

# Intersubjectivity of outcomes of quantum measurements

MASANAO OZAWA

Graduate School of Informatics, Nagoya University, Chikusa-ku, Nagoya, 464-8601, Japan  
College of Engineering, Chubu University, 1200 Matsumoto-cho, Kasugai, 487-8501, Japan

Every measurement determines a single value as its outcome, and yet quantum mechanics predicts it only probabilistically [1]. The Kochen-Specker theorem [2] and Bell's inequality [3], enforced by the recent loophole-free experimental tests [4–6], reject a realist view that any observable has its own value at any time consistent with the statistical predictions of quantum mechanics, and favor a skeptical view that measuring an observable does not mean ascertaining the value that it has, but producing the outcome, having only a personal meaning [7–11]. However, precise analysis supporting this view is unknown. Here, we show that a quantum mechanical analysis turns down this view. Suppose that two observers simultaneously measure the same observable. We ask whether they always obtain the same outcomes, or their probability distributions are the same but the outcomes are uncorrelated. Contrary to the widespread view in favor of the second, we shall show that quantum mechanics predicts that only the first case occurs. This suggests the existence of a correlation between the measurement outcome and the pre-existing value of the measured observable as a common cause for the coincidence of the outcomes. In fact, we shall show that any measurement establishes a time-like entanglement between the observable to be measured and the meter after the measurement, which causes the space-like entanglement between the meters of different observers. We also argue that our conclusion cannot be extended to measurements of so-called “generalized” observables [12], suggesting a demand for more careful analysis on the notion of observables in foundations of quantum mechanics.

## I. INTRODUCTION—CONVENTIONAL VIEW

The theorems due to Kochen-Specker [2] and Bell [3] are often considered to defy the correlation between the measurement outcome and the *pre*-measurement value of the measured observable. Accordingly, it is a standard view that the measurement outcome should only correlate to the *post*-measurement value of the measured observable, as Schrödinger stated long ago:

The rejection of realism has logical consequences. In general, a variable has no definite value before I measure it; then measuring it does not mean ascertaining the value that it has. But then what does it mean? [...] Now it is fairly clear; if reality does not determine the measured value, then at least the measured value must determine reality [...] That is, the desired criterion can be merely this: repetition of the measurement must give the same result. [8, p. 329]

The *repeatability hypothesis* mentioned above is formulated as one of the basic axioms of quantum mechanics by von Neumann:

If [a] physical quantity is measured twice in succession in a system, then we get the same value each time. [13, p. 335]

It is well known that this hypothesis is equivalent to the *collapsing hypothesis* formulated by Dirac:

A measurement always causes the system to jump into an eigenstate of the observable that is being measured, the eigenvalue this eigenstate belongs to being equal to the result of the measurement. [14, p. 36]

The repeatability hypothesis and the collapsing hypothesis formulated as above have been broadly accepted as long as measurements of observables are concerned since the inception of quantum mechanics.

Von Neumann [13, p. 440] found a measuring interaction satisfying the repeatability for an observable  $A = \sum_a a|\varphi_a\rangle\langle\varphi_a|$ . He showed that such a measurement is described by a unitary operator  $U(\tau)$  such that

$$U(\tau)|\varphi_a\rangle|\xi\rangle = |\varphi_a\rangle|\xi_a\rangle, \quad (1)$$

where  $\{|\xi_a\rangle\}$  is an orthonormal basis for the environment and the meter observable is given by  $M = \sum_a a|\xi_a\rangle\langle\xi_a|$ . If the initial system state is a superposition  $|\psi\rangle = \sum_a c_a|\varphi_a\rangle$ , by linearity we obtain

$$U(\tau)|\psi\rangle|\xi\rangle = \sum_a c_a|\varphi_a\rangle|\xi_a\rangle. \quad (2)$$

Then we have

$$\Pr\{A(\tau) = x, M(\tau) = y\} = \delta_{x,y}|c_x|^2. \quad (3)$$

Thus, this measurement satisfies the probability reproducibility condition,

$$\Pr\{M(\tau) = x\} = |c_x|^2, \quad (4)$$

and the repeatability condition,

$$\Pr\{A(\tau) = x, M(\tau) = y\} = 0 \quad \text{if } x \neq y; \quad (5)$$

see [13, p. 440].

According to the above analysis, the measurement outcome is often considered to be created, rather than reproduced, by the act of measurement [7]. Quantum Bayesian interpretation emphasizes its personal nature as one of the fundamental tenets [11]. However, if we consider the process of measurement from a more general perspective, in which only the probability reproducibility condition is required, we confront a puzzling problem.

