Gradient flow and the Wilsonian renormalization group flow

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- Hiroki Makino, O.M. and Hiroshi Suzuki, arXiv:1802.07897 [hep-th], to appear in PTEP.
- Aya Kasai, O.M. and Hiroshi Suzuki, in preparation

Gradient flow [Lüscher 2010]

- Gradient flow: an evolution of fields and physical quantities along a fictitious time
- Gradient flow equation (gauge field)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x),$$
$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

and the boundary condition

$$B_\mu(t=0,x) = A_\mu(x)$$

- t: mass dimension -2
- "Gradient" $\rightarrow \partial_t B_\mu = -g_0^2 \delta S_{YM}/\delta B_\mu$
- Diffusion equation with the diffusion length

$$x \sim \sqrt{8t}$$

Smoothing effect for the configuration as $t \rightarrow$ large

Perturbative expansion of the gradient flow

- Gradient flow with the "gauge fixing"

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x)$$

- Formal solution of this equation

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right]$$

where the heat kernel

$$K_t(x)_{\mu\nu} = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 t p^2} \right]$$

$$= \delta_{\mu\nu} \int \frac{d^d p}{(2\pi)^d} e^{ipx} e^{-tp^2} \qquad (\alpha_0 = 1)$$

- and the non-linear term R

$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]]$$

Perturbative expansion of the gradient flow

- Correlation function of the flowed gauge field

$$
\begin{aligned}
&\langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_N}(t_N, x_N) \rangle \\
&= \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \, B_{\mu_1}(t_1, x_1) \dots B_{\mu_N}(t_N, x_N) e^{-S_{\mathrm{YM}} - S_{\mathrm{gf}} - S_{c\bar{c}}}
\end{aligned}
$$

- Free propagator of the flowed field

$$
\begin{aligned}
&\left\langle B_\mu^a(t, x) B_\nu^b(s, y) \right\rangle_0 \\
&= \delta^{ab} g_0^2 \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{(p^2)^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + p_\mu p_\nu e^{-\alpha_0 (t+s) p^2} \right] \\
&= \delta^{ab} \delta_{\mu\nu} g_0^2 \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2} e^{-(t+s)p^2} \qquad (\alpha_0 = 1)
\end{aligned}
$$

- Gaussian damping factor e^{-tp^2}

  $\Rightarrow$ a simple renormalization property of the gradient flow

Flowed composite operator [Lüscher–Weisz 2011]

- Correlation function of the flowed gauge field ($t_i > 0$)

$$\langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_N}(t_N, x_N) \rangle$$

$\Rightarrow$ UV finite without the wave function renormalization

  - Momentum integral $\int d^D p \to \int d^D p \exp(-tp^2)$

- Independence of the regularization

$\to$ Probes of universal properties of theories

- Compatible with the gauge invariance
- Many interesting applications to lattice gauge theory
  - Topological charge
  - Chiral condensation
  - Energy-momentum tensor
  - Non-perturbative gauge coupling

Flow of fermion field [Lüscher 2013]

- A possible choice of flow equation

$$\partial_t \chi(t,x) = \left[\Delta - \alpha_0 \partial_\mu B_\mu(t,x)\right] \chi(t,x), \qquad \chi(t=0,x) = \psi(x),$$
$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x) \left[\overleftarrow{\Delta} + \alpha_0 \partial_\mu B_\mu(t,x)\right], \quad \bar{\chi}(t=0,x) = \bar{\psi}(x),$$

where

$$\Delta = D_\mu D_\mu \qquad\qquad D_\mu = \partial_\mu + B_\mu$$
$$\overleftarrow{\Delta} = \overleftarrow{D}_\mu \overleftarrow{D}_\mu \qquad\qquad \overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu - B_\mu$$

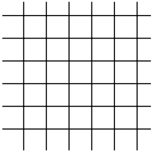
- Now, we need the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2} \chi(t,x) \qquad\qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2} \bar{\chi}(t,x)$$

- Any composite operators of χ_R, $\bar{\chi}_R$ are UV finite

Lattice gauge theory

- Lattice gauge theory: Non-perturbative formulation of gauge theory in the discretized spacetime



$\updownarrow a$: lattice spacing

- Compatible with the gauge invariance
- For $a \neq 0$, incompatible with spacetime symmetries:
  translational symmetry, SUSY, conformal symmetry, ...
- In the continuum limit $a \to 0$, the spacetime symmetries would be restored
- It is difficult to construct Noether currents associated with those because of UV divergences

Lattice energy-momentum tensor...?

