Numerical study of $\mathcal{N} = 2$ Landau–Ginzburg models

Okuto Morikawa
Collaborator: Hiroshi Suzuki

Kyushu University

2017/11/17 @Hokkaido

Introduction

- QFT on RG fixed point $\rightarrow$ scale inv. $\approx$ conformal inv.
- CFT in terms of Lagrangian: Landau–Ginzburg (LG) description
  - RG fixed point of LG model $\rightarrow$ CFT

- IR limit of 2D $\mathcal{N} = (2,2)$ Wess–Zumino ($\mathcal{N} = 2$ WZ) model $\Rightarrow \mathcal{N} = 2$ SCFT [Cappelli–Itzykson–Zuber '87, ...]
- Various evidences
  - Vafa–Warner '88, Howe–West '89, Witten '93, ...
- Application to spacetime compactification
  - $\mathcal{N} = 2$ SCFT on world sheet and Calabi–Yau (CY) manifold
  - σ-model on CY and LG model [Greene-Vafa-Warner '89,Witten '93]

- (IR) Strong coupling/divergence $\rightarrow$ Non-perturbative approach

2D $\mathcal{N} = 2$ WZ model

- WZ action ($z, \bar{z} = x_0 \pm ix_1$, and $\partial, \bar{\partial} = \frac{1}{2}(\partial_0 \mp i\partial_1)$)

$$S = \int d^2x \Big[4\partial A^* \bar{\partial} A - F^* F - F^* W'(A)^* - F W'(A) \\ + (\bar{\psi}_{\dot{1}}, \psi_2) \begin{pmatrix} 2\partial & W''(A)^* \\ W''(A) & 2\bar{\partial} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \bar{\psi}_{\dot{2}} \end{pmatrix} \Big]$$

- Superpotential

$$W(\Phi) = \frac{\lambda}{n}\Phi^n, \qquad n \geq 3$$

- Correspondence to a minimal model
  - Central charge and conformal weight of primary field

$$c = \frac{3(n-2)}{n} = 1, 1.5, 1.8, \cdots < 3,$$

$$h = \bar{h} = \frac{1}{2n} \qquad (1 - h - \bar{h} = 0.666..., 0.75, 0.8, \cdots < 1)$$

  - In UV limit, WZ model $\rightarrow$ free SCFT with $c = 3$

Preceding numerical studies

- SUSY regulator...
  - Lattice regularization breaks SUSY!
  - Lattice SUSY [Cohen–Kaplan, Sugino, D'Adda et al., Catterall,...]
- Lattice simulation of scaling dimension $h + \bar{h}$ for superpotential $W = \Phi^3$ [Kawai–Kikukawa 2009]
  - $1 - h - \bar{h} = 0.660(11)$ [exact value: $0.666\ldots$ for SCFT]
- SUSY-preserving simulation method [Kadoh–Suzuki 2009]
- Computation of $h + \bar{h}$ and central charge c for $W = \Phi^3$ [Kamata–Suzuki 2010]
  - $1 - h - \bar{h} = 0.616(25)(13)$ [exact value: $0.666\ldots$]
  - $c = 1.09(14)(31)$ [exact value: 1]

Supersymmetric formulation [Kadoh–Suzuki 2009]

- Continuum physical box $L \times L \longrightarrow$ Discretized momentum

$$\varphi(x) = \frac{1}{L^2} \sum_p e^{ipx} \varphi(p),$$

$$p_\mu = \frac{2\pi}{L} n_\mu. \qquad (n_\mu = 0, \pm 1, \pm 2, \dots)$$

  Formulation is defined in this momentum space.

- UV cutoff $\Lambda = \pi/a$ (a: "lattice spacing")

$$-\Lambda \leq p_\mu \leq \Lambda$$

  - All dimensionful quantities will be measured in *unit length* a
  - L/a: even integer, then

$$n_\mu = 0, \pm 1, \dots, \pm L/2a$$

Supersymmetric formulation [Kadoh–Suzuki 2009]

- SUSY (similarly for $\bar{Q}_{\dot{\alpha}}$),

$$
\begin{aligned}
Q_1\bar{\psi}_{\dot{1}}(p) &= -2ip_{\bar{z}}A^*(p), & Q_1A*(p) &= 0,\\
Q_1F^*(p) &= 2ip_{\bar{z}}\bar{\psi}_{\dot{2}}(p), & Q_1\bar{\psi}_{\dot{2}}(p) &= 0,\\
Q_1A(p) &= \psi_1(p), & Q_1\psi_1(p) &= 0,\\
Q_1\psi_2(p) &= F(p), & Q_1F(p) &= 0,
\end{aligned}
$$

$$
\begin{aligned}
Q_2\bar{\psi}_{\dot{2}}(p) &= -2ip_zA^*(p), & Q_2A^*(p) &= 0,\\
Q_2F^*(p) &= -2ip_z\bar{\psi}_{\dot{1}}(p), & Q_2\bar{\psi}_{\dot{1}}(p) &= 0,\\
Q_2A(p) &= \psi_2(p), & Q_2\psi_2(p) &= 0,\\
Q_2\psi_1(p) &= -F(p), & Q_2F(p) &= 0,
\end{aligned}
$$

  translational and R symmetries are conserved
  $\Rightarrow$ All symmetries under linear transformations
- Problem of locality (Perturbatively, locality restores as $\Lambda \to \infty$.)

