Lattice realization of the axial $U(1)$ non-invertible symmetry

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- Y. Honda, OM, S. Onoda and H. Suzuki, [arXiv:2401.01331v3 [hep-lat]].
- (Main) Results in latter part of this talk given by Tanizaki-san

Symmetry

- Symmetry: fundamental tool in physics
  - Universal applications to high energy physics, condensed matter physics and mathematics
- Noether theorem and conservation law

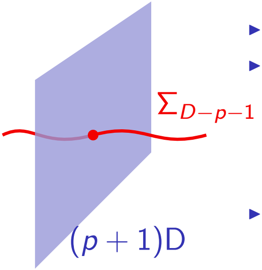
$$\text{Sym} : \phi \mapsto \phi'; S(\phi) = S(\phi'), \qquad \partial_\mu j^\mu = 0$$

$$\text{Charge } Q \equiv \int j^0 d^{D-1}x$$

- Structure in the nature: Lorentz symmetry, CPT theorem, Gauge invariance, Internal symmetries
- More aspects of Symmetry
  - Landau theory: phase transition (or vacuum structure) from viewpoint of symmetry
  - Symmetry breaking: quantum anomaly, spontaneous breaking

Recent generalization of symmetry

- Generalized global symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]
  - Coupled with topological field theory (TQFT)
    - ⋆ Changing topological structure without changing local dynamics [Kapustin–Seiberg '14]:
    - ⋆ Nontrivial information by using generalized 't Hooft anomaly matching (see later)
- An example: Higher-form symmetry



- Time slice $\to$ $(D-p-1)$-dim surface
- p-form symmetry $G^{[p]}$ (codim $p+1$)

$$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \quad U_\alpha(\Sigma) = e^{i\alpha Q}$$

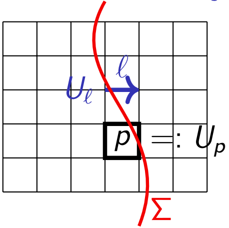
  Topological under deformation of Σ
- Transforming a "loop" operator $W(C)$

$$W(C) \mapsto U(\Sigma) W(C)$$
$$= e^{i\alpha \#(\Sigma, C)} W(C) \quad \text{w/ linking } \#$$

E.g., Center in YM theory and 't Hooft anomaly

- Center symmetry: $\mathbb{Z}_N^{[1]}$ intersection of Σ & ℓ

  $e^{\frac{2\pi i}{N}k} \in \mathbb{Z}_N \subset SU(N)$; $\quad U_\ell \mapsto e^{\frac{2\pi i}{N}k\#(\Sigma,\ell)} U_\ell$



- Gauging the center symmetry

$$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N}B_p} U_p$$

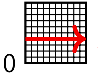
B_p: 2-form gauge field assoc. $\mathbb{Z}_N^{[1]}$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell$, $B_p \mapsto B_p + (d\lambda)_p$

  - Usually, topo. charge $Q \sim \int F\tilde{F} \in \mathbb{Z}$ under topo. sectors
    - ⋆ θ term: θQ (strong CP problem, axion physics, sign problem)
  - $Q \sim \int B\tilde{B} \in \frac{1}{N}\mathbb{Z}$; 't Hooft anomaly $\mathcal{Z}_{\theta+2\pi}[B] = e^{-2\pi i Q}\mathcal{Z}_\theta[B]$

    [Kapustin–Thorngren '13, Gaiotto–Kapustin–Komargodski–Seiberg '17]

$\overset{\text{global}}{\Rightarrow}$ 't Hooft twisted boundary condition [79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N}z_{\mu\nu}} \in \mathbb{Z}_N$$

$$z_{\mu\nu} = \sum B_p \bmod N;\ Q = \frac{1}{4N}z\tilde{z} \bmod 1$$

[van Baal 82] cf. [Edwards-Heller-Narayanan, de Forcrand-Jahn, Fodor-Holland-Kuti-Nógrádi-Schroeder, Kitano-Suyama-Yamada, Itou, . . .]

