

Perturbative ambiguities and resurgence in circle-compactified theories

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- K. Ishikawa, O.M., K. Shibata and H. Suzuki, PTEP **2020** (2020) 063B02 [arXiv:2001.07302 [hep-th]]
- O.M. and H. Takaura, PLB **807** (2020) 135570 [arXiv:2003.04759 [hep-th]]
- M. Ashie, O.M., H. Suzuki and H. Takaura, PTEP **2020** (2020) 093B02 [arXiv:2005.07407 [hep-th]]

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1 Introduction

- Factorial growth in QFT
- Resurgence structure in $\mathbb{R}^{d-1} \times S^1$

2 PFD-type ambiguity on circle compactification

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- “Renormalon precursor” and decompactification

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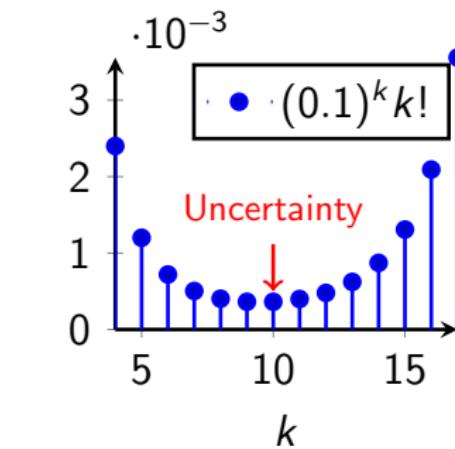
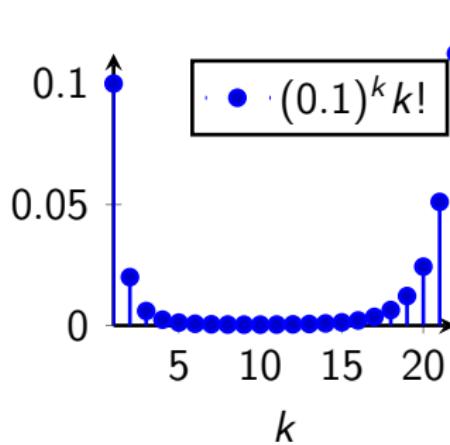
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4 Summary

Factorial growth of perturbation series

- In QM/QFT, perturbative expansions of observables are divergent series
- Typically,

$$F(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k, \quad c_k \sim k! \text{ at large } k.$$



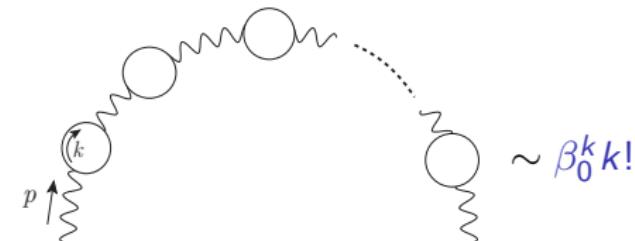
- Accuracy of perturbative predictions is limited. . .

Factorial growth in QFT

- Two sources of factorial growth of perturbative coefficients:
 - ➊ Proliferation of Feynman diagrams (PFD)

Ground state energy in QM	
Zeeman effect	$\sim (-1)^k (2k)!$
Stark effect	$\sim (2k)!$
$V(\phi) \sim \phi^3$	$\sim \Gamma(k + 1/2)$
$V(\phi) \sim \phi^4$	$\sim (-1)^k \Gamma(k + 1/2)$
Double well	$\sim k!$
Periodic cosine	$\sim k!$

- ➋ Renormalon [t Hooft '79]



(β_0 : one-loop coefficient of the beta function)

Borel resummation

- Borel resummation: summing divergent asymptotic series

$$f(g^2) \sim \sum_{k=0}^{\infty} f_k \left(\frac{g^2}{16\pi^2} \right)^{k+1} \quad \text{with } f_k \sim a^k k! \text{ as } k \rightarrow \infty$$

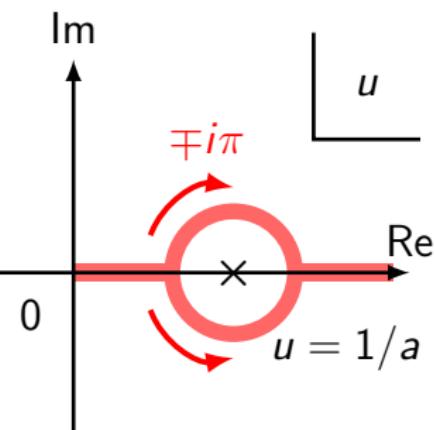
⇓ Borel transform

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k = \frac{1}{1 - au} \quad (\text{Pole singularity at } u = 1/a).$$