WZ model and Nicolai map [Nicolai 1980]

- Action of complex scalar A and fermion ψ

$$S = S_B + \frac{1}{L^2}\sum_p \left[(\bar{\psi}_{\dot 1}, \psi_2)\,(-p) \begin{pmatrix} 2ip_z & W''(A)^* * \\ W''(A)* & 2ip_{\bar z} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \bar{\psi}_{\dot 2} \end{pmatrix} (p) \right],$$

where we have integrated auxiliary field, $*$ is convolution, and

$$S_B = \frac{1}{L^2}\sum_p N(p)^* N(p), \quad N(p) \equiv 2ip_z A(p) + W'(A)^*(p).$$

- Integrate fermion, variable transf. $A(p), A^*(p) \to N(p), N^*(p)$.

$$\mathcal{Z} = \int \prod_{|p_\mu|\le\Lambda} [dA(p)dA^*(p)]\, e^{-S_B} \det \frac{\partial(N,N^*)}{\partial(A,A^*)}$$

$$= \int \prod_{|p_\mu|\le\Lambda} [dN(p)dN^*(p)]\, e^{-S_B} \sum_i \operatorname{sign} \det \left.\frac{\partial(N,N^*)}{\partial(A,A^*)}\right|_{A=A_i}.$$

  - $A_i(p)$ ($i = 1, 2, \dots$) are solutions with respect to $N(p)$.
  - Gaussian function (Nicolai map)

Algorithm

1. Generate Gaussian random numbers $(N(p), N^*(p))$
2. Solve numerically the equation

$$2ip_z A(p) + W'(A)^*(p) - N(p) = 0.$$

   $\to$ all solutions $A(p)_i$ $(i = 1, 2, \dots)$
3. Calculate following sums

$$\sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)} \mathcal{O}(A, A^*)\Big|_{A=A_i}, \quad \sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)}\Big|_{A=A_i}$$

4. Repeat (1)∼(3), and average

$$\Delta \equiv \left\langle \sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)}\Big|_{A=A_i} \right\rangle,$$

$$\langle \mathcal{O} \rangle = \frac{1}{\Delta} \left\langle \sum_i \text{sign det} \frac{\partial(N, N^*)}{\partial(A, A^*)} \mathcal{O}(A, A^*)\Big|_{A=A_i} \right\rangle$$

Algorithm

(Advantage)

- No auto-correlation among confs. $N(p)$
- Correctly normalized partition function Δ

$$\Delta \equiv \left\langle \sum_i \text{sign det} \left. \frac{\partial(N, N^*)}{\partial(A, A^*)} \right|_{A=A_i} \right\rangle$$

This is equal to Witten index $n - 1$ for $W = \Phi^n$

- No sign problem

(Disadvantage)

- How many solutions $A(p)_i$?
  Signs for $\{A(p)_i\}_N$: $(+ \cdots + - \cdots -) \Rightarrow \Delta_N = n_+ - n_-$
- *Supersymmetric boundary condition only*

Simulation parameters

- $W(\Phi) = \lambda\Phi^n/n$ ($\Delta = n - 1$)

$$a\lambda = 0.3$$

  - (sama as Kawai–Kikukawa, Kamata–Suzuki)
  - $L \gtrsim \lambda^{-1} \approx 3a$
- Solve by Newton method; Convergent init. conf.$\times$100
  - KK, KS: Initial confs.$\times$100 with divergent ones
  - Precision: maximum norm of residue ($\sim 10^{-13}$ in KS)

$$\frac{1}{\sqrt{\sum_q |N(q)|^2}} \left[2ip_z A(p) + W'(A)^*(p) - N(p)\right]$$

- Setup: L and #confs.
  - Φ^3: $L/a = 8$–36, 640 confs.
  - Φ^4: $L/a = 8$–30, 640 confs.
  - Φ^5: $L/a = 8$–18, 640 confs.

Classification of configurations

$W = \Phi^3$ $(\Delta = 2)$

$L_0 = L_1$	18	20	22	24	26
$(++)_2$	634	636	634	637	635
$(+++-)_2$	6	4	6	3	5
Δ	2	2	2	2	2
Residue(e-16)	6.27	7.02	7.14	7.70	8.48
core·hour	49	87	144	237	405

$L_0 = L_1$	28	30	32	34	36
$(++)_2$	634	626	633	628	627
$(+++-)_2$	6	14	7	12	13
Δ	2	2	2	2	2
Residue(e-16)	9.05	9.85	11.2	11.2	13.6
core·hour	650	964	1381	1936	2699

Classification of configurations

$W = \Phi^4$ $(\Delta = 3)$

$L_0 = L_1$	24	26	28	30
$(+++)_3$	625	616	614	615
$(++++-)_3$	15	23	20	22
$(+++++--)_3$	0	0	2	0
$(++++)_4$	0	1	3	2
$(+++++-)_4$	0	0	1	0
$(++)_2$	0	0	0	1
Δ	3	3.002(2)	3.006(3)	3.002(3)
Residue(e-16)	14.1	15.9	26.4	18.9
core·hour	2917	5004	8273	12905

