalon problem.*

Contents

Summary

- Resurgence program in 2D SUSY $\mathbb{C}P^{N-1}$ model
- We computed $E(\delta\epsilon)$ to NLO in $1/N$, and found $E(\delta\epsilon)$ has no well-defined weak coupling expansion.
- On the other hand,

$$E_{\text{bion}} \sim (\pm\pi i) 2N \left(R\Lambda\right)^2 \delta\epsilon^2$$

  - In our result with $NR\Lambda \gg 1$, $E^{(2)}\delta\epsilon^2 = \mathcal{O}(N^{-1})$!?
  - $NR\Lambda \ll 1$ [Fujimori *el al.*]
- What cancels the ambiguity from the bion?

Backup: Effective action in $N \to \infty$

- We consider $NR\Lambda \to \infty$ where Λ is a dynamical scale.
- Effective action S_{eff} for fluctuations of the auxiliary fields

$$A_\mu \equiv A_{\mu 0} + \delta A_\mu, \qquad f \equiv f_0 + \delta f, \qquad \sigma \equiv \sigma_0 + \delta\sigma,$$

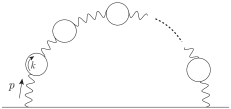
  around the large N saddle point $f_0 = 0$, $\bar{\sigma}_0 \sigma_0 = \Lambda^2$, $\eta_0 = 0$.
- $S_{\mathrm{eff}}|_{\mathbb{R}\times S^1} \overset{N\to\infty}{\to} S_{\mathrm{eff}}|_{\mathbb{R}^2}$ in [D'Adda–Di Vecchia–Lüscher '79].
- E.g., A_μ propagator is given by

$$\langle \delta A_\mu(p) \delta A_\nu(q) \rangle = \frac{4\pi}{N} \frac{\delta_{\mu\nu} + 4\Lambda^2 p_\mu p_\nu/(p^2)^2}{(p^2 + 4\Lambda^2)\mathcal{L}_\infty(p)} 2\pi\delta(p_x + q_x) 2\pi R \delta_{p_y+q_y,0}$$

  where

$$\mathcal{L}_\infty(p) \equiv \frac{2}{\sqrt{p^2(p^2+4\Lambda^2)}} \ln\left(\frac{\sqrt{p^2+4\Lambda^2}+\sqrt{p^2}}{\sqrt{p^2+4\Lambda^2}-\sqrt{p^2}}\right).$$

- Such propagators give rise to bubble chain diagrams giving renormalon



Backup: Computation of $E^{(1)}\delta\epsilon\big|_{\text{1-loop}}$

$$
\begin{aligned}
&E^{(1)}\delta\epsilon\Big|_{\text{1-loop}} \\
&= 2\delta\epsilon \sum_A m_A \left(\langle \bar{z}^A z^A \rangle_{\text{1-loop}} - 1/N \right) \\
&= 2\frac{\delta\epsilon}{N} \sum_A m_A \left[\lambda \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{1}{(p_\mu + A_{\mu 0} + m_A \delta_{\mu y})^2 + \Lambda^2} - 1 \right] \\
&= 2\frac{\delta\epsilon}{N} \sum_A m_A \left[\lambda \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip_y 2\pi R n}}{(p_\mu + A_{\mu 0} + m_A \delta_{\mu y})^2 + \Lambda^2} - 1 \right] \\
&= 2\frac{\delta\epsilon}{N} \sum_A m_A \left[\lambda \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{e^{i(p_y - A_{y0} - m_A) 2\pi R n}}{p^2 + \Lambda^2} - 1 \right] \\
&= \frac{\lambda_R \delta\epsilon_R}{4\pi R} \left(1 - \frac{1}{N} \right) \sum_{m=1}^{\infty} 4\cos(A_{y0} 2\pi R N m) K_0(2\pi R N m \Lambda) \\
&\quad + \frac{\lambda_R \delta\epsilon_R}{4\pi R} \frac{1}{N} \sum_{n \neq 0 \bmod N} 4 \frac{e^{-iA_{y0} 2\pi R n}}{e^{-2\pi n i/N} - 1} K_0(2\pi R |n| \Lambda) \xrightarrow{\int_0^1 (A_{y0} R N)(\text{vacuum moduli})} 0
\end{aligned}
$$

Backup: $F(\Lambda R)$ and $G(\Lambda R)$

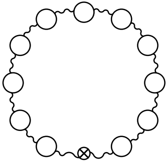
$$F(\xi) = -\frac{1}{12\pi^2}\left[\xi + c(\xi)\right], \quad c(\xi) = \lim_{N\to\infty}\frac{6}{N}\sum_{n>0,n\neq 0 \text{ mod } N}\frac{\xi^2 K_1(2\pi\xi n)}{\tan(\pi n/N)},$$