- Composite operator contains further UV divergence

$$a \times \frac{1}{a} \stackrel{a\to 0}{\to} 1$$

- i.e., EMT on a lattice [Caracciolo et al. '89]

$$\{T_{\mu\nu}\}_R(x) = Z_1 \left\{ \sum_{\rho} F_{\mu\rho} F_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu} \sum_{\rho\sigma} F_{\rho\sigma} F_{\rho\sigma} \right\} + Z_2 \delta_{\mu\nu} \sum_{\rho\sigma} F_{\rho\sigma} F_{\rho\sigma} + Z_3 \delta_{\mu\nu} \sum_{\rho} F_{\mu\rho} F_{\mu\rho} + \dots$$

  To-be-determined coefficients Z_i (for QCD, 7 coefficients)

  $\to$ Tune Z_i to satisfy the translation WT identity

- Lattice $\leftarrow$ flowed composite operator $\to$ other regularizations (dimensional)

Small flow-time expansion

- The relation between composite operators in $t > 0$ and $t = 0$ becomes tractable in the limit $t \to 0$
- Small flow-time expansion [Lüscher–Weisz 2011]

$$\tilde{\mathcal{O}}_i(t,x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Inverse of this relation:

$$\mathcal{O}_i(x) = \lim_{t\to 0}\left\{\sum_j (\zeta^{-1})_{ij}(t)\tilde{\mathcal{O}}_j(t,x)\right\}$$

- $\zeta(t)$ is independent of the renormalization scale μ,

$$\zeta(t)[g,m;\mu] = \zeta(t)\left[\bar{g}(1/\sqrt{8t}), \bar{m}(1/\sqrt{8t}); 1/\sqrt{8t}\right]$$

- Because of the asymptotic freedom, $\bar{g}(1/\sqrt{8t}) \to 0$ for $t \to 0$, and we can obtain ζ^{-1} with perturbation theory

Universal formulas and numerical studies

- Energy-momentum tensor (EMT)
  - 4D pure YM [Suzuki '13]
    - ⋆ $SU(3)$ YM: Trace anomaly, entropy density [FlowQCD '13]
  - 4D N_f-flavor gauge theory [Makino–Suzuki '14]
    - ⋆ $N_f = 2 + 1$ QCD: Trace anomaly, entropy density [WHOT '16]
    - ⋆ Topological susceptibility [WHOT '16]
    - ⋆ Latent heat, equation of state, two-point function of EMT, ... [WHOT]
- Other constructions
  - Chiral current [Endo–Hieda–Miura–Suzuki '15, Hieda–Suzuki '16]
  - (Axial $U(1)$ anomaly in curved space [O.M.–Suzuki '18])

Supercurrent of 4D $\mathcal{N} = 2$ pure YM (preliminary)

- Lattice regularization is incompatible with SUSY
  $\rightarrow$ Parameter tuning to restore the SUSY WT identity
- Construction of supercurrent by using the gradient flow
  - (Flow equation of 4D $\mathcal{N} = 1$ pure YM [Kikuchi–Onogi '14])
  - 4D $\mathcal{N} = 1$ pure YM [Hieda–Kasai–Makino–Suzuki '16]

- Next target: 4D $\mathcal{N} = 2$ pure YM [Kasai–O.M.–Suzuki]
- For any regularizations, one cannot preserve SUSY exactly...
  - Classical supercurrent (Wess–Zumino gauge)

$$
\begin{aligned}
S_\mu^{\text{classical}} &= -\frac{1}{4g_0}\sigma_{\rho\sigma}\gamma_\mu\psi^a F^a_{\rho\sigma} \\
&\quad + \frac{1}{2\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)\left(P_+ D_\nu\psi^a\varphi^a - P_- D_\nu\psi^a\varphi^{a\dagger}\right) \\
&\quad - \frac{1}{\sqrt{2}}\left(\frac{1}{3}\sigma_{\mu\nu} - \delta_{\mu\nu}\right)\left(P_+\psi^a D_\nu\varphi^a - P_-\psi^a D_\nu\varphi^{a\dagger}\right)
\end{aligned}
$$

- *We expect that this is the correctly normalized supercurrent with the dimensional regularization*