Our enterprise: Fully lattice regularized framework

- Topology on lattice without spacetime continuity?
  - Instead, what is $\mathbb{Z}_N$ field in continuum? [Kapustin–Seiberg '14]
    - ⋆ $\mathbb{Z}_N^{[q]}$ gauge field: $U(1)$ field $B^{(q)}$ with constraint $NB^{(q)} = dB^{(q-1)}$ from charge-N Higgs
    - ⋆ There are some subtleties: Many fields needed (which is essential?), different cohomological structure ($\mathbb{Z}_N$ vs $U(1)$), What happens due to divergence in QFT...
  - Lattice construction of fiber bundle and Q [Lüscher '84]
- Fully lattice regularized framework:
  - Fractional Q and 't Hooft anomaly [Abe–OM–Suzuki, Abe–OM–Onoda–Suzuki–Tanizaki]
  - Higher-group symmetry under instanton-sum modification [Kan–OM–Nagoya–Wada, Abe–OM–Onoda]
  - Magnetic operators and "Witten effect" [Abe–OM–Onoda–Suzuki–Tanizaki]
    - ⋆ [See seminar slide at KEK Theory Center 3/10/2023]

Generalized symmetry: Non-invertibility

- LIE ALGEBRAS IN PARTICLE PHYSICS by H. Georgi

  "Group theory is the study of symmetry."

- Based on recent developments of generalized symmetry, symmetry is not necessarily described by group!
  1. Associativity: $(ab)c = a(bc)$ for $\forall a$, b, c
  2. Identity element: $\exists \mathrm{e}$ s.t. $a\mathrm{e} = \mathrm{e}a = a$
  3. Inverse element: $\exists b$ s.t. $ab = ba = \mathrm{e}$ $(b = a^{-1})$

  So far, symmetry possesses invertibility.
- Now, for "naively unitary" symmetry operators $\{U_\alpha = e^{i\alpha Q}\}$, there can exist Non-invertible symmetry in some systems

$$\mathcal{D} \times \mathcal{D}^\dagger \neq 1$$

  [Many studies; See recent lectures by Schäfer-Nameki, Shao]
- Let's consider non-invertible symmetry for axial $U(1)$ transf, and then, in this talk, our aim is to realize it on lattice!

Axial $U(1)$ symmetry in continuum theory

- Quantum anomaly $\rightarrow$ loss of symmetry
  ABJ anomaly: no conserved current for axial $U(1)$ symmetry
- For fractional rotation angles, conserved, gauge-invariant and topological non-invertible symmetry exists
  [Córdova–Ohmori '22, Choi–Lam–Shao '22]
- Naive symmetry operator

$$d \star j = \frac{1}{4\pi^2} F \wedge F, \qquad U_\alpha(M) = \exp\left(\frac{i\alpha}{2}\int_M \star j\right)$$

  is not topological
- Let us consider topological modification by Chern–Simons:

$$\hat{U}_\alpha(M) = \exp\left[\frac{i\alpha}{2}\int\left(\star j - \frac{1}{4\pi^2} A \wedge dA\right)\right]$$

  But gauge non-invariant!!
- For $\alpha = \frac{2\pi}{N}$, $-i\int_M \frac{1}{4\pi N} A \wedge dA$ is still gauge non-invariant because of fractional CS level $1/N$

Fractional quantum Hall effect and non-invertibility

- TQFT (fractional quantum Hall state) as

$$i \int_M \left(\frac{N}{4\pi} a \wedge da + \frac{1}{2\pi} a \wedge dA \right)$$

which is gauge-invariant with fractional CS level

  - a: additional dynamical $U(1)$ gauge field
  - Naively, this action is (level-N CS) + $(A_\mu J^\mu)$; $a = -A/N$ "Substituting" this into action reproduces $-i \int \frac{1}{4\pi N} A \wedge dA$