## II. UNIQUENESS OF MEASUREMENT OUTCOMES

Suppose that two remote observers, I and II, simultaneously measure the same observable. Then, we can ask whether quantum mechanics predicts that they always obtain the same outcome, or quantum mechanics predicts only that their probability distributions are the same but the outcomes are uncorrelated. In the following we shall show that quantum mechanics predicts that only the first case occurs, in contrast to a common interpretation of the theorems due to Kochen-Specker and Bell.

It is fairly well-known that any measurement can be described by an interaction between the system  $\mathbf{S}$  to be measured and the environment  $\mathbf{E}$  including measuring apparatuses and that the outcome of the measurement is obtained by a subsequent observation of a meter observable in the environment by the observer [13, 15].

Let  $A$  be an observable to be measured. Let  $M_1$  and  $M_2$  be the meter observables of observers I and II, respectively. We assume that at time 0 the system  $\mathbf{S}$  is in an arbitrary state  $|\psi\rangle$  and the environment  $\mathbf{E}$  is in a fixed state  $|\xi\rangle$ , respectively. In order to measure the observable  $A$  at time 0, observers I and II locally measure their meters  $M_1$  and  $M_2$  at times  $\tau_1 > 0$  and  $\tau_2 > 0$ , respectively.

Then, the time evolution operator  $U(t)$  of the total system  $\mathbf{S} + \mathbf{E}$  determines the Heisenberg operators  $A(0)$ ,  $M_1(t)$ ,  $M_2(t)$  for any time  $t > 0$ , where  $A(0) = A \otimes I$ ,  $M_1(t) = U(t)^\dagger(I \otimes M_1)U(t)$ , and  $M_2(t) = U(t)^\dagger(I \otimes M_2)U(t)$ . For any observable  $X$ , we denote by  $P^X(x)$  the spectral projection of  $X$  corresponding to  $x \in \mathbb{R}$ , i.e.,  $P^X(x)$  is the projection onto the subspace of vectors  $|\psi\rangle$  satisfying  $X|\psi\rangle = x|\psi\rangle$ .

We pose the following two assumptions.

**Assumption 1 (Locality).** We suppose that  $M_1(\tau_1)$  and  $M_2(\tau_2)$  are mutually commuting and that the joint probability distribution of the outcomes of measurements by observers I and II are given by

$$\begin{aligned} \Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} \\ = \langle \psi, \xi | P^{M_1(\tau_1)}(x) P^{M_2(\tau_2)}(y) | \psi, \xi \rangle \end{aligned} \quad (6)$$

for all  $x, y \in \mathbb{R}$ , where  $|\psi, \xi\rangle = |\psi\rangle|\xi\rangle$  [16].

**Assumption 2 (Probability reproducibility).** The measurements of the observable  $A$  by observers I and II satisfy the probability reproducibility condition, i.e.,

$$\Pr\{M_1(\tau_1) = x\} = \Pr\{M_2(\tau_2) = x\} = \Pr\{A(0) = x\} \quad (7)$$

for any  $x \in \mathbb{R}$ .

Assumption 1 is a natural consequence from the assumption that the two local meter-measurements by observers I and II are space-like separated. Thus, the joint probability of their outcomes is well-defined by Eq. (6), and our problem is well-posed. In Assumption 2 we only require that the outcome of a measurement of an observable should satisfy the Born rule for the measured observable, and we make no assumption on the state change caused by the measurement.

We shall show under Assumptions 1 and 2 that *the outcomes of the measurements of the observable  $A$  by observers I and II are always identical, namely, we have*

$$\Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} = 0 \quad (8)$$

if  $x \neq y$ .

The proof runs as follows. From Eq. (7) we have

$$\|P^{M_1(\tau_1)}(x)|\psi\rangle|\xi\rangle\|^2 = \|P^{A(0)}(x)|\psi\rangle|\xi\rangle\|^2.$$

Since  $|\psi\rangle$  is arbitrary, replacing it by  $P^A(y)|\psi\rangle/\|P^A(y)|\psi\rangle\|$  if  $P^A(y)|\psi\rangle \neq 0$ , we obtain

$$\begin{aligned} \|P^{M_1(\tau_1)}(x)P^{A(0)}(y)|\psi\rangle|\xi\rangle\|^2 \\ = \|P^{M_1(\tau_1)}(x)(P^A(y)|\psi\rangle)|\xi\rangle\|^2 \\ = \|P^{A(0)}(x)(P^A(y)|\psi\rangle)|\xi\rangle\|^2 \\ = \delta_{x,y}\|P^{A(0)}(y)|\psi\rangle|\xi\rangle\|^2. \end{aligned}$$