Supercurrent of 4D $\mathcal{N}=2$ pure YM (preliminary)

$$
\begin{aligned}
S_\mu(x) = &-\frac{1}{4\bar{g}(1/\sqrt{8t})}\left\{1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left[-\ln\pi-\frac{9}{4}+\frac{1}{2}\ln(432)\right]\right\}\sigma_{\rho\sigma}\gamma_\mu\mathring{\chi}^a(t,x)G^a_{\rho\sigma}(t,x)\\
&-\frac{\bar{g}(1/\sqrt{8t})}{(4\pi)^2}C_2(G)\gamma_\nu\mathring{\chi}^a(t,x)G^a_{\nu\mu}(t,x)\\
&+\frac{1}{2\sqrt{2}}\left\{1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left[-\frac{19}{4}+4\ln 2+\frac{1}{2}\ln(432)\right]\right\}\\
&\qquad\times\left(\tfrac{1}{3}\sigma_{\mu\nu}-\delta_{\mu\nu}\right)\left(P_+D_\nu\mathring{\chi}^a(t,x)\mathring{\phi}^a(t,x)-P_-D_\nu\mathring{\chi}^a(t,x)\mathring{\phi}^{a\dagger}(t,x)\right)\\
&\qquad-\frac{3}{\sqrt{2}}\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left(P_+D_\mu\mathring{\chi}^a(t,x)\mathring{\phi}^a(t,x)-P_-D_\mu\mathring{\chi}^a(t,x)\mathring{\phi}^{a\dagger}(t,x)\right)\\
&-\frac{1}{\sqrt{2}}\left\{1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left[\frac{1}{2}+4\ln 2+\frac{1}{2}\ln(432)\right]\right\}\\
&\qquad\times\left(\tfrac{1}{3}\sigma_{\mu\nu}-\delta_{\mu\nu}\right)\left(P_+\mathring{\chi}^a(t,x)D_\nu\mathring{\phi}^a(t,x)-P_-\mathring{\chi}^a(t,x)D_\nu\mathring{\phi}^{a\dagger}(t,x)\right)\\
&+\frac{1}{\sqrt{2}}\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left(\tfrac{1}{3}\sigma_{\mu\nu}-\delta_{\mu\nu}\right)\gamma_5D_\nu\mathring{\chi}^a(t,x)\left(\mathring{\phi}^a(t,x)+\mathring{\phi}^{a\dagger}(t,x)\right)\\
&+\frac{1}{2\sqrt{2}}\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(G)\left(\tfrac{1}{3}\sigma_{\mu\nu}-\delta_{\mu\nu}\right)\gamma_5\mathring{\chi}^a(t,x)\left(D_\nu\mathring{\phi}^a(t,x)+D_\nu\mathring{\phi}^{a\dagger}(t,x)\right)\\
&-\frac{1}{4}f^{abc}\frac{\bar{g}(1/\sqrt{8t})^3}{(4\pi)^2}C_2(G)\gamma_5\gamma_\mu\mathring{\chi}^a(t,x)\mathring{\phi}^{b\dagger}(t,x)\mathring{\phi}^c(t,x)
\end{aligned}
$$

Non-perturbative running coupling

- Larger flow time?
- i.e., Determination of α_s, Λ-parameter
  - ALPHA ['17]: Step scaling + gradient flow
  - HPQCD
  - JLQCD
  - PACS-SC ...
- Gauge coupling

$$\bar{g}^2_{\mathrm{GF}}(\mu = 1/\sqrt{8t}) \propto t^2 \left\langle G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x) \right\rangle$$

where the normalization constant is chosen such that

$$\bar{g}^2_{\mathrm{GF}} = g_0^2 + O(g_0^4)$$

- Gradient flow acts as the "coarse-graining"
- Relation between the gradient flow and the Wilsonian RG flow

Wilsonian renormalization group

- Wilsonian RG flow [Wilson–Kogut '74] characterized by the scaling relation

$$\left\langle \mathcal{O}_1(e^{2\xi}t_1, e^{\xi}x_1)\dots\mathcal{O}_N(e^{2\xi}t_N, e^{\xi}x_N)\right\rangle_{\{g_i\}}$$
$$= Z(\xi)\left\langle \mathcal{O}_1(t_1, x_1)\dots\mathcal{O}_N(t_N, x_N)\right\rangle_{\{g_i(\xi)\}}$$

