

Lattice description of generalized symmetries and topological phenomena

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- 1 Introduction
 - Symmetry and 't Hooft anomaly
 - Generalized global symmetry
- 2 Review on topology of lattice gauge fields [Lüscher]
- 3 Fractionality of topology in lattice $SU(N)/\mathbb{Z}_N$ gauge theory
[Abe–OM–Suzuki, Abe–OM–Onoda–Suzuki–Tanizaki]
- 4 Higher-group structure in lattice gauge theory
[Kan–OM–Nagoya–Wada, Abe–OM–Onoda]
 - Higher-group symmetry under modification of instanton sum
- 5 Magnetic monopole and the Witten effect
[Abe–OM–Onoda–Suzuki–Tanizaki]
 - Magnetic operators in lattice 2D scalar theory
- 6 Summary

1 Introduction

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6 Summary

Symmetry

- Symmetry: fundamental tool in physics
 - ▶ Universal applications to high energy physics, condensed matter physics and mathematics
- Noether theorem and conservation law

$$\text{Sym} : \phi \mapsto \phi'; S(\phi) = S(\phi'), \quad \partial_\mu j^\mu = 0$$

$$\text{Charge } Q \equiv \int j^0 d^{D-1}x$$

- Structure in the nature: Lorentz symmetry, *CPT* theorem, Gauge invariance, Internal symmetries

Symmetry

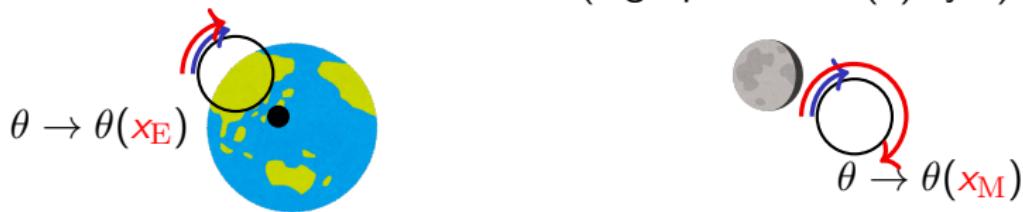
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- Structure in the nature: Lorentz symmetry, *CPT* theorem, Gauge invariance, Internal symmetries
 - ▶ Fundamental forces in the nature (e.g., photon: $U(1)$ sym)



't Hooft anomaly matching

- More aspects of Symmetry
 - ▶ Landau theory: phase transition (or vacuum structure) from viewpoint of symmetry
 - ▶ Symmetry breaking: quantum anomaly, spontaneous breaking
- 't Hooft anomaly matching for strongly coupled theories [79]
 - ▶ Assume global symmetry G in system
 - ▶ Introduce background gauge field A assoc. G (by gauging G)

$$\begin{aligned}\mathcal{Z}[A] &= \int \mathcal{D}\{\phi\} e^{-S(\{\phi\}, A)} \\ &\stackrel{?}{=} \mathcal{Z}[A^g] = \dots = e^{A[A,g]} \mathcal{Z}[A]\end{aligned}$$

$e^A \neq 1 \xrightarrow{\text{Anomalous}}$ 't Hooft anomaly which is invariant at any energy scale (renormalization group inv.)

- ▶ Restriction on low-energy dynamics: SSB, phase structure, SPT

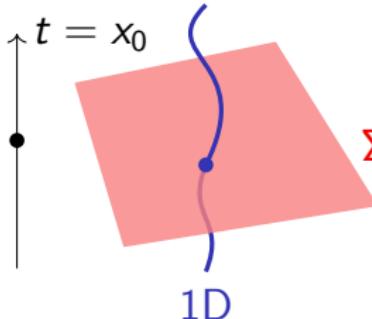
Summary: Recent generalization of symmetry

- Generalized global symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]
 - ▶ Coupled with topological field theory
 - ★ Changing topological structure without changing local dynamics [Kapustin–Seiberg '14]:
 - ★ Nontrivial information by using generalized 't Hooft anomaly matching
- Basic property: fractionality of topological charge
 - ▶ Usually, topo. charge $Q \sim \int F\tilde{F} \in \mathbb{Z}$ under topo. sectors with integer instanton # ($\pi_3(SU(N)) \in \mathbb{Z}$)
 - ★ θ term: θQ (strong CP problem, axion physics, sign problem)
 - ▶ Discrete higher-form sym \rightarrow background gauge field B :
 $Q \sim \int B\tilde{B} \in \frac{1}{N}\mathbb{Z}$, 't Hooft anomaly $\mathcal{Z}_{\theta+2\pi}[B] = e^{-2\pi i Q} \mathcal{Z}_\theta[B]$
[Kapustin–Thorngren '13, Gaiotto–Kapustin–Komargodski–Seiberg '17]
 - ▶ 't Hooft twisted boundary condition $Q \sim \int F\tilde{F} \in \frac{1}{N}\mathbb{Z}$
[van Baal '82] cf. [Edwards–Heller–Narayanan, de Forcrand–Jahn, Fodor–Holland–Kuti–Nógrádi–Schroeder, Kitano–Suyama–Yamada, Itou, ...]

global desc.
⇒

Generalization: higher-form symmetry

- (0-form) Symmetry



- ▶ Charge (codim 1)