Since  $P^{M_1(\tau_1)}(x)$  is a projection, it follows that

$$P^{M_1(\tau_1)}(x)P^{A(0)}(y)|\psi\rangle|\xi\rangle = \delta_{x,y}P^{A(0)}(y)|\psi\rangle|\xi\rangle.$$

Summing up both sides of the above equation for all  $y$ , we obtain

$$P^{M_1(\tau_1)}(x)|\psi\rangle|\xi\rangle = P^{A(0)}(x)|\psi\rangle|\xi\rangle. \quad (9)$$

Similarly,

$$P^{M_2(\tau_2)}(x)|\psi\rangle|\xi\rangle = P^{A(0)}(x)|\psi\rangle|\xi\rangle. \quad (10)$$

Therefore, we have

$$\begin{aligned} \Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} \\ = \langle \psi, \xi | P^{M_1(\tau_1)}(x) P^{M_2(\tau_2)}(y) | \psi, \xi \rangle \\ = \langle \psi, \xi | P^{A(0)}(x) P^{A(0)}(y) | \psi, \xi \rangle \\ = 0 \end{aligned}$$

if  $x \neq y$ . Thus, we conclude that the joint probability distribution of the outcomes of the simultaneous measurements of the observable  $A$  by observers I and II is given by Eq. (8), which shows that the outcomes are always identical.

It can be easily seen that the above conclusion can be extended to the assertion for  $n$  observers with any  $n > 2$ . Thus, we conclude that *if two or more remote observers simultaneously measure the same observable, then their outcomes always coincide.*

Example 1 in Appendix A illustrates a typical system-environment interaction to realize simultaneous position measurements of  $n$  observers.

## III. NON-UNIQUENESS FOR GENERALIZED OBSERVABLES

We note that the above result cannot be extended to an arbitrary ‘‘generalized observable’’  $A$  represented by a probability operator-valued measure (POVM), i.e., a family  $\{P^A(x)\}$  of

positive operators  $P^A(x) \geq 0$ , instead of projections, such that  $\sum_x P^A(x) = I$ . The optical phase is not considered as a quantum observable but typically considered as a physical quantity corresponding to a generalized observable (see Ref. [17] and the references therein).

To immediately see that our conclusion cannot be extended to the class of generalized observables, consider a generalized observable  $A$  defined by  $P^A(x) = \mu(x)I$ , where  $\mu$  is an arbitrary probability distribution, *i.e.*,  $\mu(x) \geq 0$  and  $\sum_x \mu(x) = 1$ . Then, as shown in Example 2 in Appendix B, we can construct continuously parametrized models for which Assumptions 1 and 2 are satisfied but Eq. (8) is not satisfied.

#### IV. VALUE REPRODUCIBILITY OF MEASUREMENT

The uniqueness of the measurement outcomes ensures that in quantum mechanics the phrase “the outcome of a measurement of an observable  $A$  at time  $t$ ” has an unambiguous meaning. This suggests the existence of a correlation between the measurement outcome and the pre-measurement value of the measured observable as a common cause for the coincidence of the outcomes. Then, this unambiguous value of the outcome of the measurement of an observable  $A$  is considered to be the value that the observable  $A$  has just before the measurement. Since quantum mechanics predicts a relation between values of observables only in the form of probability correlations, this fact should be best expressed by the relation

$$\Pr\{A(0) = x, M(\tau) = y\} = 0 \quad (11)$$

if  $x \neq y$ , where  $M(\tau) = M_1(\tau_1)$  or  $M(\tau) = M_2(\tau_2)$ . We call this relation the *value reproducibility condition*. However, this relation shows a difficulty. Since  $A(0)$  and  $M(\tau)$  may not commute in general, the joint probability distribution may not be well-defined.