- One-point function

$$\left\langle \mathcal{O}_1(e^{2\xi}t)\right\rangle_{\{g_i\}} = \left\langle \mathcal{O}_1(t)\right\rangle_{\{g_i(\xi)\}}$$

- Spaces of one-point functions and coupling constants

$$\{\langle \mathcal{O}_i(t)\rangle\} \Leftrightarrow \{g_i(\xi)\}$$

- Two examples that possess an IR fixed point
  - 4D N_f-flavor gauge theory
  - 3D $O(N)$ linear sigma model

4D N_f-flavor gauge theory

- 4D gauge theory with N_f-flavor Dirac fermions (mass m)
- Banks–Zaks IR fixed point ['82]
- Running coupling and mass parameter in the $\overline{\text{MS}}$ scheme

$$\left[b_0\bar{g}(\mu)^2\right]^{-b_1/(2b_0^2)}\exp\left[-\frac{1}{2b_0\bar{g}(\mu)^2}\right]=\frac{\Lambda}{\mu},$$
$$\bar{m}(\mu)=M\left[2b_0\bar{g}(\mu)^2\right]^{d_0/(2b_0)},$$

where

$$b_0=\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G)-\frac{4}{3}T(R)N_f\right],$$
$$b_1=\frac{1}{(4\pi)^2}\left\{\frac{34}{3}C_2(G)^2-\left[4C_2(R)+\frac{20}{3}C_2(G)\right]T(R)N_f\right\},$$
$$d_0=\frac{1}{(4\pi)^2}6C_2(R)$$

(Λ and M are RG invariant mass scales)

4D N_f-flavor gauge theory

- Dimensionless operators

$$\mathcal{O}_1(t,x) \equiv \frac{8(4\pi)^2 t^2}{3\dim(G)} \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)$$

$$\mathcal{O}_2(t,x) \equiv \frac{\bar{\chi}(t,x)\chi(t,x)}{t^{1/2}\left\langle \bar{\chi}(t,x) \overleftrightarrow{\not{D}} \chi(t,x) \right\rangle}$$

- $\mathcal{O}_1$ does not receive any multiplicative renormalization
- $\mathcal{O}_2$ also does not ... because of the division by $\left\langle \bar{\chi} \overleftrightarrow{\not{D}} \chi \right\rangle$

4D N_f-flavor gauge theory

- Two-loop approximation in $\overline{\text{MS}}$ scheme [Harlander–Neumann]

$$\langle \mathcal{O}_1(t)\rangle = \bar{g}(1/\sqrt{8t})^2\left[1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}K_1(t)+\frac{\bar{g}(1/\sqrt{8t})^4}{(4\pi)^4}K_2\right],$$

where (leading mass correction only)

$$\begin{aligned}K_1(t) &= \left(\frac{11}{3}\gamma_E+\frac{52}{9}-3\ln 3\right)C_2(G)\\ &\quad\left[-\frac{4}{3}\gamma_E-\frac{8}{9}+\frac{8}{3}\ln 2+16\bar{m}(1/\sqrt{8t})^2 t\right]T(R)N_f,\\ K_2 &= 8(4\pi)^2\{-0.0136423(7)C_2(G)^2\\ &\quad+[0.006440134(5)C_2(R)-0.0086884(2)C_2(G)]T(R)N_f\\ &\quad+0.000936117\,T(R)^2N_f^2\}\end{aligned}$$

- One-loop approximation [Makino–Suzuki]

$$\langle \mathcal{O}_2(t)\rangle = \bar{m}(1/\sqrt{8t})t^{1/2}\left[1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}K_1'\right],$$

where

$$K_1' = [3\gamma_E+4+2\ln 2-\ln(432)]\,C_2(R)$$

4D N_f-flavor gauge theory

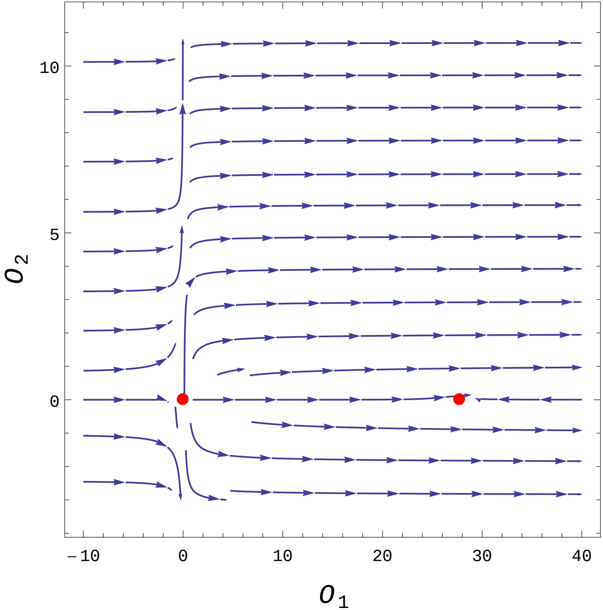
- Running parameters → One-point functions