- More difficult to solve equation for the case Φ^4
- $\Delta \cong 3$ almost within 0.5%

Classification of configurations

$W = \Phi^5$ $(\Delta = 4)$

$L_0 = L_1$	14	16	18	20
$(++++)_4$	613	602	589	570
$(++++++-)_4$	17	19	20	12
$(+++++++--)_4$	1	0	1	0
$(+++)_3$	6	12	21	40
$(++++-)_3$	0	4	2	4
$(++++++--)_3$	0	0	0	1
$(+++++)_5$	3	3	6	7
$(+++++++-)_5$	0	0	0	1
$(++)_2$	0	0	0	1
$(+++-)_2$	0	0	1	3
$(+)_1$	0	0	0	1
Δ	3.995(5)	3.980(7)	3.970(9)	3.925(13)
Residue(e-16)	19.2	37.8	28.2	24.2
core·hour	508	1340	3287	7860

SUSY Ward–Takahashi identity

- Numerical confirmation of exactly SUSY invariant
- One-point SUSY WT identity [Catterall–Karamov]

$$\delta = \langle S_B \rangle / (L_0 + 1)(L_1 + 1) - 1 = 0$$

W	L/a	δ
Φ^3	36	0.0007(11)
Φ^4	30	$-0.0010(15)$
Φ^5	20	$-0.0002(39)$

- Two-point SUSY WT identity

$$\left\langle Q_1(A(p)\bar{\psi}_{\bar{1}}(-p))\right\rangle = 0$$
$$\Rightarrow 2ip_{\bar{z}} \langle A(p)A^*(-p)\rangle = -\left\langle \psi_1(p)\bar{\psi}_{\bar{1}}(-p)\right\rangle$$

SUSY Ward–Takahashi identity

$$p_1 \langle A(p) A^*(-p) \rangle = \mathrm{Re} \langle \psi_1(p) \bar{\psi}_1(-p) \rangle$$

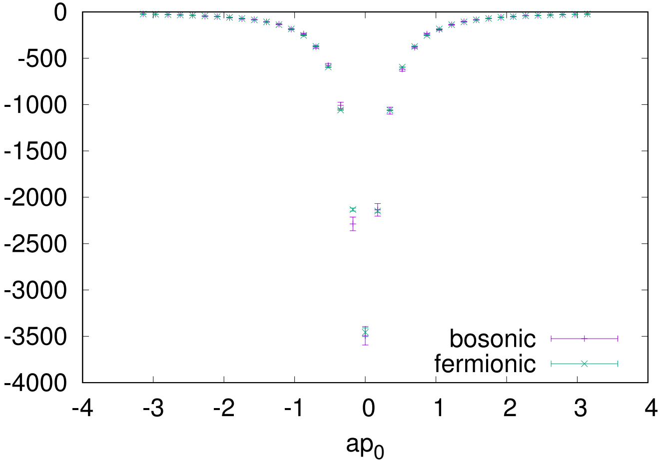


Figure: Φ^3, $L/a = 36$, $ap_1 = 2\pi/L = \pi/18$

SUSY Ward–Takahashi identity

$$p_0 \langle A(p) A^*(-p) \rangle = -\,\mathrm{Im} \left\langle \psi_1(p) \bar{\psi}_{\dot{1}}(-p) \right\rangle$$

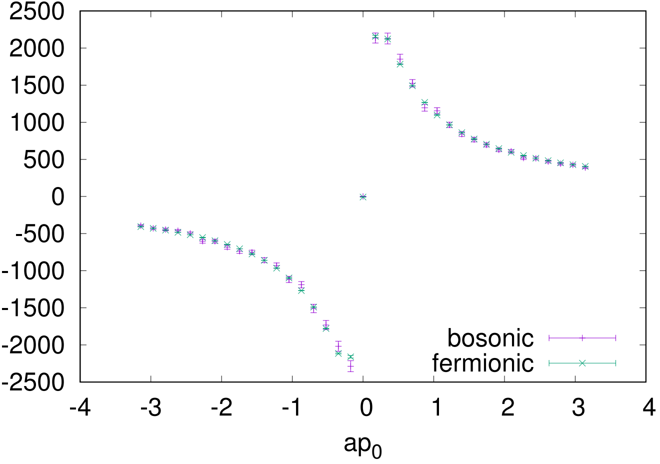


Figure: Φ^3, $L/a = 36$, $ap_1 = 2\pi/L = \pi/18$

SUSY Ward–Takahashi identity

$$p_1 \langle A(p) A^*(-p) \rangle = \mathrm{Re} \left\langle \psi_1(p) \bar{\psi}_1(-p) \right\rangle$$