- The Borel sum is given by

$$f(g^2) \equiv \int_0^{\infty} du B(u) e^{-16\pi^2 u/g^2}.$$

- $a < 0$ (alternating series) → convergent
- $a > 0$ → ill-defined due to the pole
⇒ Imaginary ambiguity $\sim \pm e^{-16\pi^2/(ag^2)}$



Resurgence theory and renormalon problem

- Perturbative ambiguities (Borel resummation)

$$\text{PFD : } \sum_k k! \left(g^2/16\pi^2\right)^k \Rightarrow \delta \sim \exp(-16\pi^2/g^2)$$

$$\text{Renormalon : } \sum_k \beta_0^k k! \left(g^2/16\pi^2\right)^k \Rightarrow \delta \sim \exp[-16\pi^2/(\beta_0 g^2)]$$

- Resurgence structure [Bogomolny '80, Zinn-Justin '81]

$$\delta_{\text{PFD}} \sim \exp(-16\pi^2/g^2) = \exp(-2S_I)$$

\Updownarrow Cancellation

$$\delta_{\text{Instanton}} \sim \exp(-2S_I) \quad (\text{Instanton action } S_I = 8\pi^2/g^2)$$

- But, what cancels the renormalon ambiguity $\delta_{\text{Renormalon}}$?
 - Non-trivial configuration with $S = S_I/\beta_0$ ($\delta_{\text{NP}} \sim e^{-2S_I/\beta_0}$)?
 - Not known (cf. cancellation between terms of OPE [David '82])
 - Higher-order perturbation theory?

Renormalon cancellation in $\mathbb{R}^{d-1} \times S^1$?

- A possible candidate [Argyres–Ünsal '12, Dunne–Ünsal '12, ...]:
Bion [Ünsal '07] = fractional instanton/anti-instanton pair
on $\mathbb{R}^{d-1} \times S^1$ with \mathbb{Z}_N -twisted boundary conditions (BC)

$$\underbrace{\varphi^A(x, x_d + 2\pi R)}_{N\text{-component field}} = e^{2\pi i m_A R} \varphi^A(x, x_d), \quad m_A R = \begin{cases} A/N & \text{for } A \neq N \\ 0 & \text{for } A = N \end{cases}$$

- $S_B = S_I / N \leftarrow$ similar N dependence! ($\beta_0 = \frac{11}{3} N$ for $SU(N)$ GT)
- Resurgence on S^1 compactified spacetime?

	PT	Semi-classical object
\mathbb{R}^4	$\delta_{\text{Renormalon}} \sim e^{-16\pi^2/(\beta_0 g^2)}$?
$\mathbb{R}^3 \times S^1$	$\underline{\delta_{\text{Renormalon}} ?}$	$\delta_{\text{Bion}} \sim e^{-2S_B} = e^{-2S_I / N}$

► QFT on $\mathbb{R}^4 \stackrel{\text{Def.}}{=} \text{QFT on } \mathbb{R}^3 \times S^1 \text{ in } R \rightarrow \infty$

Resurgence structure in $\mathbb{R}^{d-1} \times S^1$

- *No renormalons* in $SU(2)$ and $SU(3)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$
[Anber–Sulejmanpasic '14]
- Questions:
 - ① What cancels the *bion ambiguity*? (PFD?)
 - ② Are there no renormalons in other theories? ($SU(\forall N)$?)
 - ③ How does $\delta_{\text{Renormalon}}$ on \mathbb{R}^4 emerge in $R \rightarrow \infty$?
- Answers [O.M.–Takaura, Ashie–O.M.–Suzuki–Takaura]:
 - ① Enhancement of PFD due to \mathbb{Z}_N twisted BC: $k! \rightarrow N^k k!$

	PT	Semi-classical object
\mathbb{R}^4	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{Instanton}} \sim e^{-2S_I}$
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{Bion}} \sim e^{-2S_I/N}$

- *No renormalons* in $SU(\forall N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$
- “Renormalon precursor” $\xrightarrow{R \rightarrow \infty}$ Renormalon on \mathbb{R}^4
- Helpful in giving a unified understanding on resurgence in QFT

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Enhancement of PFD ambiguity

- Enhanced PFD-type ambiguities cancel bion ambiguities

$$\text{PFD: } k! \Rightarrow N^k k!$$

$$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2} \Rightarrow \delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)} \leftrightarrow \delta_{\text{bion}}|_{\mathbb{R}^3 \times S^1}$$

- Mechanism (\sim Linde problem ['80] in finite-temperature QFT):
 - S^1 compactification \rightarrow IR divergences

$$\int d^d p \dots = \text{finite} \rightarrow \int d^{d-1} p \frac{1}{2\pi R} \sum_{p_d \in \mathbb{Z}/R} \dots = \infty \text{ at IR}$$