$$
\begin{aligned}
t\frac{d}{dt}\langle\mathcal{O}_1(t)\rangle &= b_0\langle\mathcal{O}_1(t)\rangle^2 + b_1\langle\mathcal{O}_1(t)\rangle^3 \\
&\quad + \frac{1}{(4\pi)^2}16T(R)N_f\langle\mathcal{O}_1(t)\rangle^2\langle\mathcal{O}_2(t)\rangle^2 \\
t\frac{d}{dt}\langle\mathcal{O}_2(t)\rangle &= \frac{1}{2}\left[1 + d_0\langle\mathcal{O}_1(t)\rangle\right]\langle\mathcal{O}_2(t)\rangle
\end{aligned}
$$

- In the IR limit $t \to \infty$, $\langle\mathcal{O}_2(t)\rangle \to \infty$: relevant coupling
- Bank–Zaks fixed point:

$$
(\langle\mathcal{O}_1(t)\rangle, \langle\mathcal{O}_2(t)\rangle) = (-b_0/b_1, 0)
$$

4D N_f-flavor gauge theory



3D $O(N)$ linear σ-model at large N

- Action of the 3D $O(N)$ linear σ-model

$$S = \int d^3x \left\{ \frac{1}{2}\partial_\mu \phi_i(x)\partial_\mu \phi_i(x) + \frac{1}{2} m_0^2 \phi_i(x)\phi_i(x) + \frac{\lambda_0}{8N}[\phi_i(x)\phi_i(x)]^2 \right\}$$

$\rightarrow$ Wilson–Fisher IR fixed point ['72]

- For simplicity, we will work out the large-N approximation
  - Gradient flow in this system was studied in [Aoki et al. '17]
- "Physical" mass M given by the gap equation

$$M^2 + \frac{\lambda_0}{8\pi} M = m_0^2 + \frac{1}{4\pi^2}\lambda_0 \Lambda$$

(Λ: momentum cutoff)

- Flow of $\phi_i(x)$

$$\partial_t \varphi_i(t,x) = \partial_\mu \partial_\mu \varphi_i(t,x), \qquad \varphi_i(t=0,x) = \phi_i(x)$$

3D $O(N)$ linear σ-model at large N

- Dimensionless operators

$$\mathcal{O}_1(t,x) \equiv -\frac{N\left[\varphi_i(t,x)\varphi_i(t,x)\right]^2}{\langle\varphi_j(t,x)\varphi_j(t,x)\rangle^2} + (N+2),$$

$$\mathcal{O}_2(t,x) \equiv \frac{4t\partial_\mu\varphi_i(t,x)\partial_\mu\varphi_i(t,x)}{(2\pi)^{1/2}\langle\varphi_j(t,x)\varphi_j(t,x)\rangle} - \frac{1}{(2\pi)^{1/2}}$$

- At the leading order of the large-N expansion,

$$\begin{aligned}\langle\mathcal{O}_1(t)\rangle &= \lambda_0 t^{1/2}\left[\int\frac{d^3p}{(2\pi)^3}\frac{e^{-2p^2}}{p^2+M^2t}\right]^{-2} \\ &\quad\times\prod_{i=1}^{4}\left(\int\frac{d^3p_i}{(2\pi)^3}\frac{e^{-p_i^2}}{p_i^2+M^2t}\right)(2\pi)^3\delta(p_1+p_2+p_3+p_4) \\ &\quad\times\left[1+\frac{\lambda_0 t^{1/2}}{8\pi}\frac{1}{\sqrt{(p_1+p_2)^2}}\arctan\left(\frac{1}{2}\sqrt{\frac{(p_1+p_2)^2}{M^2t}}\right)\right]^{-1}\end{aligned}$$

$$\langle\mathcal{O}_2(t)\rangle = \frac{1}{8\pi^2}\left(\int\frac{d^3p}{(2\pi)^3}\frac{e^{-2p^2}}{p^2+M^2t}\right) - \left(\frac{8}{\pi}\right)^{1/2}M^2t - \frac{1}{(2\pi)^{1/2}}$$