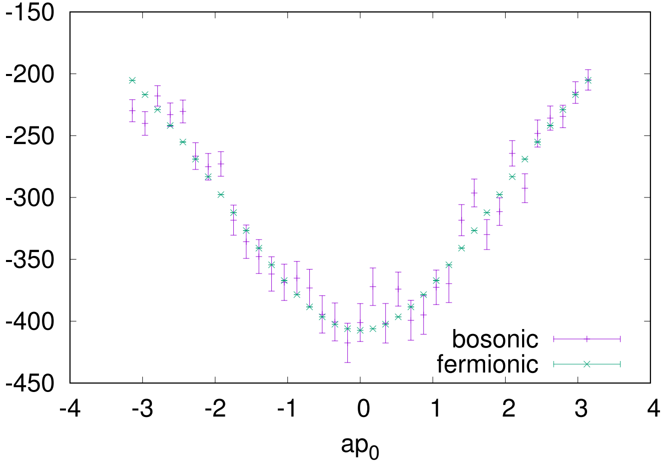


Figure: Φ^3, $L/a = 36$, $ap_1 = \pi$

SUSY Ward–Takahashi identity

$$p_0 \langle A(p) A^*(-p) \rangle = -\operatorname{Im} \langle \psi_1(p) \bar{\psi}_{\dot{1}}(-p) \rangle$$

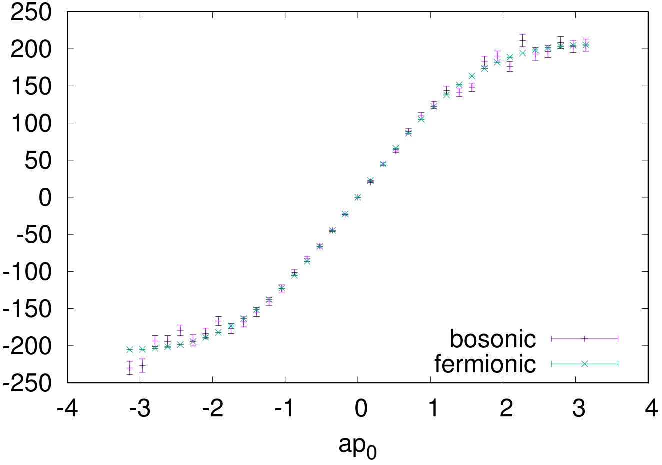


Figure: Φ^3, $L/a = 36$, $ap_1 = \pi$

Correlation function

- Simulation of scaling dimension and central charge
- Two-point function

$$\langle \varphi_1(p)\varphi_2(-p)\rangle = L^2 \int d^2x\, e^{-ipx}\underline{\langle \varphi_1(x)\varphi_2(0)\rangle}$$

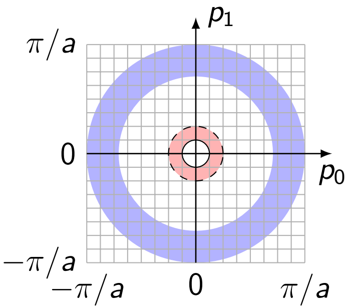
- Ex) In the case of scalar field $A(x)$ ($\rightarrow$ Chiral primary field)

$$\langle A(x)A^*(0)\rangle \propto 1/z^{2h}\bar{z}^{2\bar{h}} \quad \Rightarrow \langle A(p)A^*(-p)\rangle \propto 1/(p^2)^{1-h-\bar{h}}$$

IR behavior of $\langle \varphi_1(p)\varphi_2(-p)\rangle$
$\Longrightarrow$ Scaling dimension $h+\bar{h}$
Central charge c

- Fitting $\langle \varphi_1(p)\varphi_2(-p)\rangle$
  - IR: $2\pi/L \leq |p| < 2\pi/L \times 2$

    (UV: free SCFT)