- To modify naive symmetry operator $U_\alpha(M)$, we use this action at boundary M instead of naive CS term:

$$\mathcal{D}_{1/N}(M) = \exp \left[i \int \left(\frac{\pi}{N} \star j + \frac{N}{4\pi} a \wedge da + \frac{1}{2\pi} a \wedge dA \right) \right]$$

- Gauge invariant and topological; no inverse operator

$$\mathcal{D}_{1/N} \times \mathcal{D}^\dagger_{1/N} = \mathcal{C} \neq 1$$

  - $\mathcal{C}$: condensation operator

Lattice realization?

- We want to realize lattice axial $U(1)$ non-invertible symmetry
- Problems:
  - Lattice Chern–Simons theory?
    - ★ cf. Villain formulation of $U(1)$ GT [Jacobson–Sulejmanpasic '23]
      $\not\Rightarrow$ non-Abelian generalization
  - (Anomaly-free) Chiral lattice gauge theory?
    - ★ For $U(1)$ gauge theory, Lüscher's construction ['98]
    - ★ See [Intensive lecture by Kikukawa-san @YITP, 2/19-21]
  - Anomalous gauge theory? (Continuation of studying it?)
    - ★ E.g., [Forster–Nielsen–Ninomiya '80, Harada–Tsutsui '87, ...]
    - ★ Lattice framework: e.g., [Kikukawa–Suzuki '07]

? Prescription by Karasik ['22] instead of [Choi–Lam–Shao '22]
  - "Gauge average" (similar to Harada–Tsutsui formalism)

$$\tilde{U}_\alpha(M) = \int \mathrm{D}\phi\, \hat{U}_\alpha(M)|_{A\to A-d\phi}$$

  - There are some subtleties $\to$ other method by Tanizaki-san

- Construct (fermion measure in) anomalous chiral gauge theory

$U(1) \times U(1)'$ lattice gauge theory: Gauge

- Vector $U(1) \ni u(x,\mu)$ (physical) and Chiral $U(1)' \ni U(x,\mu)$ (non-dynamical) gauge group
- Expectation value with magnetic flux m:

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}u \, e^{-S_{\mathrm{G}}} \langle \mathcal{O} \rangle_{\mathrm{F}} \,, \quad \langle \mathcal{O} \rangle_{\mathrm{F}} = w[m] \int \mathrm{D}\psi \mathrm{D}\bar{\psi} \, e^{-S_{\mathrm{F}}} \mathcal{O}.$$

- (Unconventional) Gauge action S_{G} following [Lüscher],

$$S_{\mathrm{G}} = \frac{1}{4g_0^2} \sum_{x \in \Gamma} \sum_{\mu,\nu} \mathcal{L}_{\mu\nu}(x)$$

$$\mathcal{L}_{\mu\nu}(x) = \begin{cases} [f_{\mu\nu}(x)]^2 \left\{ 1 - \frac{[f_{\mu\nu}(x)]^2}{\epsilon^2} \right\}^{-1} & \text{if } |f_{\mu\nu}(x)| < \epsilon, \\ \infty & \text{otherwise.} \end{cases}$$

for $0 < \epsilon < \pi/3$, where the field strength of $u(x,\mu)$ is

$$f_{\mu\nu}(x) = \frac{1}{i} \ln u(x,\mu) u(x+\hat{\mu},\nu) u(x+\hat{\nu},\mu)^{-1} u(x,\nu)^{-1}.$$

$U(1) \times U(1)'$ lattice gauge theory: Fermion

- Two left-handed Weyl fermions $\psi_{1,2}$ possess $U(1)$ charges

$$(\mathrm{e}_1, \mathrm{e}'_1) = (+1, -1), \qquad (\mathrm{e}_2, \mathrm{e}'_2) = (-1, -1).$$