- twisted BC \rightarrow twist angles as IR regulators

$$\frac{1}{p^2} \rightarrow \frac{1}{\mathbf{p}^2 + (p_d + m_A)^2} \xrightarrow[p_d=0, A=1]{\text{Amplitude}} \left(\frac{g^2}{m_A R} \right)^k = \underbrace{(Ng^2)^k}_{\text{Enhancement!}}$$

- Let us study the $\mathbb{C}P^N$ model on $\mathbb{R} \times S^1$
 - Bion calculus [Fujimori-Kamata-Misumi-Nitta-Sakai '18]

2-dimensional $\mathbb{C}P^N$ model with twisted BC

- 2D $\mathbb{C}P^{N-1}$ model with \mathbb{Z}_N -twisted BC ($A = 1, 2, \dots, N$)

$$S = \frac{1}{g^2} \int d^2x \left[\partial_\mu \bar{z}^A \partial_\mu z^A - j_\mu j_\mu + f(\bar{z}^A z^A - 1) \right]$$

where $j_\mu = (1/2i)\bar{z}^A \overleftrightarrow{\partial}_\mu z^A$, f is an auxiliary field ($\bar{z}^A z^A = 1$)

- # of vacuum bubble diagrams, T_k ($j_\mu \rightarrow \bar{z}z$, propagator $\rightarrow 1$)

$$T_k = \frac{1}{(2k)!} \left(\frac{\delta}{\delta \bar{z}} \frac{\delta}{\delta z} \right)^{2k} \frac{1}{k!} [(\bar{z}z)^2]^k \sim 4^k \Gamma(k+1/2)$$

- Amplitude of $(k+1)$ -loop connected diagram

$$\xrightarrow{p_2=0, A=1} \frac{V_2(g^2)^k}{(2\pi R)^{k+1}} \int \left(\prod_{i=1}^{k+1} \frac{dp_{i,1}}{2\pi} \right) \frac{F^{2k}(p_{i,1}, m_A)}{\prod_{i=1}^{2k} [q_{i,1}^2 + m_A^2 + f_0]}$$

(F^{2k} : $2k$ th-order polynomial, q : linear combination of $\{p_i\}$, $f_0 = \langle f \rangle$)

- IR divergence in massless limit $m_A^2 + f_0 \rightarrow 0$ ($\int d^2p \rightarrow \int dp$)

IR structure and enhancement mechanism

- $m_A^2 + f_0 = 1/(NR)^2 + f_0$ works as an IR regulator

- ▶ $m_A^2 \gg f_0$

$$\sim \frac{V_2}{R^2} \frac{1}{(m_A R)^{k-1}} \left(\frac{g^2}{4\pi}\right)^k \xrightarrow{A=1} \frac{V_2}{R^2} \frac{1}{N} \left(\frac{Ng^2}{4\pi}\right)^k \quad \text{Enhancement!}$$

- ▶ $m_A^2 \ll f_0$

$$\sim \frac{V_2}{R^2} \sum_{\alpha \geq 0} \frac{(m_A R)^\alpha}{(f_0 R^2)^{(k+\alpha-1)/2}} \left(\frac{g^2}{4\pi}\right)^k \quad \text{Enhancement}$$

- Dependence on $NR\Lambda$ ($\Lambda = \mu e^{-2\pi/(\beta_0 g^2)}$: dynamical scale)

- ▶ $NR\Lambda \ll 1$ (Bion calculus is valid)

$$\sqrt{f_0}R \sim \frac{g^2}{4\pi} \Rightarrow [m_A^2 = \mathcal{O}(g^0)] \gg [f_0 = \mathcal{O}(g^4)] \quad \text{Enhancement!}$$

- ▶ $NR\Lambda \gg 1$ ("Large N ")

$$f_0 \sim \Lambda^2 \quad \Rightarrow \quad \frac{m_A^2}{f_0} = \frac{1}{(NR\Lambda)^2} \ll 1 \quad \text{Enhancement}$$

Discussions on enhancement mechanism

	PT	Semi-classical object
\mathbb{R}^4	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{Instanton}} \sim e^{-2S_I}$
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{Bion}} \sim e^{-2S_I/N}$

- Enhancement phenomenon is consistent with bion
- Borel singularities at $m_A R = A/N = \underbrace{1/N, 2/N, \dots}_{\text{1-bion}} \underbrace{\dots}_{\text{2-bion}}$
- Disconnected diagram (n connected) is suppressed by $1/N^{n-1}$
 - ▶ # of connected diagrams $\sim \ln [\sum_k T_k(g^2)^k]$
- Sum over flavor indices gives rise to further N factor
- Vacuum energy of 2D SUSY $\mathbb{C}P^N$ model (following slides)
 - ▶ $N R \Lambda \gg 1$ [Ishikawa–O.M.–Shibata–Suzuki]
 - ▶ $N R \Lambda \ll 1$ [Fujimori–Kamata–Misumi–Nitta–Sakai '18]