Scaling dimension

$$\langle A(p)A^*(-p)\rangle \sim 1/(p^2)^{1-h-\bar{h}}$$

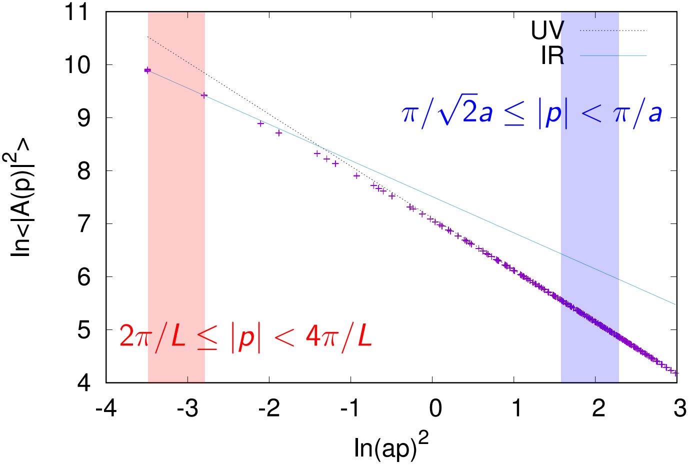


Figure: Φ^3, $L/a = 36$

Scaling dimension

$$\langle A(p)A^*(-p)\rangle \sim 1/(p^2)^{1-h-\bar{h}}$$

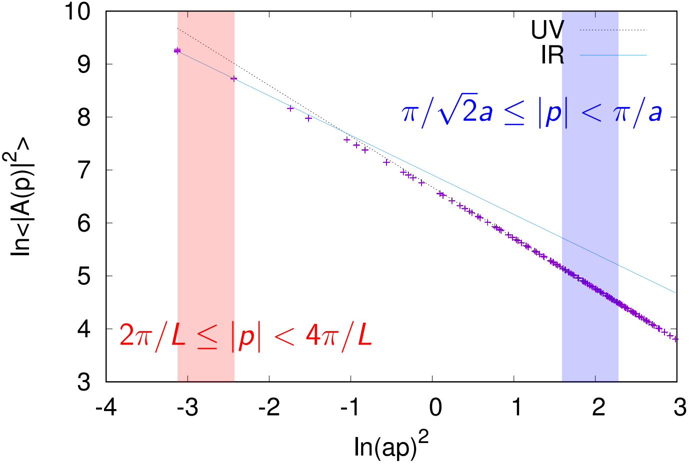


Figure: Φ^4, $L/a = 30$

Scaling dimension

$$\langle A(p)A^*(-p)\rangle \sim 1/(p^2)^{1-h-\bar{h}}$$

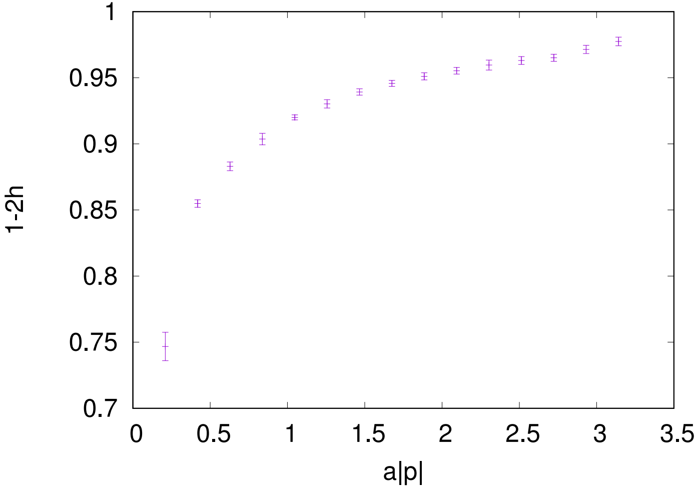
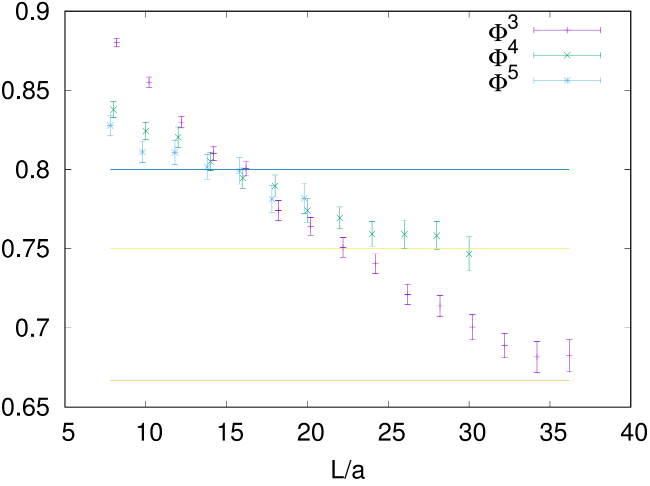


Figure: Φ^4, $L/a = 30$. (IR$\to$ 0.75, UV$\to$ 1)

Scaling dimension $h + \bar{h}$



Scaling dimension $h + \bar{h}$

- Scaling dimensions for $W = \Phi^n$ $(n = 3, 4, 5)$

W	L/a	$1 - h - \bar{h}$	expected value
Φ^3	36	0.682(10)	$0.666\ldots$
Φ^4	30	0.747(11)	0.75
Φ^5	20	0.7818(95)	0.8

  - Kawai–Kikukawa (Φ^3): 0.660(11)
  - Kamata–Suzuki (Φ^3): 0.616(25)(13)
    (KK and KS computed susceptibility χ_ϕ of scalar field)

$$\chi_\phi = \frac{1}{a^2} \int d^2x \, \langle A(x) A^*(0) \rangle \propto (L^2)^{1-h-\bar{h}}$$

- As $L \to$ large, $1 - h - \bar{h}$ approaches expected values for SCFT
  - For $W = \Phi^5$, max of L is small?