- Since overlap Dirac operator fulfills Ginsparg–Wilson relation

$$\gamma_5 D + D\gamma_5 = D\gamma_5 D,$$

  chirality projection operators are defined by

$$\hat{\gamma}_5 = \gamma_5(1 - D), \qquad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5), \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5).$$

- Fermion action is

$$S_{\mathrm{F}} = \sum_{x\in\Gamma} \bar{\psi}(x) D\psi(x) = \sum_{x\in\Gamma} \bar{\psi}(x) P_+ D \hat{P}_- \psi(x)$$

- Fermion integration measure is

$$\mathrm{D}\psi\mathrm{D}\bar{\psi} = \prod_j \mathrm{d}c_j \prod_k \mathrm{d}\bar{c}_k, \; \psi(x) = \sum_j v_j(x) c_j, \; \bar{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

  Basis $v_j(x)$ are in projected space $\hat{P}_- v_j = v_j$, $(v_k, v_j) = \delta_{kj}$

Construction of fermion integration measure

- Basis vectors depend on gauge field due to $\hat{P}_-$
- Under variation δ of gauge field, δv_j is not determined...
  - Requirements: (single-valued, gauge-inv) smooth, & local
- Under infinitesimal variations $\delta_\eta u(x,\mu) = i\eta_\mu(x)u(x,\mu)$, $\delta_\eta U(x,\mu) = i\eta'_\mu(x)U(x,\mu)$,

$$\delta_\eta \ln\det \bar{v}Dv = (\text{term from } \delta_\eta D) + (\text{term from } \delta_\eta v).$$

Second term is called measure term (measure currents)

$$-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j) := -i \sum_{x\in\Gamma} \left[\eta_\mu(x) j_\mu(x) + \eta'_\mu(x) J_\mu(x)\right].$$

  - Consistent/covariant anomaly: $\mathcal{A}_{\text{cons}} = \frac{1}{3}\mathcal{A}_{\text{cov}}$ for Abelian GT
  - To remedy this issue, we need to include counterterm [E.g., see Fujikawa–Suzuki Chap. 6]

Construction of measure currents

- For $\eta_\mu(x) = -\partial_\mu \omega(x)$, $\eta'_\mu(x) = -\partial_\mu \Omega(x)$, $(\gamma = -\frac{1}{32\pi^2})$

$$\delta_\eta \langle \mathcal{O} \rangle_{\mathrm{F}} = \langle \delta_\eta \mathcal{O} \rangle_{\mathrm{F}} - 2i\gamma \sum_{x \in \Gamma} \Omega(x) \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x + \hat{\mu} + \hat{\nu}) \langle \mathcal{O} \rangle_{\mathrm{F}}. \tag{$\star$}$$

- To prove this, measure currents fulfill the following conditions:
  1. depend smoothly and locally on gauge fields
  2. satisfy "integrability condition" $([\delta_\eta, \delta_\zeta])$
  3. satisfy "anomalous conservation law" $(\partial^* j,\ \partial^* J)$
- We find non-local dependence of $U(x, \mu)$; but $U(x, \mu)$ is external
- In infinite volume, one can construct the currents explicitly w.r.t.

$$u(x, \mu) = \mathrm{e}^{i\mathfrak{a}_\mu(x)}, \quad |\mathfrak{a}_\mu(x)| \leq \pi(1 + 8\|x\|), \quad f(x) = d\mathfrak{a}(x),$$
$$U(x, \mu) = \mathrm{e}^{i\mathfrak{A}_\mu(x)}, \quad |\mathfrak{A}_\mu(x)| \leq \pi(1 + 8\|x\|), \quad F(x) = d\mathfrak{A}(x),$$