Vacuum energy of 2D SUSY $\mathbb{C}P^{N-1}$ model

- 2D SUSY $\mathbb{C}P^{N-1}$ model with SUSY breaking term

$$S = \frac{N}{\lambda} \int d^2x \left[-\bar{z}^A D_\mu D_\mu z^A + \bar{\sigma} \sigma + \bar{\chi}^A (\not{D} + \bar{\sigma} P_+ + \sigma P_-) \chi^A \right] \\ + \int d^2x \frac{\delta\epsilon}{\pi R} \sum_A m_A \left(\bar{z}^A z^A - \frac{1}{N} \right)$$

where $D_\mu = \partial_\mu + i\gamma_5$, $\gamma_5 = -i\gamma_x\gamma_y$, $P_\pm = (1 \pm \gamma_5)/2$,
and we impose $\bar{z}^A z^A = 1$, $\bar{\chi}^A \chi^A = 0$ and $\bar{z}^A \chi^A = 0$.

- Vacuum energy as a function of $\delta\epsilon$

$$E(\delta\epsilon) = \underbrace{E^{(0)}}_{=0} + E^{(1)} \delta\epsilon + E^{(2)} \delta\epsilon^2 + \dots$$

- Vacuum energy at $NR\Lambda \ll 1$ [Fujimori-Kamata-Misumi-Nitta-Sakai]

$$E_{\text{Bion}} = 2\Lambda \sum_{b=1}^{N-1} (-1)^b \frac{b}{(b!)^2} (NR\Lambda)^{2b-1} \\ \times \left\{ \delta\epsilon + [-2\gamma_E - 2 \ln(4\pi b/\lambda_R) \mp \pi i] \delta\epsilon^2 + \dots \right\}$$

Vacuum energy in terms of $1/N$ expansion

- Vacuum energy ($N\Lambda \gg 1$)
[Ishikawa–O.M.–Shibata–Suzuki]

$$\begin{aligned} E^{(1)}\delta\epsilon &= 2\delta\epsilon \sum_A m_A \left\langle \bar{z}^A z^A - 1/N \right\rangle_{\delta\epsilon=0} \\ &= 0 \cdot N^0 + 0 \cdot N^{-1} + \mathcal{O}(N^{-2}) \end{aligned}$$

$$\begin{aligned} RE^{(2)}\delta\epsilon^2 &= -\frac{\delta\epsilon^2}{\pi} \int d^2x \sum_{A,B} m_A m_B \\ &\quad \times \left\langle \bar{z}^A z^A(x) \bar{z}^B z^B(0) \right\rangle_{\delta\epsilon=0} \\ &= N^{-1}(\lambda_R \delta\epsilon_R)^2 (\Lambda R)^{-2} F(\Lambda R) \\ &\quad + N^{-2}(\lambda_R \delta\epsilon_R)^2 (\Lambda R)^{-3} G(\Lambda R) + O(N^{-3}) \end{aligned}$$

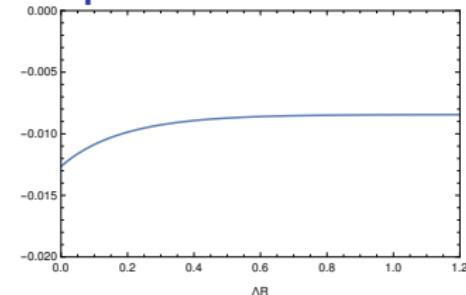


Figure: F

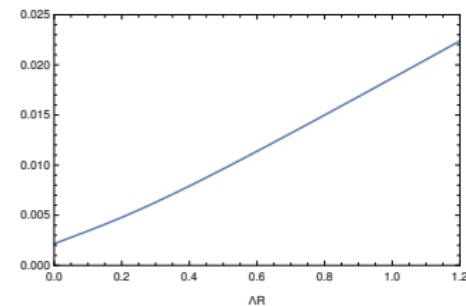


Figure: G

- $RE^{(2)} \rightarrow \infty$ as $\Lambda \rightarrow 0$; no well-defined weak coupling expansion

N -enhancement versus $1/\Lambda$ (m_A^2 vs f_0)

- Vacuum energy for $NR\Lambda \gg 1$ and $NR\Lambda \ll 1$

$$E_{\text{Large } N} \sim N^{-1}(\Lambda R)^{-2}\delta\epsilon^2 \quad \Rightarrow f_0 = \Lambda^2 \text{ in denominator}$$