Central charge (Supercurrent)

- The supercurrent is given by

$$S_z^+(p) = \frac{4\pi}{L_0 L_1} \sum_q i(p-q)_z A(p-q) \bar{\psi}_{\dot{2}}(q),$$

$$S_z^-(p) = -\frac{4\pi}{L_0 L_1} \sum_q i(p-q)_z A^*(p-q) \psi_2(q),$$

with a requirement

$$S_{\bar{z}}^+(p), S_{\bar{z}}^-(p) \to 0 \quad \text{in the UV limit (free SCFT).}$$

- Two-point function of supercurrent $S^\pm$ in SCFT

$$\left\langle S_z^+(z) S_z^-(0) \right\rangle = \frac{2c}{3z^3} \quad \Rightarrow \left\langle S_z^+(p) S_z^-(-p) \right\rangle = L^2 \frac{i\pi c}{3} \frac{p_z^2}{p_{\bar{z}}}$$

Central charge (Supercurrent)

Real part of $\langle S_z^+(p) S_z^-(-p) \rangle = L^2 i\pi c p_z^2 / 3p_{\bar{z}}$

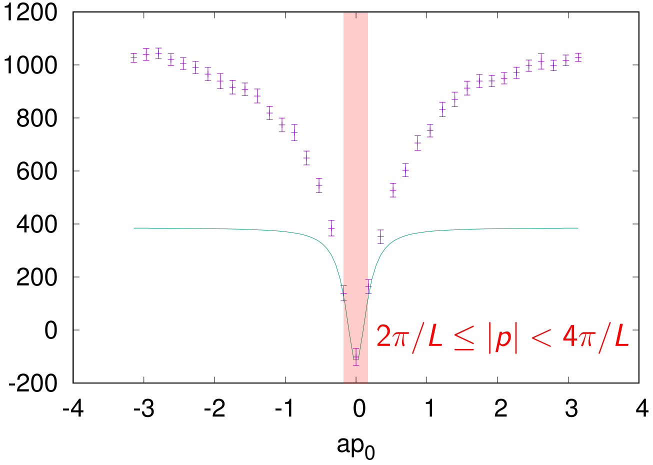


Figure: Φ^3, $L/a = 36$, $ap_1 = \pi/18$

Central charge (Supercurrent)

Imaginary part of $\langle S_z^+(p) S_z^-(-p) \rangle = L^2 i\pi c p_z^2 / 3 p_{\bar{z}}$

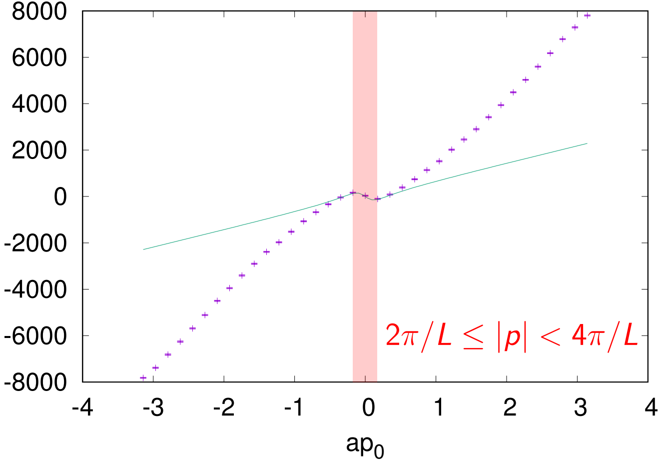


Figure: Φ^3, $L/a = 36$, $ap_1 = \pi/18$

Central charge (Supercurrent)

$$\langle S_z^+(p) S_z^-(-p) \rangle = L^2 i\pi c p_z^2 / 3p_{\bar{z}}$$

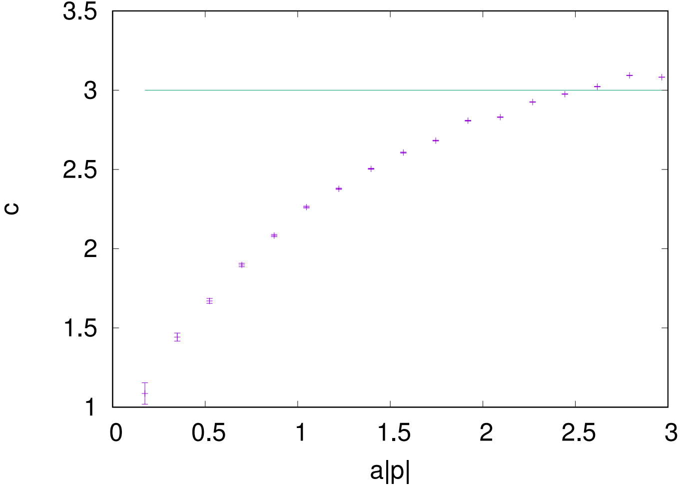
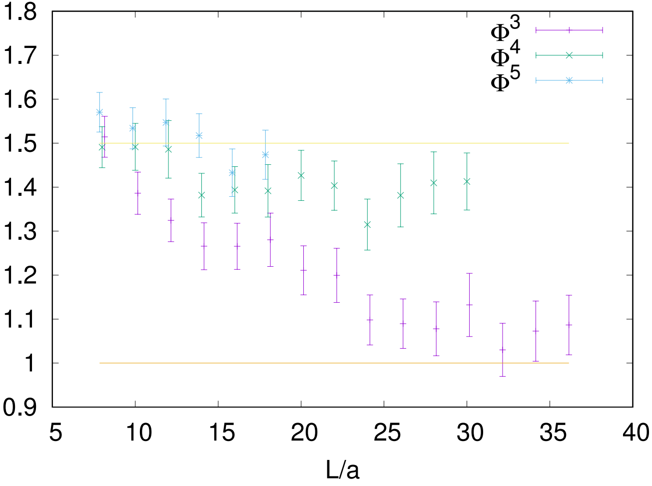


Figure: Φ^3, $L/a = 36$. SUSY-analogue of Zamolodchikov's c-theorem.