In what follows, we shall show that the above relation is actually well-defined to hold, even though  $A(0)$  and  $M(t)$  do not commute as operators. To see this, recall the notion of *partial commutativity* or *state-dependent commutativity* introduced by von Neumann [13, p. 230]: *If a state  $|\Psi\rangle$  is a superposition of common eigenstates  $|X = x, Y = y\rangle$  of observables  $X$  and  $Y$  of the form*

$$|\Psi\rangle = \sum_{x,y \in S} c_{x,y} |X = x, Y = y\rangle,$$

where  $S \subseteq \mathbb{R}^2$ , then the joint probability distribution  $\Pr\{X = x, Y = y\}$  of  $X$  and  $Y$  in  $|\Psi\rangle$  is well-defined as

$$\Pr\{X = x, Y = y\} = \begin{cases} |c_{x,y}|^2 & \text{if } (x,y) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

In this case,  $X$  and  $Y$  actually commute on the subspace  $\mathcal{M}$  generated by  $\{|X = x, Y = y\rangle\}_{(x,y) \in S}$ , and we say that  $X$  and  $Y$  commute in the state  $|\Psi\rangle$ . In fact, the joint probability distribution  $\Pr\{X = x, Y = y\}$  satisfies the relation

$$\langle \Psi | f(X, Y) | \Psi \rangle = \sum_{x,y} f(x, y) \Pr\{X = x, Y = y\}$$

for every real polynomial  $f(X, Y)$  of  $X$  and  $Y$ .

Under the above definition of joint probability distributions, we can show that the joint probability distribution  $\Pr\{A(0) = x, M(\tau) = y\}$  is well-defined in the state  $|\psi\rangle|\xi\rangle$  and satisfies Eq. (11) as shown in Theorem 1 (iii) in Appendix C.

From Eq. (11), we conclude that *every probability reproducible measurement of an observable  $A$  is value reproducible*. We call Eq. (9) or Eq. (10) the *time-like entanglement condition*. It should also be pointed out that the uniqueness of the outcomes for multiple observers is an immediate consequence of the time-like entanglement condition as shown in the derivation of Eq. (8). In Theorem 1 in Appendix C, we show that those three conditions, the probability reproducibility condition, the time-like entanglement condition, and the value reproducibility condition, are all equivalent.

#### V. VALUE REPRODUCIBILITY OF THE CONVENTIONAL MODEL

In the conventional approach to quantum measurements of an observable  $A = \sum_a a |\varphi_a\rangle\langle\varphi_a|$  the measurement is required to satisfy both the probability reproducibility, Eq. (4), and the repeatability, Eq. (5), whereas the rejection of the realism in quantum mechanics is considered to defy the value reproducibility, Eq. (11), of the measurement [18]. However, according to our analysis discussed so far, we should point out that *the conventional analysis of measuring process given by Eq. (2) has failed to unveil the fact that the measurement actually satisfies the value reproducibility*, that is, a measurement *reproduces* the value of the observable to be measured by establishing the time-like entanglement between  $A(0)$  and  $M(\tau)$ . To see this by direct computations, first note that Eq. (2) can also be rewritten as

$$|\psi\rangle|\xi\rangle = \sum_a c_a |A(\tau) = a, M(\tau) = a\rangle, \quad (12)$$

showing that the measurement establishes the space-like entanglement between  $A(\tau)$  and  $M(\tau)$ . Now, we shall show that Eq. (2) can be rewritten as

$$|\psi\rangle|\xi\rangle = \sum_a c_a |A(0) = a, M(\tau) = a\rangle, \quad (13)$$

which in turn shows that the measurement establishes the time-like entanglement between  $A(0)$  and  $M(\tau)$ . In fact, we have

$$\begin{aligned} A(0)|\varphi_a\rangle|\xi\rangle &= (A \otimes I)|\varphi_a\rangle|\xi\rangle \\ &= a|\varphi_a\rangle|\xi\rangle, \\ M(\tau)|\varphi_a\rangle|\xi\rangle &= U(\tau)^\dagger (I \otimes M) U(\tau) |\varphi_a\rangle|\xi\rangle \\ &= U(\tau)^\dagger (I \otimes M) |\varphi_a\rangle|\xi_a\rangle \\ &= aU(\tau)^\dagger |\varphi_a\rangle|\xi_a\rangle \\ &= a|\varphi_a\rangle|\xi\rangle. \end{aligned}$$

Thus, Eq. (13) holds with

$$|A(0) = a, M(\tau) = a\rangle = |\varphi_a\rangle|\xi\rangle. \quad (14)$$

Note that the above argument equally holds only assuming the probability reproducibility as

$$U(\tau)|\varphi_a\rangle|\xi\rangle = |\varphi'_a\rangle|\xi_a\rangle, \quad (15)$$

where  $\{|\varphi'_a\rangle\}$  is an arbitrary family of states in  $\mathcal{H}$  instead of Eq. (1).