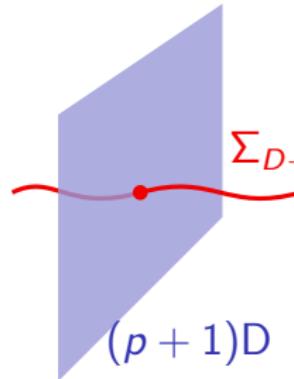
$$Q \equiv \int_{\Sigma_{D-1}} j_0 dx_1 \wedge \cdots \wedge dx_{D-1}$$

- ▶ Symmetry operator

$$U_\alpha(\Sigma_{D-1}) = e^{i\alpha Q}$$

Topological under deformation of Σ_{D-1}

- Higher-form symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]



- ▶ p -form symmetry $G^{[p]}$ (codim $p+1$)

$$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \quad U_\alpha(\Sigma) = e^{i\alpha Q}$$

- ▶ Transforming a "loop" operator $W(C)$

$$\begin{aligned} W(C) &\mapsto U(\Sigma)W(C) \\ &= e^{i\alpha \#(\Sigma, C)} W(C) \quad \text{w/ linking } \# \end{aligned}$$

Center symmetry in YM theory

- Lattice $SU(N)$ YM theory
 - ▶ link variable $U_\ell \in SU(N)$

- Center symmetry: $\mathbb{Z}_N^{[1]}$

$$e^{\frac{2\pi i}{N} k} \in \mathbb{Z}_N \subset SU(N); \quad U_\ell \mapsto e^{\frac{2\pi i}{N} k \#(\Sigma, \ell)} U_\ell$$

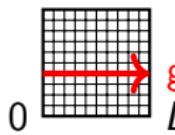
Intersection # of Σ & link ℓ ; $U_p \mapsto U_p$

- Gauging the center symmetry

$$S \sim \sum \text{Tr } e^{-\frac{2\pi i}{N} B_p} U_p \quad B_p: \text{2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N} \lambda_\ell} U_\ell$, $B_p \mapsto B_p + (d\lambda)_p$

- ▶ Recall 't Hooft twisted b.c. [79]: $U_{n+\hat{L}\hat{\nu}, \mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu}, \nu}$



$$g_{n+\hat{L}\hat{\nu}, \mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+\hat{\mu}, \nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

$$'t \text{ Hooft flux } z_{\mu\nu} = \sum B_p \bmod N$$

Aiming at transparent understanding

- Wise but *not transparent* understanding
 - ▶ Topological objects from **lattice** viewpoint as center sym
 - ▶ Formal discussion in **continuum** theory
 - ★ $\mathbb{Z}_N^{[q]}$ gauge field: **$U(1)$** field $B^{(q)}$
with constraint $NB^{(q)} = dB^{(q-1)}$ from charge- N Higgs
 - ▶ $Q \sim \frac{1}{N} \int B \wedge B$? $\xrightarrow[\text{cohomology op}]{\text{Swapping } \wedge w/}$ $\frac{1}{N} \int P_2(B) \sim -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N}$
 - ★ Global nature described by Čech cohomology (discrete group!)
 - ▶ Indicating mixed 't Hooft anomaly with chiral sym/ θ -periodicity

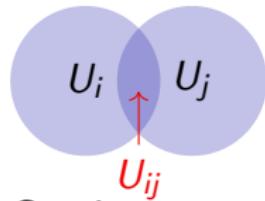
$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q} \mathcal{Z}_\theta[B_p] \quad \text{not } 2\pi \text{ periodic}$$

- Fully **regularized framework**: Lattice regularization
 - ▶ Lattice construction of fiber bundle and Q [**Lüscher '84**] (§2)
 - ▶ Coupled with *higher-form* lattice gauge fields (§3)
- Application to **higher-group symmetry** (§4),
How to construct **magnetic monopole** on lattice (§5)

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Principal fiber bundle

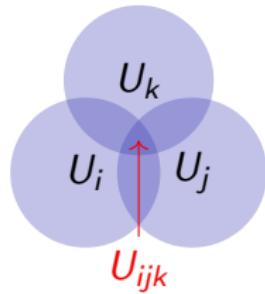
- Recall Dirac's discussion
 - ▶ Gauge fields cannot be defined globally in spacetime
 - ▶ Defined on each subspace: northern/southern hemisphere
- Manifold (spacetime) X ; open covering (patches) $\{U_i\}$
 - ▶ Gauge group G , gauge field a_i on U_i
 - ▶ Relation between a_i and a_j ?