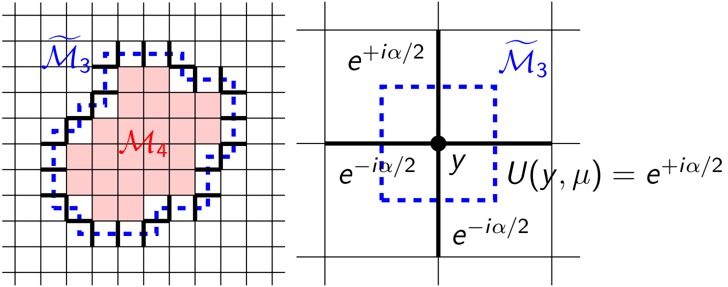
as follows:

Backup: Construction of measure currents $L \to \infty$

$$
\begin{aligned}
\mathfrak{L}_\eta^{\star\text{inv}} &= i\int_0^1 ds\ \mathrm{Tr}(\hat{P}_-[\partial_s \hat{P}_-, \delta_\eta \hat{P}_-]) + \int_0^1 ds \sum_{x\in\mathbb{Z}^4} [\eta_\mu(x) k_\mu(x) + \mathfrak{a}_\mu(x)\delta_\eta k_\mu(x)] \\
&\quad + \int_0^1 ds \sum_{x\in\mathbb{Z}^4} \left[\eta'_\mu(x) K_\mu(x) + \mathfrak{A}_\mu(x)\delta_\eta K_\mu(x)\right] \\
&\quad - \frac{4}{3}\gamma \sum_{x\in\mathbb{Z}^4} \epsilon_{\mu\nu\rho\sigma} \{\eta'_\mu(x)\mathfrak{a}_\nu(x+\hat{\mu}) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \\
&\qquad\qquad + \mathfrak{A}_\mu(x)\delta_\eta [\mathfrak{a}_\nu(x+\hat{\mu}) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})]\} \\
&:= \sum_{x\in\mathbb{Z}^4} \left[\eta_\mu(x) j_\mu^{\star\text{inv}}(x) + \eta'_\mu(x) J_\mu^{\star\text{inv}}(x)\right], \\
\mathfrak{L}_\eta^{\star\text{non-inv}} &= 4\gamma \sum_{x\in\mathbb{Z}^4} \epsilon_{\mu\nu\rho\sigma} \{\eta'_\mu(x)\mathfrak{a}_\nu(x+\hat{\mu}) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu}) \\
&\qquad\qquad + \mathfrak{A}_\mu(x)\delta_\eta [\mathfrak{a}_\nu(x+\hat{\mu}) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})]\} \\
&:= \sum_{x\in\mathbb{Z}^4} \left[\eta_\mu(x) j_\mu^{\star\text{non-inv}}(x) + \eta'_\mu(x) J_\mu^{\star\text{non-inv}}(x)\right].
\end{aligned}
$$

Introduction of topological defect

- $U(x,\mu) = e^{\pm i\alpha/2}$ for 3D closed surface $\widetilde{\mathcal{M}}_3$; otherwise 1



- From Eq. ($\star$), anomalous Ward–Takahashi identity is

$$\langle \mathcal{O}\rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} = \exp\left[-i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right] \langle \mathcal{O}^{\alpha}\rangle_{\mathrm{F}},$$

where $\psi(x)^{\alpha} = e^{-i\alpha/2}\psi(x)$, $\bar{\psi}(x)^{\alpha} = \bar{\psi}(x)e^{i\alpha/2}$ for $x \in \mathcal{M}_4$

- Symmetry operator may be written by

$$\left\langle U_{\alpha}(\widetilde{\mathcal{M}}_3)\mathcal{O}\right\rangle_{\mathrm{F}} := \langle \mathcal{O}\rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \exp\left[i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]$$

Gauge average and projection

- U_α is not invariant under gauge transf on boundary variables
- An operator defined by average over gauge transf