## VI. DISCUSSION

Schrödinger [8, p. 329] argued that a measurement does not ascertain the pre-existing value of the observable and is only required to be repeatable. Since the inception of quantum mechanics, this view has long been supported as one of the fundamental tenets of quantum mechanics. In contrast, we have shown that any probability reproducible measurement indeed ascertains the value that the observable has, whether the repeatability is satisfied or not.

It is an interesting problem to what extent a probability reproducible measurement of a generalized observable can be value reproducible. Theorem 2 in Appendix D answers this question rather surprisingly as that only conventional observables can be measured value-reproducibly. This suggests a demand for more careful analysis on the notion of observables in foundations of quantum mechanics. In this area, generalized probability theory [19, 20] has recently been studied extensively. However, the theory only has the notion of “generalized” observable, but does not have the counter part of “conventional” observables being value-reproducibly measurable.

In this paper, we have considered the notion of measurement of observables “state-independently”, and we take it for granted that a measurement of an observable is accurate if and only if it satisfies the probability-reproducibility in all states. However, this does not mean that “state-dependent” definition of an accurate measurement of an observable should only require the probability-reproducibility in a given state, since requiring the probability-reproducibility for all the state is logically equivalent to requiring the value-reproducibility for all the state as shown in this paper. In the recent debate on the formulation of measurement uncertainty relations, some authors have claimed that the state-dependent approach to this problem is not tenable, based on the state-dependent probability-reproducibility requirement [21, 22]. In contrast, we have recently shown that state-dependent approach to measurement uncertainty relations is indeed tenable, based on the state-dependent value-reproducibility requirement [23]. The debate suggests that the value-reproducibility is more reasonable requirement for the state-dependent accuracy of measurements of observables.

The cotextuality in assigning the values to observables shown by the theorems due to Kochen-Specker [2] and Bell [3] is often considered as the rejection of realism. However, it should be emphasized that what is real depends on a particular philosophical premise, and it is not completely determined by physics. Here, we have revealed a new probability correlation, Eq. (8), predicted solely by quantum mechanics ensuring that the outcome of a measurement of an observable

is unambiguously defined in quantum mechanics worth communicating intersubjectively. Further, we have shown that the intersubjectivity of outcomes of measurements is an immediate consequence from another new probability correlation, Eq. (11), the value-reproducibility of measurements. Since the value-reproducibility is in an obvious conflict with the rejection of realism, it would be an interesting problem to interpret quantum reality taking into account both the contextuality of value-assignments of observables and the intersubjectivity of outcomes of measurements.

### Appendix A: Uniqueness for simultaneous position measurements

**Example 1.** The system  $\mathbf{S}$  to be measured has canonically conjugate observables  $Q, P$  on an infinite dimensional state space with  $[Q, P] = i\hbar$ . Consider the measurement of the observable  $A = Q$ . The environment  $\mathbf{E}$  consists of  $n$  sets of canonically conjugate observables  $Q_j, P_j$  with  $[Q_j, P_k] = \delta_{jk}$ ,  $[Q_j, Q_k] = [P_j, P_k] = 0$  for  $j, k = 1, \dots, n$ . Here, the part of the actual environment not effectively interacting with  $\mathbf{S}$  can be neglected without any loss of generality. Consider  $n$  observers with their meters  $M_j = Q_j$  for  $j = 1, \dots, n$ . Suppose that the system  $\mathbf{S}$  is in an arbitrary state  $|\psi\rangle$  and the environment  $\mathbf{E}$  is in the joint eigenstate  $|\xi\rangle = |Q_1 = 0, \dots, Q_n = 0\rangle$ . The interaction between  $\mathbf{S}$  and  $\mathbf{E}$  is given by

$$H = KQ \otimes (P_1 + \dots + P_n),$$

where the coupling constant  $K$  is large enough to neglect the other term in the total Hamiltonian of the composite system  $\mathbf{S} + \mathbf{E}$ . Then, we have

$$\begin{aligned} \frac{d}{dt}Q_j(t) &= \frac{1}{i\hbar}[Q_j(t), H(t)] = KQ(t), \\ Q_j(t) &= Q_j(0) + KtQ(0). \end{aligned}$$

Assumptions 1 and 2 are satisfied for  $\tau_j = 1/K$  with  $j = 1, \dots, n$ , *i.e.*,

$$\begin{aligned} [M_j(\tau_j), M_k(\tau_k)] &= 0, \\ \langle \psi, \xi | P^{M_j(\tau_j)}(x) | \psi, \xi \rangle dx &= \langle \psi | P^Q(x) | \psi \rangle dx. \end{aligned}$$