\uparrow

$\downarrow (m_A^2 \text{ vs } f_0)$

$$\text{Im } E_{\text{1-bion}} \sim \pm N(\Lambda R)^2\delta\epsilon^2 \quad \Leftarrow \text{enhancement of } N$$

- cf. ground state energy of SUSY $\mathbb{C}P^1$ QM
[Fujimori–Kamata–Misumi–Nitta–Sakai '17]

$$E = m \sum_k A_k (g^2/m)^k$$

- “Large N ” means $NR\Lambda \gg 1$
 - ▶ “ $1/N$ expansion” $\equiv 1/(NR\Lambda)$ expansion
 - ▶ Higher orders in $1/N$ depend on negative powers of Λ
 - ▶ $f_0 = \Lambda^2$ dominates the IR structure

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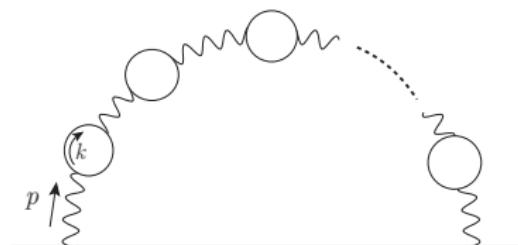
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(Definition of) Renormalon ambiguity on \mathbb{R}^4

- Renormalon on \mathbb{R}^4 can appear from



$$\sim g^2(\mu^2) \sum_k \int \frac{d^4 p}{(2\pi)^4} (p^2)^\alpha \Pi(p^2)^k,$$

$$\Pi(p^2) = \beta_0 \frac{g^2(\mu^2)}{16\pi^2} \ln \frac{e^C \mu^2}{p^2}$$

(α, C : constants)

- As $p \rightarrow 0$, logarithmic factor in vacuum polarization is crucial

$$\int_p (p^2)^\alpha (\ln p^2)^k \xrightarrow{p \sim 0} k! \quad \stackrel{\text{Borel}}{\Rightarrow} \quad \pm i\pi \frac{1}{\beta_0} (e^C \Lambda^2)^{\alpha+2}$$

where dynamical scale $\Lambda^2 = \mu^2 e^{-16\pi^2/(\beta_0 g^2)}$

► Borel transform \rightarrow momentum integral

$$B(u) = \int d^4 p (p^2)^\alpha \left(\frac{e^C \mu^2}{p^2} \right)^u \xrightarrow{u \geq \alpha+2} \text{IR divergence}$$

$SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$

- Renormalon analysis for $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$
($N = 2, 3$ [Anber-Sulejmanbasic '14], $\forall N$ [Ashie-O.M.-Suzuki-Takaura '20])
- $SU(N)$ gauge theory with n_W -flavor adjoint Weyl fermions

$$S = -\frac{1}{2g_0^2} \int d^4x \operatorname{tr}(F_{\mu\nu}F_{\mu\nu}) - 2 \int d^4x \operatorname{tr}\bar{\psi}\gamma_\mu(\partial_\mu + [A_\mu, \psi])$$

- \mathbb{Z}_N twisted BC for adjoint representation

$$\psi(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega\psi(x)\Omega^{-1}$$

$$A_\mu(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega A_\mu(x)\Omega^{-1}$$

where $\Omega = e^{i\pi \frac{N+1}{N}} \operatorname{diag}(e^{-i\frac{2\pi}{N}}, e^{-i\frac{2\pi}{N}2}, \dots, e^{-i\frac{2\pi}{N}N})$.

► Equivalently, $e^{2\pi R A_3^{(0)}} = \Omega$ under periodic BC

- $\underbrace{\text{Cartan part of } A_\mu}_{N-1}$: “**photon**”, the others of A_μ : $\underbrace{\text{massless } U(1)}_{m \sim |p_3| \geq 1/(NR)}$, the others of A_μ : “**W-boson**”
- W-boson cannot give rise to renormalon; consider $U(1)^{N-1}$

1-loop effective action

- 1-loop effective action of photon, A_μ^ℓ ($\ell = 1, 2, \dots, N - 1$)

$$\frac{1}{2g^2} \int d^4x d^4y A_\mu^\ell(x) A_\nu^r(y) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} e^{-ip(x-y)} \\ \times \left[p^2 \mathcal{P}_{\mu\nu}^L (\delta_{\ell r} - L_{\ell r}) + p^2 \mathcal{P}_{\mu\nu}^T (\delta_{\ell r} - T_{\ell r}) + \delta_{\ell r} \xi p_\mu p_\nu \right]$$

where projection operators, \mathcal{P}^L in S_1 , \mathcal{P}^T in \mathbb{R}^3

- Vacuum polarization

$$L_{\ell r} = \frac{\beta_0 g^2}{16\pi^2} \left[\delta_{\ell r} \ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) + \underbrace{f_0(p^2 R^2)_{\ell r} - f_2(p^2 R^2)_{\ell r}}_{\text{finite volume}} \right],$$

$$T_{\ell r} = \frac{\beta_0 g^2}{16\pi^2} \left[\delta_{\ell r} \ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) + \underbrace{f_0(p^2 R^2)_{\ell r}}_{\text{finite volume}} \right]$$