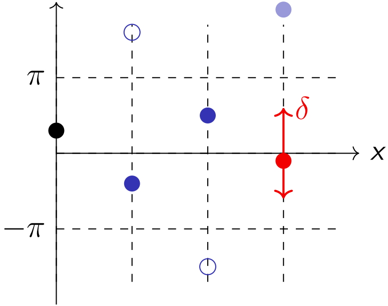
$$\left\langle \tilde{U}_\alpha(\widetilde{\mathcal{M}}_3)\mathcal{O}\right\rangle_{\mathrm{F}} := \langle\mathcal{O}\rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \int \mathrm{D}\lambda\, e^{\left[i\alpha\gamma\sum_{x\in\mathcal{M}_4}\epsilon_{\mu\nu\rho\sigma}f_{\mu\nu}(x)f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]^\lambda}$$

$[\quad]^\lambda$ indicates

$u(x,\mu) \to \lambda(x)u(x,\mu)\lambda(x+\hat{\mu})^{-1}$

- Smoothness condition

$\frac{1}{i}\ln\lambda(x)\lambda(x+\hat{\mu})^{-1}$



- Winding k_ν can be defined under this gauge non-inv condition
- $|k_\nu| < \frac{\delta}{2\pi}L$ with lattice size L; if $L\to\infty$,
  $\sum_{k=-\infty}^{\infty} e^{ikx} \propto \sum_{n=-\infty}^{\infty}\delta(x-2\pi n)$
- Gauge average implies (S^1_μ)
  $\delta\left(\frac{\alpha}{2\pi}\frac{1}{4\pi}\sum_{x\in\mathcal{M}_2^\nu}\epsilon_{\mu\nu\rho\sigma}f_{\rho\sigma}(x)-\mathbb{Z}\right)$
  - $\frac{\alpha}{2\pi}$ irrational: no magnetic flux
  - $\frac{\alpha}{2\pi}=\frac{p}{N}$: $\frac{1}{2\pi}\int_{\mathcal{M}_2} da = N\mathbb{Z}$
- $\tilde{U}(\widetilde{\mathcal{M}}_3) = U(\widetilde{\mathcal{M}}_3)P(\widetilde{\mathcal{M}}_3)$
  - P for allowed magnetic fluxes

Some subtleties in Karasik prescription on lattice

- We constructed non-invertible symmetry operator $\tilde{U}$ following Karasik in lattice gauge theory
  - Non-locality of $U(x,\mu)$ in fermion integration measure
  - Gauge non-invariant constraint for gauge transf; but irrelevant in continuum limit?
  - Non-intrinsically 3D construction in reference to auxiliary 4D bulk
  - (Are relative weights correct? Really topological?)
- Instead of CS, lattice $\mathbb{Z}_N$ TQFT (thanks to Tanizaki-san)
  - Set rotation angles α as $2\pi p/N$ (p, $N \in \mathbb{Z}$)
  - It is natural to consider 3D level-N BF theory

$$S_{\mathrm{BF}} = -\frac{ip\pi}{N} \sum_{x \in \mathcal{M}_3} \epsilon_{\mu\nu\rho} \left\{ b_\mu(\tilde{x}) \left[\partial_\nu c(x+\hat{\mu}) - \frac{1}{2} z_{\nu\rho}(x+\hat{\mu}) \right] - \frac{1}{2} z_{\mu\nu}(x) c_\rho(x+\hat{\mu}+\hat{\nu}) \right\}$$

  - ★ Dual lattice $\tilde{x}$; $f = \delta a + 2\pi z$, $-\pi < a_\mu(x) \equiv \frac{1}{i} \ln u(x,\mu) \leq \pi$
  - For simplicity, $S_{\mathrm{BF}} = \frac{-ip\pi}{N} \sum [b(\delta c - z) - z \cup c]$