Gauge transformation:

$$a_j = g_{ij}^{-1} a_i g_{ij} + g_{ij}^{-1} d g_{ij} \quad \text{at } U_{ij}$$

- ▶ Consistency condition for transition function g_{ij} :

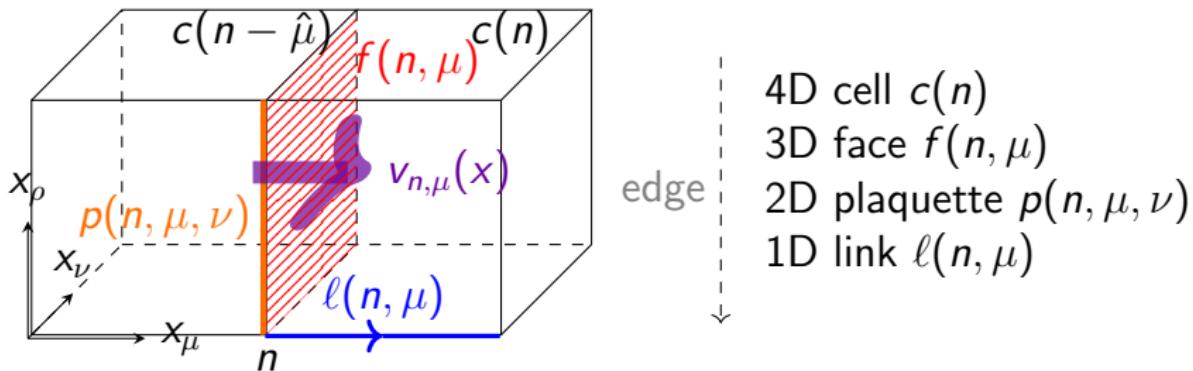


$$g_{ii} = \text{id}, \quad g_{ji} = g_{ij}^{-1}$$

Cocycle condition: $g_{ij} g_{jk} g_{ki} = 1 \quad \text{at } U_{ijk}$

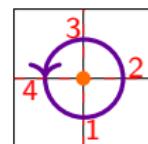
Bundle structure on lattice?

- No continuity for lattice fields?
 - ▶ Any configuration can be deformed continuously to others
 - ▶ We can observe topological structure even on lattice by Lüscher
- Setup/strategy: Lattice Λ divides X into 4D hypercubes



- ① Regard $\{c(n)\}$ as patches
- ② Define transition function $v_{n, \mu}(x)$ at $f(n, \mu)$ from data as U_ℓ

- ▶ Difficult to define it at $x \neq n$ s.t.
cocycle condition is kept intact



$$v_1 v_2 v_3^{-1} v_4^{-1} = 1 \quad \text{at } x \in p(n)$$

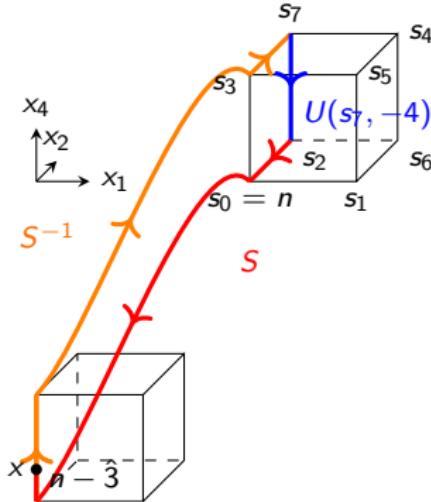
Construction of topo. sectors on lattice [Lüscher]

- Parallel transporter (interpolation):

$$U_\ell \rightarrow v_f(n) \rightarrow v_f(x) \Leftarrow \text{Cocycle}$$

$$v_{n,\mu}(n) = U(n - \hat{\mu}, \mu)$$

$$v_{n,\mu}(x) \equiv S_{n,\mu}^{n-\hat{\mu}}(x)^{-1} v_{n,\mu}(n) S_{n,\mu}^n(x)$$



- Topo. sectors on lattice so that $[Q \text{ in terms of } v_f(x)] \in \mathbb{Z}$

$$Q = \sum_{n \in \Lambda} q(n), \quad q(n) = -\frac{1}{24\pi^2} \sum_{\mu, \nu, \rho, \sigma} \epsilon_{\mu\nu\rho\sigma} \left\{ 3 \int_{p_n, \mu\nu} d^2x \operatorname{Tr} \left[(v_{n,\mu} \partial_\rho v_{n,\mu}^{-1}) (v_{n-\hat{\mu},\nu}^{-1} \partial_\sigma v_{n-\hat{\mu},\nu}) \right] \right. \\ \left. + \int_{f_n, \mu} d^3x \operatorname{Tr} \left[(v_{n,\mu}^{-1} \partial_\nu v_{n,\mu}) (v_{n,\mu}^{-1} \partial_\rho v_{n,\mu}) (v_{n,\mu}^{-1} \partial_\sigma v_{n,\mu}) \right] \right\}$$

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$$w^n(x) = U_{n,4}^{y_4} U_{n+y_4\hat{4},3}^{y_3} U_{n+y_4\hat{4}+y_3\hat{3},2}^{y_2} U_{n+y_4\hat{4}+y_3\hat{3}+y_2\hat{2},1}^{y_1}$$

$$f_{n,\mu}^m(x_\gamma) = (u_{s_3 s_0}^m)^{y\gamma} (u_{s_0 s_3}^m u_{s_3 s_7}^m u_{s_7 s_2}^m u_{s_2 s_0}^m)^{y\gamma} u_{s_0 s_2}^m (u_{s_2 s_7}^m)^{y\gamma},$$

$$g_{n,\mu}^m(x_\gamma) = (u_{s_5 s_1}^m)^{y\gamma} (u_{s_1 s_5}^m u_{s_5 s_4}^m u_{s_4 s_6}^m u_{s_6 s_1}^m)^{y\gamma} u_{s_1 s_6}^m (u_{s_6 s_4}^m)^{y\gamma},$$