3D $O(N)$ linear σ-model at large N

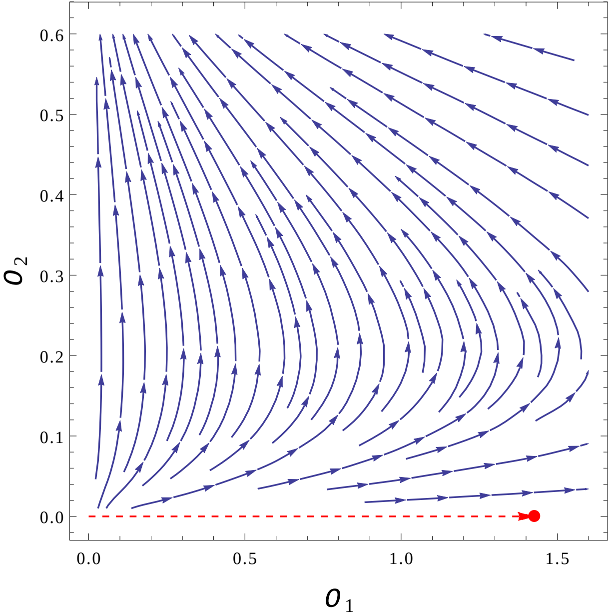
- Asymptotic behaviors:

$$
\begin{aligned}
\langle \mathcal{O}_1(t) \rangle &\stackrel{t\to 0}{\to} K\lambda_0 t^{1/2} \\
&\stackrel{t\to\infty}{\to} \frac{1}{(4\pi)^{3/2}} \frac{\lambda_0}{M} \left(1 + \frac{1}{16\pi} \frac{\lambda_0}{M} \right) \frac{1}{M^3 t^{3/2}}, \\
\langle \mathcal{O}_2(t) \rangle &\stackrel{t\to 0}{\to} M t^{1/2} \\
&\stackrel{t\to\infty}{\to} \left(\frac{2}{\pi} \right)^{1/2} - \frac{3}{(8\pi)^{1/2}} \frac{1}{M^2 t},
\end{aligned}
$$

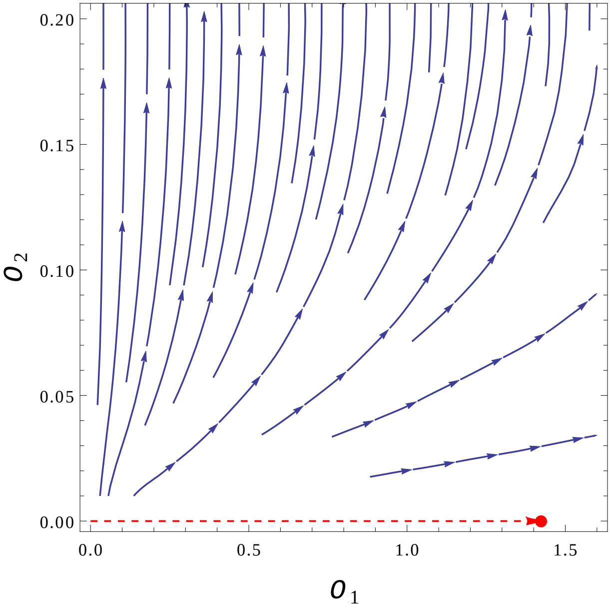
  where $K \simeq 0.289432$

- Set $M \to 0$ first; $\langle \mathcal{O}_1(t) \rangle \stackrel{t\to 0}{\to} K\lambda_0 t^{1/2}$, $\langle \mathcal{O}_1(t) \rangle \stackrel{t\to\infty}{\to} 1.42596...$, and $\langle \mathcal{O}_2(t) \rangle \equiv 0$

3D $O(N)$ linear σ-model



3D $O(N)$ linear σ-model



Summary

- Gradient flow possesses a simple renormalization property
- Many interesting applications
  - Construction of EMT, supercurrent, ...
  - Non-perturbative running coupling
  - ...
- Gradient flow can be identified with the renormalization group

- Further applications of the gradient flow?
- Relation with the holographic renormalization group?
- What is the gradient flow for QFT?