Symmetry operator in terms of $\mathbb{Z}_N$ TQFT

- Consider $\mathcal{Z}_{\mathcal{M}_3}[z] = \frac{1}{N^s} \int \mathrm{D}b\mathrm{D}c\, e^{-S_{\mathrm{BF}}}$, s: # of sites $\in \mathcal{M}_3$
- From summation over b, $\mathcal{Z}_{\mathcal{M}_3}[z] = 0$ if $\sum_{\mathcal{M}_2} z \neq 0 \bmod N$
  - If $z = 0$, $\mathcal{Z}_{\mathcal{M}_3}[0] = N^{b_2 - 1}$, where b_2: 2nd Betti number of $\mathcal{M}_3$
  - If $z = \delta\nu \bmod N$,

$$\mathcal{Z}_{\mathcal{M}_3}[z] = \exp\left(-\frac{ip\pi}{N} \sum_{\mathcal{M}_3} \delta\nu \cup \nu\right) \mathcal{Z}_{\mathcal{M}_3}[0]$$

- Redefine symmetry operator on arbitrary 3-cycle $\widetilde{\mathcal{M}}_3$:

$$\left\langle \tilde{U}_{\frac{2\pi p}{N}}(\widetilde{\mathcal{M}}_3)\mathcal{O} \right\rangle_{\mathrm{F}} = \langle \mathcal{O} \rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \exp\left[-\frac{ip}{4\pi N} \sum_{\mathcal{M}_3} (a \cup f + 2\pi z \cup a)\right] \mathcal{Z}_{\mathcal{M}_3}[z]$$

  - Then, topological: $\left\langle \tilde{U}_{\frac{2\pi p}{N}}(\widetilde{\mathcal{M}'}_3)\mathcal{O} \right\rangle_{\mathrm{F}} = \left\langle \tilde{U}_{\frac{2\pi p}{N}}(\widetilde{\mathcal{M}}_3)\mathcal{O} \right\rangle_{\mathrm{F}}$
  - Gauge invariant: cancellation between exp and $\mathcal{Z}_{\mathcal{M}_3}$ under $a \mapsto a - \delta\phi - 2\pi l$ and $z \mapsto z + \delta l$

Fusion rules

- From $\mathcal{Z}^{(p,N)}_{\mathcal{M}_3}[z]\mathcal{Z}^{(p,N)}_{-\mathcal{M}_3}[z] = \mathcal{Z}_{\mathcal{M}_3}[0]\mathcal{C}_{\mathcal{M}_3}[z]$, condensation operator:

$$\mathcal{C}_{\mathcal{M}_3}[z] = \frac{1}{N^s}\int \mathrm{D}b\mathrm{D}c\, e^{b(\delta c - z)}$$

- For different p_1 and p_2 (assuming $\gcd(p_1+p_2, N) = 1$)

$$\tilde{U}_{2\pi p_1/N}\tilde{U}_{2\pi p_2/N} = \mathcal{Z}_{\mathcal{M}_3}[0]\tilde{U}_{2\pi(p_1+p_2)/N}$$

- More generally ($\gcd(p_1, N_1) = 1$, $\gcd(p_2, N_2) = 1$, $\gcd(N[p_1/N_1 + p_2/N_2], N) = 1$ with $N = \mathrm{lcm}(N_1, N_2)$),

$$\tilde{U}_{2\pi p_1/N_1}\tilde{U}_{2\pi p_2/N_2} = \frac{\mathcal{Z}^{(N_1)}_{\mathcal{M}_3}[0]\mathcal{Z}^{(N_2)}_{\mathcal{M}_3}[0]}{\mathcal{Z}^{(N)}_{\mathcal{M}_3}[0]}\tilde{U}_{2\pi(p_1/N_1+p_2/N_2)}$$

Summary

- Generalized symmetries have been developed in this decade
  - Axial $U(1)$ non-invertible symmetry
- Standing on a fully regularized framework: lattice gauge theory
  - Generalized Lüscher's construction of chiral lattice gauge theory
  - Construction of fermion measure: for dynamical fields, smooth and local; for external $U(x, \mu)$, non-local (unphysical)
- Karasik prescription
  - Under gauge non-invariant constraint and in terms of 4D bulk, we constructed symmetry operator
- Level-N BF theory (thanks to Tanizaki-san)
  - By using his technique, we can construct symmetry operator for rational angles, and evaluate fusion rules
- Questions
  - Physical phenomena (e.g., monopole?)
  - Generalization to non-Abelian gauge theory
  - Some aspects of anomalous/chiral lattice gauge theory (w/ boundary)