Central charge (Supercurrent)



Central charge (Supercurrent)

- Central charges for $W = \Phi^n$ ($n = 3, 4, 5$)

W	L/a	c	expected value
Φ^3	36	1.087(68)	1
Φ^4	30	1.413(65)	1.5
Φ^5	18	1.474(56)	1.8

  - Kamata–Suzuki (Φ^3): 1.09(14)(31)
- As $L \to$ large, c approaches expected values for SCFT
  - But, $W = \Phi^5$ is not the case (L is too small?)
- Note: sometimes, χ^2/d.o.f. is quit different from unity.
  - Φ^4: χ^2/d.o.f. $= 4.606$

$\Rightarrow$ Energy-momentum tensor

Central charge (Energy-momentum tensor)

- Energy-momentum tensor is given by

$$
\begin{aligned}
T_{zz}(x) &= -4\pi\partial A^*(x)\partial A(x) \\
&\quad - \pi\psi_2(x)\bar{\psi}_{\dot{2}}(x) + \pi\partial\psi_2(x)\bar{\psi}_{\dot{2}}(x), \\
T_{\bar{z}\bar{z}}(x) &= -4\pi\bar{\partial} A^*(x)\bar{\partial} A(x) \\
&\quad - \pi\bar{\psi}_{\dot{1}}(x)\bar{\partial}\psi_1(x) + \pi\bar{\partial}\bar{\psi}_{\dot{1}}\psi_1(x),
\end{aligned}
$$

  with a requirement

$$T_{z\bar{z}}(x) = T_{\bar{z}z} \to 0 \quad \text{in the UV limit (free SCFT).}$$

- Two-point function of EMT in SCFT

$$\langle T_{zz}(x) T_{zz}(0)\rangle = \frac{c}{2z^4} \quad \Rightarrow \langle T_{zz}(p) T_{zz}(-p)\rangle = L^2 \frac{\pi c}{12}\frac{p_z^3}{p_{\bar{z}}}.$$

Central charge (Energy-momentum tensor)

- Usually, numerical computation of EMT is very noisy.
- Thus, it is difficult to simulate its two-point function.
  - In the formulation, four-point function $\langle A^* A^* A A \rangle \to$ noisy.

$$TT \sim (\partial A^* \partial A + \dots)(\partial A^* \partial A + \dots)$$

- Note that

$$T_{zz} = \frac{1}{4} Q_2 S_z^+ - \frac{1}{4} \bar{Q}_{\dot{2}} S_z^- .$$

- SUSY WT identity gives a less noisier form of $\langle TT \rangle$

$$\langle T_{zz}(p) T_{zz}(-p) \rangle = -\frac{2ip_z}{16} \left\langle S_z^+(p) S_z^-(-p) + S_z^-(p) S_z^+(-p) \right\rangle .$$

  - The structure respects to (anti-)symmetry under $p \to -p$. Then, $\chi^2/\text{d.o.f.} \approx 1$.

Central charge (Energy-momentum tensor)

Real part of $\langle T_{zz}(p) T_{zz}(-p) \rangle = L^2 \pi c p_z^3 / 12 p_{\bar{z}}$

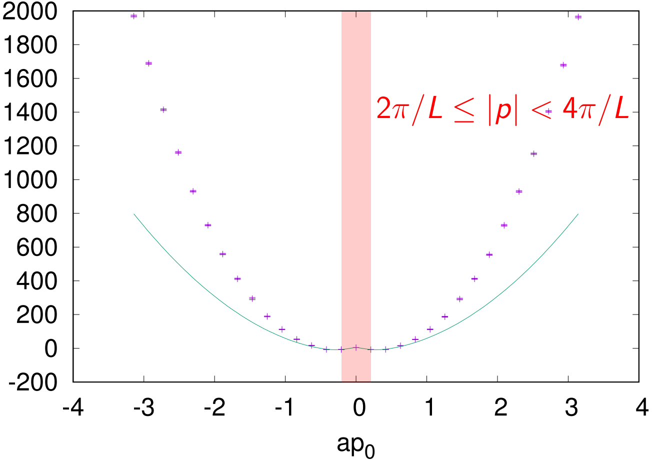


Figure: Φ^4, $L/a = 30$, $ap_1 = \pi/15$

Central charge (Energy-momentum tensor)

Imaginary part of $\langle T_{zz}(p)T_{zz}(-p)\rangle = L^2\pi c p_z^3/12p_{\bar{z}}$