$$G(\xi) = -\frac{1}{12\pi^2}\left\{-\frac{3}{2}\xi + \lim_{N\to\infty}\left[6\sum_{n>0,n\neq 0 \text{ mod } N}\frac{\xi^2 K_1(2\pi\xi n)}{\tan(\pi n/N)} - Nc(\xi)\right]\right\}$$
$$-\frac{1}{6\pi^3}\xi^3\int_{-\infty}^{\infty} d\ell_x \sum_{\ell_y\in\mathbb{Z}}\Bigg(\frac{4-2(\ell^2+2\xi^2)\mathcal{L}_\infty(\ell,\xi)}{\ell^2(\ell^2+4\xi^2)^2\mathcal{L}_\infty(\ell,\xi)}$$
$$+\lim_{N\to\infty}\int_0^1 dx \sum_{n>0,n\neq 0 \text{ mod } N}\left[\frac{6}{N}\frac{\cos(x\ell_y 2\pi n)}{\tan(\pi n/N)} - 6\sin(x\ell_y 2\pi n)\right]\frac{1}{(\ell^2+4\xi^2)\mathcal{L}_\infty(\ell,\xi)}$$
$$\times\Bigg\{-\frac{(2\pi n)^2}{\sqrt{x(1-x)\ell^2+\ell^2}}K_1(z)x(1-x) - \frac{2\pi n}{x(1-x)\ell^2+\xi^2}K_2(z)(1-x)$$
$$+\frac{\ell_y^2}{\ell^2}\left[\frac{(2\pi n)^2}{\sqrt{x(1-x)\ell^2+\xi^2}}K_1(z)x(1-x) + 2\pi n K_0(z)\frac{2}{\ell^2}\right]\Bigg\}\Bigg)$$

Backup: IR renormalon in gluon condensate

- We compute the gluon (photon) condensate in $N \to \infty$, and study Borel singularities associated with it.
- Gluon condensate in the large N limit

$$\begin{aligned}&\langle F_{\mu\nu}(x) F_{\mu\nu}(x) \rangle \\ &= \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{2p^2}{(p^2 + 4\Lambda^2)\mathcal{L}_\infty(p)}\end{aligned}$$



- Positive powers of $\Lambda^2 = \mu^2 e^{-4\pi/\lambda_R(\mu^2)}$ are regarded as the non-perturbative part; $\langle FF \rangle$ in PT is given by

$$\langle F_{\mu\nu}(x) F_{\mu\nu}(x) \rangle_{\mathrm{PT}} = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{p^2}{\ln(p^2/\Lambda^2)} \bigg|_{\text{expansion in } \lambda_R(\mu^2)} .$$

Note that $\mathcal{L}_\infty(p) = \frac{2}{p^2} \ln(p^2/\Lambda^2) + \mathcal{O}(\Lambda^2)$.

Backup: Renormalon ambiguity in $\mathbb{R}^d$

- Noting $\lambda_R(p^2) = 4\pi/\ln(p^2/\Lambda^2)$, $\langle FF\rangle_{\mathrm{PT}}$ is a typical form from which a renormalon appears. Generally, on $\mathbb{R}^d$, we have

$$\int \frac{d^d p}{(2\pi)^d}(p^2)^\alpha \frac{\lambda_R(p^2)}{(4\pi)^{d/2}} = \int \frac{d^d p}{(2\pi)^d}(p^2)^\alpha \sum_{n=0}^{\infty} \ln^n\left(\frac{\mu^2}{p^2}\right)\left[\frac{\lambda_R(\mu^2)}{(4\pi)^{d/2}}\right]^{n+1}.$$

  Note that $\ln(p^2/\Lambda^2) = \ln(p^2/\mu^2) + 4\pi/\lambda_R(\mu^2)$.

- Focusing on the IR region by introducing a cutoff q ($p^2 \leq q^2$),

$$B(u) = \int_{p^2\leq q^2} \frac{d^d p}{(2\pi)^d}(p^2)^\alpha \left(\frac{\mu^2}{p^2}\right)^u = \frac{\mu^{2u}}{(4\pi)^{d/2}\Gamma(d/2)}\frac{q^{2\alpha+d-2u}}{\alpha + d/2 - u}.$$

- The Borel singularity at $u = \alpha + d/2 = 2$ gives rise to

$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle_{\text{renormalon on } \mathbb{R}^2 \text{ (at } u=2)} = \pm\pi i\Lambda^4/N.$$

Backup: Renormalon ambiguity in $\mathbb{R}^{d-1} \times S^1$

- On the other hand, on $\mathbb{R}^{d-1} \times S^1$,

$$\int \frac{d^d p}{(2\pi)^d} \to \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \frac{1}{2\pi R} \sum_{p_d = n/R,\, n \in \mathbb{Z}} .$$

- Now, only the $p_d = 0$ term can be singular; the dimension of the momentum integration is effectively reduced:

$$u = \alpha + \frac{d}{2} \to \alpha + \frac{d-1}{2}$$

- The Borel singularity at $u = 3/2$ gives rise to the renormalon:

$$\langle F_{\mu\nu}(x) F_{\mu\nu}(x) \rangle_{\text{renormalon}} = \pm \pi i \frac{\Lambda^3}{\pi R N}.$$

Peculiar to the compactified space $\mathbb{R} \times S^1$!

Backup: Discussion

- The Borel singularity is generally shifted by $-1/2$ under the S^1 compactification and the following assumptions:
  1. volume independence of a loop integrand of a renormalon diagram
  2. loop momentum variable along S^1 associated with the periodic BC (not twisted!)
- Then, in the large-N (SUSY) $\mathbb{C}P^{N-1}$ model, we find an unfamiliar renormalon singularity at $u = 3/2$.
- But bion calculus $\rightarrow$ $u = 2$ [Fujimori *et al.*]. Thus, no obvious semi-classical interpretation so far.