Backup: Gauge average and *smooth* gauge transf

- U_α is not invariant under gauge transf on boundary variables
- An operator defined by average over gauge transf

$$\left\langle \tilde{U}_\alpha(\widetilde{\mathcal{M}}_3)\mathcal{O} \right\rangle_{\mathrm{F}}$$
$$:= \langle\mathcal{O}\rangle_{\mathrm{F}}^{\widetilde{\mathcal{M}}_3} \int \mathrm{D}\lambda\ \exp\left[i\alpha\gamma \sum_{x\in\mathcal{M}_4} \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x+\hat{\mu}+\hat{\nu})\right]^\lambda$$

- $[\quad]^\lambda$ indicates gauge transf on boundary variables

$$u(x,\mu) \to \lambda(x)u(x,\mu)\lambda(x+\hat{\mu})^{-1},\ \lambda(x) = e^{-i\phi(x)},\ -\pi < \phi(x) \leq \pi.$$

- Impose smoothness (gauge non-inv) condition on possible λ: for $-\pi < \frac{1}{i}\ln\left[e^{-i\phi(x)}e^{i\phi(x+\hat{\mu})}\right] = \partial_\mu\phi(x) + 2\pi l_\mu(x) \leq \pi$,

$$\sup_{x,\mu} |\partial_\mu\phi(x) + 2\pi l_\mu(x)| < \delta, \qquad 0 < \delta < \pi/6.$$

Then $f_{\mu\nu}(x) \to f_{\mu\nu}(x) + \partial_\nu\phi(x+\hat{\mu}) + 2\pi l_\nu(x+\hat{\mu})$.

- Gauge-inv of CS term is realized by this condition on the lattice.

Backup: Sum over winding number and projection

- Assume $\widetilde{\mathcal{M}}_3 = T^3$ is perpendicular to $\hat{\mu}$; $\widetilde{\mathcal{M}}_3 = S^1 \times \widetilde{\mathcal{M}}_2^\nu$
- Introduce "scalar potential" $\varphi(x)$ as $l_\nu(x) = \partial_\nu \varphi(x)$
- Winding on a cycle (ν direction) provides additional integer k_ν to φ; $|k_\nu| < \frac{\delta}{2\pi} L$ with lattice size L
- $\sum ff$ acquires $4\pi k_\nu \sum_{x+\hat{\mu}\in\mathcal{M}_3, x_\nu=0} \epsilon_{\mu\nu\rho\sigma} \left[f_{\rho\sigma}(x+\hat{\mu}) + f_{\rho\sigma}(x)\right]$
- Gauge average implies that

$$\int \mathrm{D}\lambda\, e^{8\pi i\alpha\gamma k_\nu \sum_{x\in\mathcal{M}_2^\nu} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}(x)} \text{ and } \sum_{k=-\infty}^{\infty} e^{ikx} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n),$$

and then $$\delta\left(\frac{\alpha}{2\pi}\frac{1}{4\pi}\sum_{x\in\mathcal{M}_2^\nu} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}(x) - \mathbb{Z}\right) \quad \text{if } L \to \infty.$$

  - $\alpha/(2\pi)$ is irrational: no magnetic flux
  - $\alpha/(2\pi)$ is rational (p/N): $\frac{1}{2\pi}\int_{\mathcal{M}_2} da = N\mathbb{Z}$
- $\tilde{U}(\widetilde{\mathcal{M}}_3) = U(\widetilde{\mathcal{M}}_3)P(\widetilde{\mathcal{M}}_3)$, P is a projection operator for allowed magnetic fluxes