- ▶ Large- β_0 approximation [Beneke–Braun '94]: Radiative corrections of A_μ are included by $-\frac{2}{3}n_W \rightarrow \beta_0 = (\frac{11}{3} - \frac{2}{3}n_W)N$
- If $\Pi(p^2 \rightarrow 0) \sim \ln p^2$, there exists the renormalon.

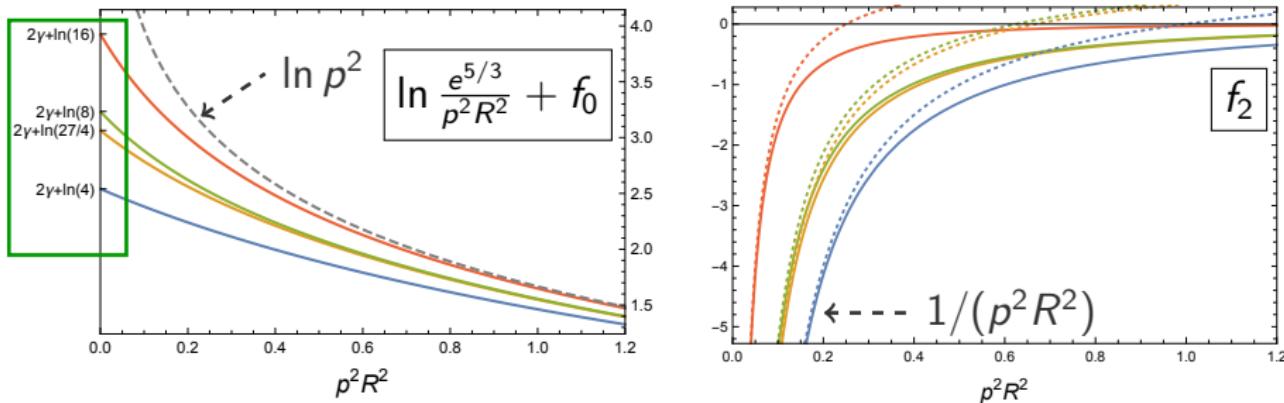
Absence of renormalons on $\mathbb{R}^3 \times S^1$

- Asymptotic behavior of L and T in $SU(\mathbb{V}N)$ QCD(adj.)

$$L_{\ell r}(p^2 R^2) = -\frac{m_{sc}^2}{p^2} \delta_{\ell r} + \text{const.} + \mathcal{O}(p^2 R^2 \ln(p^2 R^2)),$$

$$T_{\ell r}(p^2 R^2) = \text{const.} + \mathcal{O}(p^2 R^2 \ln(p^2 R^2)), \quad m_{sc}^2 \propto \frac{\beta_0 g^2}{16\pi^2 R^2}$$

- $N = 2 \& 3$: identical to [Anber-Sulejmanpasic] ($\beta_0 \leftrightarrow \frac{2}{3}(1 - n_W)$)



- No logarithmic factor \rightarrow no renormalons (no factorial growth)

Twisted BC, large N , and volume (in)dependence

- Identity for KK sum with twisted BC:

$$\begin{aligned} \frac{1}{2\pi R} \sum_{p_d \in \mathbb{Z}/R} F(\mathbf{p}, p_d + m_A) &= \sum_{n \in \mathbb{Z}} \int \frac{dp_d}{2\pi} \underbrace{e^{ip_d 2\pi R n}}_{F(\mathbf{p}, p_d + m_A)} \\ &= \sum_{n \in \mathbb{Z}} \int \frac{dp_d}{2\pi} \underbrace{e^{i(p_d - m_A) 2\pi R n}}_{F(\mathbf{p}, p_d)} \end{aligned}$$

- Noting that $\sum_A e^{2\pi i m_A R n} \times 1 = \begin{cases} N, & \text{for } n = 0 \bmod N \text{ (leading),} \\ 0, & \text{for } n \neq 0 \bmod N, \end{cases}$

the effective radius becomes NR :

$$\sum_{\substack{\text{over } A \\ n' \in \mathbb{Z}}} N \int \frac{dp_d}{2\pi} \underbrace{e^{ip_d 2\pi NR n'}}_{F(\mathbf{p}, p_d)} = \frac{N}{2\pi R} \sum_{p_d \in \mathbb{Z}/(NR)} F(\mathbf{p}, p_d)$$