In this case,  $M_j(t) - M_k(t) = Q_j(t) - Q_k(t)$  are the constant of the motion for all  $j, k$ , *i.e.*,

$$\frac{d}{dt}(Q_j(t) - Q_k(t)) = \frac{1}{i\hbar}[Q_j(t) - Q_k(t), H(t)] = 0.$$

Thus, the outcomes are identical for all the observers, *i.e.*,

$$\begin{aligned} \langle \psi, \xi | P^{M_j(\tau_j)}(x) P^{M_k(\tau_k)}(y) | \psi, \xi \rangle dx dy \\ = \delta(x - y) |\psi(x)|^2 dx dy. \end{aligned}$$

### Appendix B: Measurements of generalized observables

**Example 2.** Let  $A$  be a generalized observable on a system  $\mathbf{S}$  described by a Hilbert space  $\mathcal{H}$  such that  $P^A(x) = \mu(x)I$ ,

where  $\mu$  is a probability distribution. Let  $X = \{x \in \mathbb{R} \mid \mu(x) > 0\}$ . Suppose that the environment  $\mathbf{E}$  consists of two subsystems so that the environment is described by a Hilbert space  $\mathcal{K} = \mathcal{L} \otimes \mathcal{L}$ , where  $\mathcal{L}$  is a Hilbert space spanned by an orthonormal basis  $\{|x\rangle\}_{x \in X}$ . Suppose that observers I and II measure the meter observables  $M_1(\tau_1) = I \otimes M \otimes I$  and  $M_2(\tau_2) = I \otimes I \otimes M$ , respectively, which may be constants of motion, where  $M = \sum_x x|x\rangle\langle x|$ , so that Assumption 1 is satisfied. The initial state of the environment  $\mathbf{E}$  can be represented by

$$|\xi\rangle = \sum_x c_{x,y}|x\rangle|y\rangle.$$

Then, the joint probability distribution of  $M_1(\tau_1)$  and  $M_2(\tau_2)$  is given by

$$\begin{aligned} \Pr\{M_1(\tau_1) = x, M_2(\tau_2) = y\} \\ = \langle \psi, \xi | P^{M_1(\tau_1)}(x) P^{M_2(\tau_2)}(y) | \psi, \xi \rangle = |c_{x,y}|^2. \end{aligned}$$

Thus, Assumption 2 is satisfied if and only if  $\mu(x) = \sum_y |c_{x,y}|^2 = \sum_y |c_{y,x}|^2$ . Thus, under Assumptions 1 and 2, the joint probability distribution  $\mu(x, y) = \Pr\{M_1(\tau) = x, M_2(\tau) = y\}$  can be an arbitrary 2-dimensional probability distribution such that  $\sum_y \mu(x, y) = \sum_y \mu(y, x) = \mu(x)$ . In this case, Eq. (8) is satisfied if and only if  $|c_{x,y}|^2 = \delta_{x,y}\mu(x)$ . Thus, we have continuously parametrized models for which Assumptions 1 and 2 are satisfied but Eq. (8) is not satisfied.

### Appendix C: Equivalence

We call a quadruple  $(\mathcal{K}, |\xi\rangle, U(\tau), M)$  a *measuring process* for a Hilbert space  $\mathcal{H}$  (describing the measured system  $\mathbf{S}$ ) if  $\mathcal{K}$  is a Hilbert space (describing the environment  $\mathbf{E}$ ),  $|\xi\rangle$  is a state vector in  $\mathcal{K}$  (describing the initial state of  $\mathbf{E}$ , say, at time 0),  $U(\tau)$  is a unitary operator on  $\mathcal{H} \otimes \mathcal{K}$  (describing the time evolution of  $\mathbf{S} + \mathbf{E}$  from time 0 to  $\tau$ ), and  $M$  is an observable on  $\mathcal{K}$  (describing the meter to observe at time  $\tau$ ) [15]. In this case, we define  $A(0) = A \otimes I$  and  $M(\tau) = U(\tau)^\dagger (I \otimes M) U(\tau)$ . Then, we have

**Theorem 1.** *Let  $A$  be an observable on a Hilbert space  $\mathcal{H}$ . Let  $(\mathcal{K}, |\xi\rangle, U(\tau), M)$  be a measuring process for  $\mathcal{H}$ . Then the following conditions are all equivalent.*

(i) (*Probability reproducibility*) For any  $x \in \mathbb{R}$  and state  $|\psi\rangle \in \mathcal{H}$ ,

$$\langle \psi, \xi | P^{M(\tau)}(x) | \psi, \xi \rangle = \langle \psi | P^A(x) | \psi \rangle. \quad (\text{C1})$$