$$h_{n,\mu}^m(x_\gamma) = (u_{s_3 s_0}^m)^{y\gamma} (u_{s_0 s_3}^m u_{s_3 s_5}^m u_{s_5 s_1}^m u_{s_1 s_0}^m)^{y\gamma} u_{s_0 s_1}^m (u_{s_1 s_5}^m)^{y\gamma},$$

$$k_{n,\mu}^m(x_\gamma) = (u_{s_7 s_2}^m)^{y\gamma} (u_{s_2 s_7}^m u_{s_7 s_4}^m u_{s_4 s_6}^m u_{s_6 s_2}^m)^{y\gamma} u_{s_2 s_6}^m (u_{s_6 s_4}^m)^{y\gamma},$$

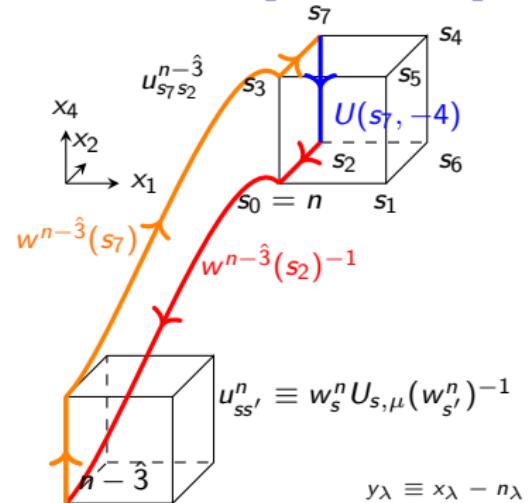
$$l_{n,\mu}^m(x_\beta, x_\gamma) = [f_{n,\mu}^m(x_\gamma)]^{-1} {}^{y\beta} [f_{n,\mu}^m(x_\gamma) k_{n,\mu}^m(x_\gamma) g_{n,\mu}^m(x_\gamma)]^{-1} h_{n,\mu}^m(x_\gamma)^{-1} {}^{y\beta} h_{n,\mu}^m(x_\gamma) [g_{n,\mu}^m(x_\gamma)]^{y\beta},$$

$$S_{n,\mu}^m(x_\alpha, x_\beta, x_\gamma) = (u_{s_0 s_3}^m)^{y\gamma} [f_{n,\mu}^m(x_\gamma)]^{y\beta} [l_{n,\mu}^m(x_\beta, x_\gamma)]^{y\alpha}.$$

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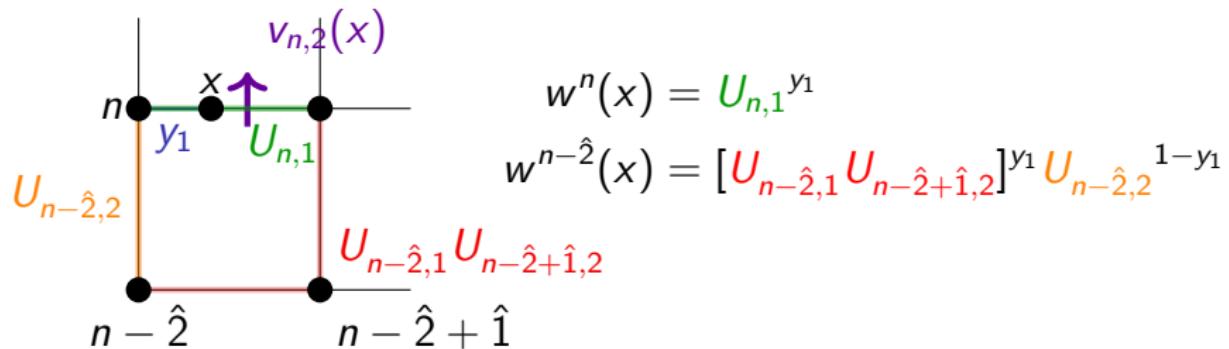
$$\left. + \int_{f_n, \mu} d^3x \operatorname{Tr} \left[(v_{n,\mu}^{-1} \partial_\nu v_{n,\mu}) (v_{n,\mu}^{-1} \partial_\rho v_{n,\mu}) (v_{n,\mu}^{-1} \partial_\sigma v_{n,\mu}) \right] \right\}$$



$$y_\lambda \equiv x_\lambda - n_\lambda$$

Exercise: Lattice 2D $U(1)$ gauge theory

- $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x)w^n(x)^{-1}$ [Lüscher '98, Fujiwara et al. '00]



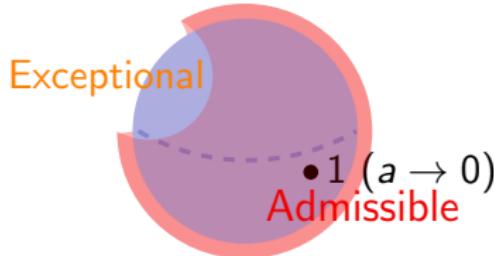
- Explicit expression of v :

$$\begin{aligned} v_{n,1}(x) &= U_{n-\hat{1},1} & v_{n,2}(x) &= U_{n-\hat{2},2} [U_{n-\hat{2},1} U_{n-\hat{2}+\hat{1},2} U_{n,1}^{-1} U_{n-\hat{2},2}^{-1}]^{y_1} \\ && &= U_{n-\hat{2},2} \exp[iy_1 F_{12}(n - \hat{2})] \end{aligned}$$