- Assume “large N limit” as $NR \Lambda \rightarrow \infty \Rightarrow$ Decompactification
- Volume dependent: $\underbrace{\text{other } m_A \text{ dependence,}}_{E \text{ in SUSY } \mathbb{C}P^N} \underbrace{\text{twisted loop momenta}}_{\text{e.g., } U(1) \text{ gauge field}}$

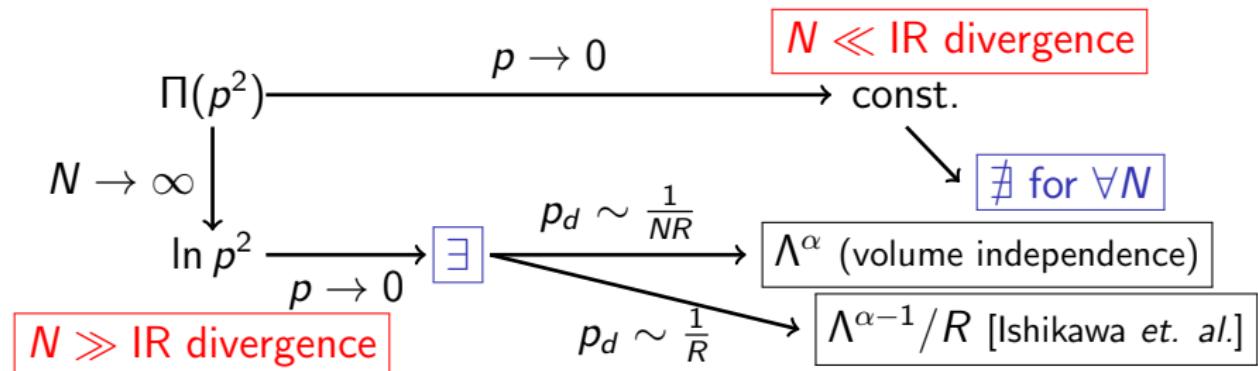
Subtlety of large N limit on $\mathbb{R}^3 \times S^1$

- Subtlety of naive (apparent) $1/N$ expansion: e.g.,

$$|f_0| \lesssim \frac{1}{N} \frac{1}{(p^2 R^2)^{3/2}} \quad \Rightarrow \quad \ln p^2 + f_0(p^2) \stackrel{\text{finite } p}{\sim} \ln p^2 + \mathcal{O}(1/N)$$

$\Pi(p^2) \xrightarrow{N \rightarrow \infty} \ln p^2$; \exists renormalons [Ashie-O.M.-Suzuki-Takaura-Takeuchi]

- Does there exist renormalons in “ $N \rightarrow \infty$ ”?



- Volume independence: “ $N \gg \infty$ ” and twisted loop momentum
- Definition of “renormalon” \longleftrightarrow $1/N$ expansion Incompatible?

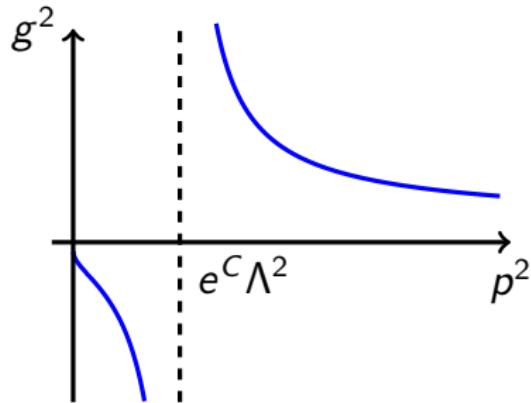
Renormalon from momentum integration on \mathbb{R}^4

- $\delta_{\text{renormalon}}|_{\mathbb{R}^3 \times S^1} = 0$ for $\forall R$, but $\delta_{\text{renormalon}}|_{\mathbb{R}^4} = e^{-16\pi^2/(\beta_0 g^2)}$
- How does renormalon on \mathbb{R}^4 emerge under $R \rightarrow \infty$?
- Reconsider renormalon diagram; resumming the geometric series,

$$\begin{aligned} & g^2(\mu^2) \int \frac{d^4 p}{(2\pi)^4} \frac{(p^2)^\alpha}{1 - \Pi(p^2)} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{(p^2)^\alpha}{\frac{\beta_0}{16\pi^2} \ln \frac{p^2}{e^C \Lambda^2}} = \int \frac{d^4 p}{(2\pi)^4} (p^2)^\alpha g^2(p^2 e^{-C}) \end{aligned}$$

- Pole singularity at $p^2 = e^C \Lambda^2$
 - Contour deformation in $p^2 \in \mathbb{C}$
 - Renormalon $\sim \pm i\pi \frac{1}{\beta_0} (e^C \Lambda^2)^{\alpha+2}$