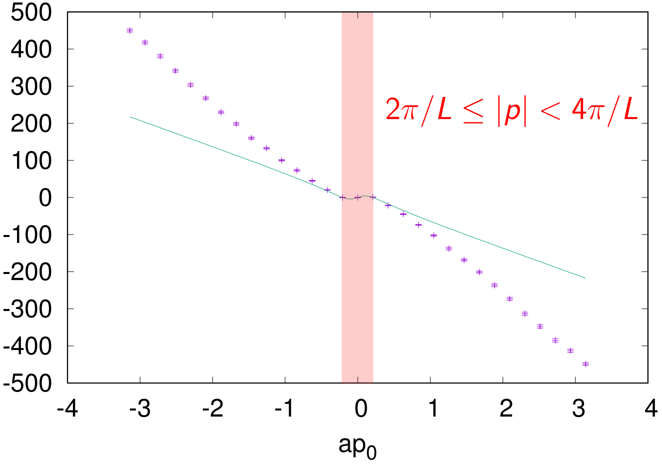


Figure: Φ^4, $L/a = 30$, $ap_1 = \pi/15$

Central charge (Energy-momentum tensor)

$$\langle T_{zz}(p)T_{zz}(-p)\rangle = L^2\pi c p_z^3/12p_{\bar{z}}$$

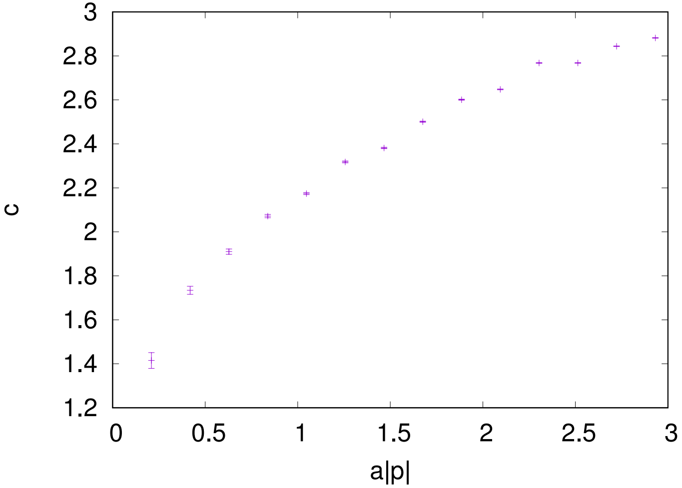
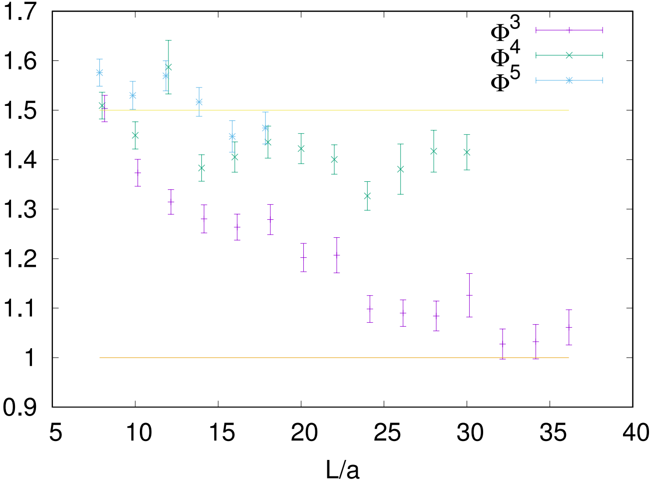


Figure: Φ^4, $L/a = 30$. Main part of Zamolodchikov's C-function.

Central charge (Energy-momentum tensor)



Central charge (Energy-momentum tensor)

- Central charges for $W = \Phi^n$ $(n = 3, 4, 5)$

W	L/a	c	expected value
Φ^3	36	1.087(68)	1
Φ^4	30	1.413(65)	1.5
Φ^5	18	1.474(56)	1.8

  - From a supercurrent correlator,
    Φ^3: , Φ^4: , Φ^5:
- As $L \to$ large, c approaches expected values for SCFT
  - But, $W = \Phi^5$ is not the case (L is too small?)

Summary

- Numerical simulation of 2D $\mathcal{N} = 2$ WZ model $\xrightarrow{\text{IR limit}}$ Non-perturbatively emerging SCFT
- Scaling dimension and central charge

W	$1 - h - \bar{h}$		c	
Φ^3	0.682(10)	0.666...	1.15(48)	1
Φ^4	0.747(11)	0.75	1.43(11)	1.5
Φ^5	0.7818(89)	0.8	1.	1.8

$\Longrightarrow$ These results for $W = \Phi^3$ and Φ^4 are consistent with SCFT.

- Future work
  - Precision: larger L, finite size scaling
  - Multi-superfield models (ADE classification)
  - Application to Calabi–Yau space

Backup: C-function

- General forms ($\tau = \ln z\bar{z}$)

$$\langle T_{zz}(x) T_{zz}(0) \rangle = F(\tau)/z^4,$$
$$\langle T_{zz}(x) T_{z\bar{z}}(0) \rangle = G(\tau)/4z^3\bar{z},$$
$$\left\langle T_{z\bar{z}(x)} T_{z\bar{z}}(0) \right\rangle = H(\tau)/z^2\bar{z}^2.$$

- Conservation laws and reflection positivity imply

$$\frac{d}{d\tau}(2F - G - 3H/8) \leq 0.$$

- Zamolodchikov's C-function:

$$C = 2F - G - \frac{3}{8}H.$$

Monotonically decreasing function along RG flow $\to c$.