- ▶ Field strength: $F_p \equiv \frac{1}{i} \ln U_p$ for $-\pi < F_p \leq \pi$
- ▶ (nD) To ensure Bianchi identity $dF_p = 0$, we should impose $\sup_p |F_p| < \epsilon$, $0 < \epsilon < \frac{\pi}{3} \rightarrow$ Admissibility condition (see also §5)

Admissibility condition

- In general, admissibility = well-defined-ness of u^y ($0 \leq y \leq 1$)
 - ★ $U(1)$: $F_p = \frac{1}{i} \ln U_p$ for plaquette U_p
 - ★ $S_{n,\mu}^m(x)$ is written in terms of $(u_{ss'}^n)^y$ where u is a loop
 $n \rightarrow s \rightarrow s' \rightarrow n$
- ▶ E.g., u^y is ill-defined at $u = -1$; ill-def regions separate **sectors**
- Admissibility condition $\text{tr}(1 - U_p) < \epsilon$ [Lüscher '84]



- ▶ Admissible lattice gauge fields:
well-defined conf space \sim disk
- ▶ Exceptional region
 - ★ Topological freezing
 - ★ Monopole as lattice artifact

- Under the admissibility condition, we can prove that $Q \in \mathbb{Z}$; we observe topo. sectors even on lattice!
- How about index theorem for finite a ?

$$\text{Index}(D) = \underbrace{-\frac{a}{2} \text{Tr } \gamma_5 D_{\text{ov}}}_{\text{Admissibility } \epsilon_{\text{ov}}} = \underbrace{n_+ - n_-}_{\in \mathbb{Z}} \stackrel{?}{=} \frac{1}{32\pi^2} \int_x \varepsilon_{\mu\nu\rho\sigma} \text{tr}[F_{\mu\nu} F_{\rho\sigma}]$$

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Cocycle condition relaxed by higher-form sym

- Lüscher's construction *Coupled* with higher-form gauge fields
 - ▶ $SU(N)$ YM theory coupled with \mathbb{Z}_N 2-form gauge field

$$S \sim \sum \text{Tr} e^{-\frac{2\pi i}{N} \mathbf{B}_p} U_p \quad B_p: \text{2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N} \lambda_\ell} U_\ell, B_p \mapsto B_p + (d\lambda)_p$

- ▶ 't Hooft twisted b.c. [’79]: $U_{n+\hat{L}\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$

$$\begin{array}{c} \text{grid} \\ \xrightarrow{\text{gauge transf}} \\ 0 \end{array} \quad L \quad g_{n+\hat{L}\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

- Cocycle condition can take a \mathbb{Z}_N value:

$$\tilde{v}_{n-\hat{\nu},\mu}(x) \tilde{v}_{n,\nu}(x) \tilde{v}_{n,\mu}(x)^{-1} \tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = e^{\frac{2\pi i}{N} B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}$$

- ▶ \mathbb{Z}_N blind matters: adjoint repr.
 - ▶ $\mathbb{Z}_N^{[1]}$ gauge inv. if $\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N} \lambda_{n-\hat{\mu},\mu}} \tilde{v}_{n,\mu}(x)$
- What is definition of $\tilde{v}_{n,\mu}(x)$?

$\mathbb{Z}_N^{[1]}$ gauge invariant construction

- Gauge invariant plaquette

$$\tilde{U}_p \equiv e^{-\frac{2\pi i}{N} B_p} U_p$$

- Recall $U_\ell \mapsto e^{\frac{2\pi i}{N} \lambda_\ell} U_\ell$
- Admissibility $\text{tr}(1 - \tilde{U}_p) < \epsilon$

- u : product of plaquettes $\rightarrow \tilde{u}$

$$\tilde{u}_{s_7 s_2}^{n-\hat{3}} = e^{\frac{2\pi i}{N} B_{34}(n-\hat{3})} e^{\frac{2\pi i}{N} B_{24}(n)} u_{s_7 s_2}^{n-\hat{3}}$$

- Similarly, \tilde{v} is defined in terms of \tilde{u}

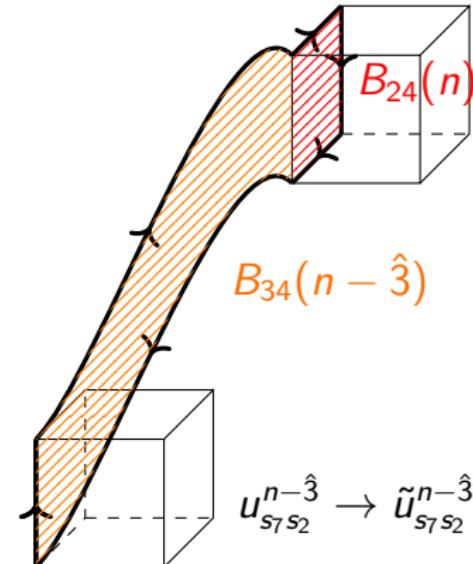
- Gauge covariance

$$\tilde{v}_{n,\mu}(x) \mapsto e^{\frac{2\pi i}{N} \lambda_\mu(n-\hat{\mu})} \tilde{v}_{n,\mu}(x)$$