- ▶ On \mathbb{R}^4 , ambiguity of $\int d^4 p$
= ambiguity in Borel sum



“Renormalon precursor” and decompactification

- In $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$,

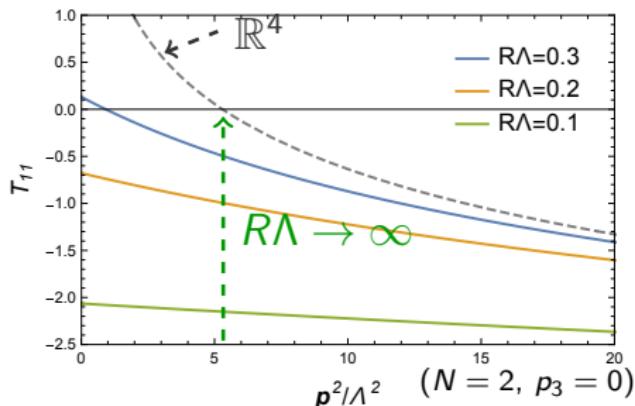
$$\begin{aligned} g^2(\mu^2) \int \frac{d^3 p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p_3 \in \mathbb{Z}/R} \frac{(p^2)^\alpha}{1 - \Pi(p^2)} \\ = \int \frac{d^3 p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p_3 \in \mathbb{Z}/R} \frac{\frac{16\pi^2}{\beta_0} (p^2)^\alpha}{\ln \frac{p^2}{e^{5/3}\Lambda^2} + f_{\text{finite}}} \end{aligned}$$

- “Renormalon precursor” comes from $\ln \frac{p^2}{e^{5/3}\Lambda^2} + f_{\text{finite}} = 0$

► For $p_3 \lesssim \Lambda$, it can exist
(no Borel singularities)

► $R \rightarrow \infty$ $f_{\text{finite}} \rightarrow 0$
(pole| $\mathbb{R}^3 \times S^1$ → pole| \mathbb{R}^4)
and $\sum_{p_3} \rightarrow \int dp_3$

Renormalon on \mathbb{R}^4 emerge!



Backup: toy model (large N volume independence)

- Difficult to compute the renormalon precursor directly
- A simplification by large N volume independence: For $|p_3| < e^{5/6}\Lambda$, integrand possesses a simple pole

$$\begin{aligned} p^2 \sim e^{5/3}\Lambda^2 \rightarrow & \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{|p_3| < e^{5/6}\Lambda} \frac{16\pi^2}{\beta_0} (p^2)^\alpha \frac{e^{5/3}\Lambda^2}{p^2 - e^{5/3}\Lambda^2} \\ & \sim \pm i\pi \frac{4}{\beta_0} \frac{(e^{5/3}\Lambda^2)^{\alpha+2}}{2\pi} \frac{1}{e^{5/6}R\Lambda} \sum_{|n| < e^{5/6}R\Lambda} \sqrt{1 - \frac{n^2}{(e^{5/6}R\Lambda)^2}} \end{aligned}$$

- Decompactification $R\Lambda \rightarrow \infty$

$$\rightarrow \pm i\pi \frac{4}{\beta_0} \frac{(e^{5/3}\Lambda^2)^{\alpha+2}}{2\pi} \int_{-1}^1 dx \sqrt{1-x^2} = \text{Renormalon on } \mathbb{R}^4$$

- $[1 - \Pi(p \rightarrow 0)] \times [1 - \Pi(p \rightarrow \infty)] < 0$: \exists Renormalon precursor

Contents

1 Introduction

- Factorial growth in QFT
- Resurgence structure in $\mathbb{R}^{d-1} \times S^1$

2 PFD-type ambiguity on circle compactification

- Enhancement mechanism and bion
- Vacuum energy of SUSY $\mathbb{C}P^N$ model

3 Renormalon on circle compactification

- $\#$ Renormalon ambiguities
- Subtlety of large N limit on circle compactification
- “Renormalon precursor” and decompactification

4 Summary

Summary

- (Our) Current understanding on resurgence structure in QFT

	PT	Semi-classical object
\mathbb{R}^4	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{instanton}} \sim e^{-2S_I}$
\mathbb{R}^4	$\delta_{\text{renormalon}} \sim e^{-16\pi^2/(\beta_0 g^2)}$?
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{bion}} \sim e^{-2S_I/N}$
$\mathbb{R}^3 \times S^1$	$\delta_{\text{renormalon}} = 0$	No

- Renormalon precursor smoothly reduces renormalon in $R \rightarrow \infty$
 - ▶ Not associated with Borel singularities
 - ▶ Analytic continuation of geometric series
- There remains renormalon puzzle
 - ▶ What definition do you prefer?
- Helpful in giving a unified understanding on resurgence