(ii) (*Time-like entanglement*) For any  $x \in \mathbb{R}$  and state  $|\psi\rangle \in \mathcal{H}$ ,

$$P^{M(\tau)}(x) | \psi, \xi \rangle = P^{A(0)}(x) | \psi, \xi \rangle. \quad (\text{C2})$$

(iii) (*Value reproducibility*) For any state  $|\psi\rangle \in \mathcal{H}$ , we have that the observables  $A(0)$  and  $M(\tau)$  commute in the state  $|\psi\rangle|\xi\rangle$ , and for any  $x, y \in \mathbb{R}$  we have

$$\langle \psi, \xi | P^{A(0)}(x) P^{M(\tau)}(y) | \psi, \xi \rangle = 0 \quad (\text{C3})$$

if  $x \neq y$ , namely, Eq. (11) holds.

*Proof.* (i) $\Rightarrow$ (ii): Suppose that condition (i) holds. Condition (ii) follows from the argument in the main text deriving Eq. (9).

(ii) $\Rightarrow$ (iii): Suppose that condition (ii) holds. To derive Eq. (11), it suffices to show that the state  $|\psi\rangle|\xi\rangle$  is a superposition of common eigenstates  $|A(0) = x, M(\tau) = x\rangle$  of the form

$$|\psi\rangle|\xi\rangle = \sum_{(x,x) \in S} c_{x,x} |A(0) = x, M(\tau) = x\rangle, \quad (\text{C4})$$

since if this holds, we have  $\sum_{(x,x) \in S} |c_{x,x}|^2 = 1$  so that  $(x, y) \notin S$  if  $x \neq y$ . Let

$$|\Psi_{x,y}\rangle = P^{A(0)}(x) P^{M(\tau)}(y) |\psi\rangle|\xi\rangle.$$

Then, we have  $|\psi\rangle|\xi\rangle = \sum_{x,y} |\Psi_{x,y}\rangle$ . It follows from Eq. (C2) that  $|\Psi_{x,y}\rangle = 0$  if  $x \neq y$ , and

$$A(0) |\Psi_{x,x}\rangle = M(\tau) |\Psi_{x,x}\rangle = x |\Psi_{x,x}\rangle.$$

Thus,  $|\Psi_{x,x}\rangle$  is a common eigenstate of  $A(0)$  and  $M(\tau)$  with common eigenvalue  $x$  if  $|\Psi_{x,x}\rangle \neq 0$ . Thus, we obtain Eq. (C4) with  $c_{x,x} = \frac{\|\Psi_{x,x}\rangle\|}{\|\Psi_{x,x}\rangle\|}$  and

$$|A(0) = x, M(\tau) = x\rangle = \frac{|\Psi_{x,x}\rangle}{\|\Psi_{x,x}\rangle\|}.$$

Therefore, we conclude Eq. (11).

(iii) $\Rightarrow$ (i): Suppose that (iii) holds. It is easy to see that the relation

$$\begin{aligned} \sum_y \langle \psi, \xi | P^{A(0)}(y) P^{M(\tau)}(x) | \psi, \xi \rangle \\ = \sum_y \langle \psi, \xi | P^{A(0)}(x) P^{M(\tau)}(y) | \psi, \xi \rangle \end{aligned}$$

holds, and Eq. (C1) follows.  $\square$

### Appendix D: Value reproducibility of measuring processes

A generalized observable  $A$  on a Hilbert space  $\mathcal{H}$  is called *value-reproducibly measurable* if there exists a measuring process  $(\mathcal{K}, |\xi\rangle, U(\tau), M)$  for a Hilbert space  $\mathcal{H}$  satisfying condition (iii) of Theorem 1 (for the generalized observable  $A$ ), where  $P^{A(0)}(x) = P^A(x) \otimes I$ . Then, we have

**Theorem 2.** *A generalized observable is value-reproducibly measurable if and only if it is an observable (in the conventional sense).*