- Fractional topological charge $Q = \sum_n q(n) \in -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N} + \mathbb{Z}$

$$q(n) = -\frac{1}{8N} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} B_{\mu\nu}(n) B_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + \check{q}(n)$$

where $\sum_n \check{q}(n) \in \mathbb{Z}$ (each term in q is not inv. under $\mathbb{Z}_N^{[1]}$)



Mixed 't Hooft anomaly in $SU(N)$ or $U(1)$ GT

- 1-form inv. action: $S_{\text{YM}} + S_{\text{matter}} - i\theta Q$

$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q[B_p]} \mathcal{Z}_\theta[B_p]$$

Anomaly between $\mathbb{Z}_N^{[1]}$ gauge inv. and θ periodicity

- In case of $U(1)$ gauge theory, gauging $\mathbb{Z}_q^{[1]} \subset U(1)_E$ symmetry

$$Q \in \frac{1}{8q^2} \varepsilon_{\mu\nu\rho\sigma} Z_{\mu\nu} Z_{\rho\sigma} + \frac{1}{4q} (\varepsilon_{\mu\nu\rho\sigma} Z_{\mu\nu})_{\mu\nu \leftrightarrow \rho\sigma} \mathbb{Z} + \mathbb{Z}$$

Modified as $-iq\theta Q$ by Witten effect [Honda–Tanizaki '20]

- ▶ If $-i\check{\theta}Q$, $\check{\theta} \sim \check{\theta} + 2\pi q$; $\check{\theta} = q\theta$, then $\theta \sim \theta + 2\pi$
- ▶ Electric charge of dyon $= qe + \frac{1}{2\pi}\check{\theta}$ (matters with charge $q\mathbb{Z}$)
- ▶ 't Hooft anomaly $= e^{-2\pi iqQ}$ with $qQ \in \frac{1}{q}\mathbb{Z}$
- ▶ Under some kind of modification (see next §), there exist $\mathbb{Z}_N^{[1]}$ if multiple of q

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Generalization: higher-group structure

- In general suppose $\otimes_{i,p} G_i^{[p]}$ global symmetry
- After gauging, a naive direct product of symmetry groups?
 - ▶ Can each symmetry be gauged *individually*?
- Gauging $G^{[0]} \times H^{[1]}$ global symmetry, then gauge transf.:

$$A \mapsto A + d\lambda^{(0)}, \quad B \mapsto B + d\lambda^{(1)} + \textcolor{red}{Ad}\lambda^{(0)}$$

2-group symmetry (cf. superstring theory [Green–Schwarz '84])

▶ p -group symmetry: $G_0^{[0]} \tilde{\times} \dots \tilde{\times} G_{p-1}^{[p-1]}$

- E.g., 4D $SU(N)$ gauge theory with instanton number $p\mathbb{Z}$
 - ▶ For any $p \in \mathbb{Z}$, local and unitary [Seiberg '10]
 - ▶ Global symmetry: $\underbrace{\mathbb{Z}_N^{[1]} \text{ center sym}}_{\text{gauging}} \times \mathbb{Z}_p^{[3]} \text{ sym} \xrightarrow{\text{gauging}} 4\text{-group}$ [Tanizaki–Ünsal '19]
cf. [Hidaka–Nitta–Yokokura '21]

- How to modify instanton sum & realize higher-group on lattice?

Modified instanton-sum: $\mathbb{Z}_N^{[1]} \times \mathbb{Z}_p^{[3]}$ gauge sym?

- Inserting the delta function (or introducing Lagrange multiplier)

$$\delta(q_n - pc_n) \text{ or } \left[\sum_n \chi_n(q - \dots) \right] \rightarrow Q = \underbrace{\color{red}p \sum_n c_n}_{\in \mathbb{Z}}$$

where $Q = \sum_n q_n$, $U(1)$ 4-form field strength c_n

- ▶ $c = dc^{(3)}$; Charged object under $\mathbb{Z}_p^{[3]}$

$$V^{(3)} = e^{\int_{M_3} c^{(3)}} \rightarrow e^{i\chi(x)} V^{(3)} = e^{\frac{2\pi i}{p} \#(x, M_3)} V^{(3)}$$

- ▶ θ term, $i\theta Q + i\hat{\theta} \sum_n c_n$, indicates the $2\pi/p$ periodicity of θ

- Just by counting numbers, obviously no nontrivial configurations for B_p :

$$\frac{1}{8N} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} = \sum_n \frac{1}{8N} \varepsilon_{\mu\nu\rho\sigma} B_{n,\mu\nu} B_{n+\hat{\mu}+\hat{\nu},\rho\sigma} \bmod 1 \in \mathbb{Z}$$

$\mathbb{Z}_N^{[1]} \times \mathbb{Z}_p^{[3]}$ global symmetry $\rightarrow \cancel{\text{gauge symmetry}}$

Modified instanton-sum: higher-group symmetry

- Introducing new field Ω_n ($\Omega_n \in \mathbb{R}$ and $\sum_n \Omega_n \in \mathbb{Z}$)
 - ▶ Replacement: $c_n \rightarrow c_n - \frac{1}{Np} \Omega_n$: 3-form gauge inv
$$q_n - pc_n + \frac{1}{N} \Omega_n = 0 \quad : \text{fractionality allowed}$$
 - ▶ Redefine Ω_n as $\tilde{\Omega}_n \equiv \frac{1}{N} \Omega_n - \underbrace{\frac{1}{8N} \varepsilon_{\mu\nu\rho\sigma} B_{n,\mu\nu} B_{n+\hat{\mu}+\hat{\nu},\rho\sigma}}_{\text{fractional part of } Q}$
Again $\sum_n \tilde{\Omega}_n \in \mathbb{Z}$
$$\check{q}_n - pc_n + \tilde{\Omega}_n = 0 \quad \text{where } \check{q}_n: \text{integral part of } Q$$
- 1-form and 3-form gauge transf with $\Omega_n^{(3)} \in \mathbb{R}$:
$$B_p \mapsto B_p + (d\lambda)_p, \quad c_n \mapsto c_n + \frac{1}{p} d\Omega_n^{(3)} (+\mathbb{Z}),$$

$$\tilde{\Omega}_n \mapsto \tilde{\Omega}_n + d\Omega_n^{(3)} (+p\mathbb{Z}) + \textcolor{red}{\left(\frac{2}{N} B \wedge d\lambda + \frac{1}{N} d\lambda \wedge d\lambda \right) (+\mathbb{Z})}$$

Finally, $w_n \equiv \text{integral part of } \tilde{\Omega}_n$

- ▶ defined by using 1-form and *continuum* 3-form gauge transf
- ▶ Theory possesses “mixed 1-form” \times discrete $\mathbb{Z}_p^{[3]}$ gauge sym

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Admissibility \rightarrow absence of monopole?

- Magnetic monopole
 - ▶ Magnetic defect operators provide nontrivial topology
 - ▶ Quite heavy but significant in nonperturbative dynamics
 - ▶ Maxwell equation w/ monopole current j_m : $d \star F = j_e$, $dF = j_m$.
- Admissibility $dF = 0$ to reinstate topo. structure on lattice fields
 - ▶ Exercise: $0 < \epsilon < \pi/3$?
 - $F_p \equiv \frac{1}{i} \ln U_p$, $a_\ell \equiv \frac{1}{i} \ln U_\ell$ in $(-\pi, \pi]$; $F_p = (da)_p + 2\pi N_p$ since

$$(da)_p = \Delta_\mu a_{n,\nu} - \Delta_\nu a_{n,\mu} = [a_{n+\hat{\mu},\nu} - a_{n,\nu}] - [a_{n+\hat{\nu},\mu} - a_{n,\mu}]$$

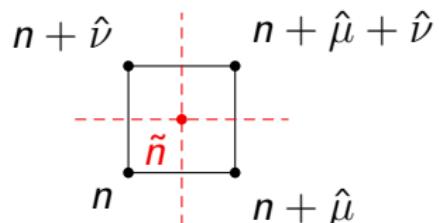
$$|da| \leq 4|a| \leq 4\pi; \quad \text{Introduce } N \text{ as } |da + 2\pi N| \leq \pi.$$

Imposing $|F_p| < \epsilon$, $6\epsilon > |\underbrace{\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu F_{\rho\sigma}}_{\rightarrow 0}| = 2\pi |\underbrace{\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}}_{< 1}|$;
therefore $6\epsilon < 2\pi$

- How to discuss monopole properties on lattice w/ admissibility?
 - ▶ 2D compact bosons: θ term and magnetic operators
 - ▶ Observation of analogue of Witten effect
 - ▶ Future study: 4D gauge theory

Lattice formulation of θ angle and Witten effect

- Compact scalar fields: $\phi_1(n)$, $\phi_2(\tilde{n})$
 - ▶ Dual lattice $\tilde{n} = n + \frac{1}{2}\hat{1} + \frac{1}{2}\hat{2}$
 - ▶ $\partial\phi_a(s, \mu) \equiv \frac{1}{i} \ln e^{-i\phi_a(s)} e^{i\phi_a(s+\hat{\mu})}$
 - ▶ $\sup_{\ell} |\partial\phi_{a,\ell}| < \epsilon$, $0 < \epsilon < \pi/2$;
then Bianchi identity $d\partial\phi = 0$



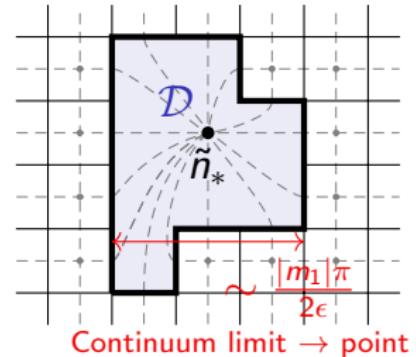
- Lattice action with θ angle

$$S = \frac{R^2}{2\pi} \sum [1 - \cos \partial\phi_{a,\ell}] - i\theta Q, \quad Q = -\frac{1}{4\pi^2} \sum \varepsilon_{\mu\nu} \partial\phi_{2,(\tilde{n},\mu)} \partial\phi_{1,(n+\hat{\mu},\nu)}$$

- Usually, $\mathcal{Q}_{m1}(C) \equiv \frac{1}{2\pi} \sum_{\ell \in C} \partial\phi_{1,\ell} = 0$
- **Excision method:** sites&links eliminated inside \mathcal{D} ; put one dual site \tilde{n}_* in \mathcal{D}

$$m_1 = \mathcal{Q}_{m1}(\partial\mathcal{D}) \in \mathbb{Z}$$

- $\langle M_1(\tilde{n}_*) \dots \rangle_{\theta+2\pi} = \langle M_1(\tilde{n}_*) e^{im_1 \phi_2(\tilde{n}_*)} \dots \rangle_\theta$
for magnetic defect operator $M_1(x)$



EM symmetry and 't Hooft anomaly

- $U(1)$ electric/magnetic global symmetries: $j_e = \star d\phi, j_m = d\phi$
 - ▶ Electric charged object $e^{i\phi}$;
Monopole $M_1(x)$ with $(0, m_1) \xrightarrow{\text{Witten}}$ dyon with (m_1, m_1)
- Background gauging $U_{p, \tilde{p}}^{\text{em}, a}$:
 $F_p^{(e,1)}$ and $F_{\tilde{p}}^{(m,1)}$ for ϕ_1 , $F_{\tilde{p}}^{(e,2)}$ and $F_p^{(m,2)}$ for ϕ_2
 - ▶ $\sup |F| < \delta$ with $0 < \delta < \min(\pi, 2\pi - 4\epsilon)$
 - ▶ $F_{\mu\nu}^e = \Delta_\mu D\phi_{n,\nu} - \Delta_\nu D\phi_{n,\mu}$
- Analogue of θ shift is given by
 $\theta \rightarrow \theta + 2\pi, F_{\tilde{p}}^{(m,1)} \rightarrow F_{\tilde{p}}^{(m,1)} - F_{\tilde{p}}^{(e,2)}, F_p^{(m,2)} \rightarrow F_p^{(m,2)} + F_p^{(e,1)}$
 - ▶ Mixing \mathcal{Q}_{m1} with \mathcal{Q}_{e2} on $\tilde{\Lambda}$; \mathcal{Q}_{m2} with \mathcal{Q}_{e1} on Λ (**Witten effect**)
- Under above shift, we observe mixed 't Hooft anomaly

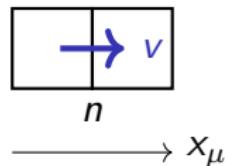
$$\mathcal{Z}_{\theta+2\pi}[A^{(e,a)}, A^{(m,a)} - \varepsilon_{ab} A^{(e,b)}] = e^{-\frac{i}{2\pi} \sum \varepsilon_{\mu\nu} A_\mu^{(e,2)}(\tilde{n}) A_\nu^{(e,1)}(n+\hat{\mu})} \mathcal{Z}_\theta[A]$$

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Summary [1/2]

- Generalized symmetries have been developed in this decade
 - ▶ Higher-form sym, higher-group sym, noninvertible sym, subsystem sym, ...
 - ▶ Through the use of 't Hooft anomaly matching, new insights about *nontrivial dynamics & classification of phases*
- Standing on a **fully regularized framework: lattice gauge theory**
 - ▶ Generalization of Lüscher's construction of topology on lattice
 - ▶ Maintaining locality, $SU(N)$ gauge inv & higher-form gauge inv
 - ▶ There exists interpolation to smooth enough bundle structure

- ★ Transition function $v_f(n) \rightarrow v_f(x)$
- ★ Q is written in terms of $v_f(x)^{-1} \partial_\nu v_f(x)$



$$Q \in \mathbb{Z} \xrightarrow{\text{Gauging } \mathbb{Z}_N^{[1]}} \frac{1}{N} \mathbb{Z}$$

- ▶ Mixed 't Hooft anomaly between $\mathbb{Z}_N^{[1]}$ & θ periodicity

Summary [2/2]

- Robust discussion on higher-group from lattice
 - ▶ [Continuum case] $SU(N) \rightarrow U(N)$ & many $U(1)$ fields: $F \rightarrow \tilde{F}$, $\text{tr } \tilde{F} = B^{(2)}$, $NB^{(2)} = dB^{(1)}$, $pD^{(4)} = dD^{(3)} + \frac{N}{4\pi} B^{(2)} \wedge B^{(2)}$
What is topological object? $\Rightarrow \int ND^{(4)} \in \frac{2\pi}{p} \mathbb{Z}$
 - ▶ [Lattice case] Counting integers and fractional numbers; mixture of symmetries
- Lattice construction of magnetic operators and observation of Witten effect
 - ▶ Lattice field theories with admissibility condition
 - ▶ Excision region $\mathcal{D} \rightarrow$ a point (monopole) in continuum limit
 - ▶ Arranging lattice/dual-lattice for mixing magnetic charges with electric ones
- Future works
 - ▶ Monopole, 't Hooft line in gauge theory
 - ▶ Other kinds: subsystem sym & non-invertible (categorical) sym