*Proof.* In the main text it has been shown that every observable is value-reproducibly measurable. It suffices to show the converse. Let  $A$  be a generalized observable, which is value-reproducibly measurable with a measuring process  $(\mathcal{K}, |\xi\rangle, U(\tau), M)$  satisfying Eq. (C3) if  $x \neq y$ . By the Naimark-Holevo dilation theorem [12], there exist a Hilbert space  $\mathcal{G}$ , a state vector  $|\eta\rangle \in \mathcal{G}$ , and an observable  $B$  on  $\mathcal{G} \otimes \mathcal{H}$  such that

$$P^A(x) = \langle \eta | P^B(x) | \eta \rangle$$

for all  $x$ . Let  $B(0) = B \otimes I_{\mathcal{K}}$  and  $N(\tau) = I_{\mathcal{G}} \otimes M(\tau)$ . Let  $|\eta_1\rangle, |\eta_2\rangle, \dots$  be an orthonormal basis of  $\mathcal{G}$  such that  $|\eta\rangle = |\eta_1\rangle$ . Let  $|\psi_1\rangle, |\psi_2\rangle, \dots$  be an orthonormal basis of  $\mathcal{H}$  such that  $|\psi\rangle = |\psi_1\rangle$ . Let  $|\xi_1\rangle, |\xi_2\rangle, \dots$  be an orthonormal basis of  $\mathcal{K}$  such that  $|\xi\rangle = |\xi_1\rangle$ . Then, we have

$$\langle \eta_j, \psi_k, \xi_l | P^{N(\tau)}(y) | \eta, \psi, \xi \rangle = 0$$

if  $j \neq 1$ . Thus, we have

$$\begin{aligned} & \langle \eta, \psi, \xi | P^{B(0)}(x) P^{N(\tau)}(y) | \eta, \psi, \xi \rangle \\ &= \sum_{j,k,l} \langle \eta, \psi, \xi | P^{B(0)}(x) | \eta_j, \psi_k, \xi_l \rangle \\ & \quad \times \langle \eta_j, \psi_k, \xi_l | P^{N(\tau)}(y) | \eta, \psi, \xi \rangle \\ &= \sum_{k,l} \langle \eta, \psi, \xi | P^{B(0)}(x) | \eta, \psi_k, \xi_l \rangle \\ & \quad \times \langle \eta, \psi_k, \xi_l | P^{N(\tau)}(y) | \eta, \psi, \xi \rangle \\ &= \sum_{k,l} \langle \psi, \xi | P^{A(0)}(x) | \psi_k, \xi_l \rangle \langle \psi_k, \xi_l | P^{M(\tau)}(y) | \psi, \xi \rangle \\ &= \langle \psi, \xi | P^{A(0)}(x) P^{M(\tau)}(y) | \psi, \xi \rangle. \end{aligned}$$

Let  $|\Psi\rangle = |\eta\rangle|\psi\rangle|\xi\rangle$ . From the value reproducibility of  $A$ , we have

$$\langle \Psi | P^{B(0)}(x) P^{N(\tau)}(y) | \Psi \rangle = 0. \quad (\text{D1})$$

if  $x \neq y$ . It follows that

$$\begin{aligned} \langle \Psi | P^{B(0)}(x) P^{N(\tau)}(x) | \Psi \rangle &= \|P^{B(0)}(x) | \Psi \rangle\|^2 \\ &= \|P^{N(\tau)}(x) | \Psi \rangle\|^2, \end{aligned}$$

and hence

$$\begin{aligned} & \|P^{B(0)}(x) | \Psi \rangle - P^{N(\tau)}(x) | \Psi \rangle\|^2 \\ &= \|P^{B(0)}(x) | \Psi \rangle\|^2 + \|P^{N(\tau)}(x) | \Psi \rangle\|^2 \\ & \quad - 2 \langle \Psi | P^{B(0)}(x) P^{N(\tau)}(x) | \Psi \rangle \\ &= 0 \end{aligned}$$

Therefore, we obtain

$$P^{B(0)}(x) | \eta \rangle | \psi \rangle | \xi \rangle = P^{N(\tau)}(x) | \eta \rangle | \psi \rangle | \xi \rangle,$$

so that

$$\begin{aligned} P^{A(0)}(x) | \psi \rangle | \xi \rangle &= \langle \eta | P^{B(0)}(x) | \eta \rangle | \psi \rangle | \xi \rangle \\ &= \langle \eta | P^{N(\tau)}(x) | \eta \rangle | \psi \rangle | \xi \rangle \\ &= P^{M(\tau)}(x) | \psi \rangle | \xi \rangle. \end{aligned}$$

Since  $|\psi\rangle$  is arbitrary, replacing  $|\psi\rangle$  by  $P^A(y)|\psi\rangle$  we have

$$P^{A(0)}(x) (P^A(y) | \psi \rangle) | \xi \rangle = P^{M(\tau)}(x) (P^A(y) | \psi \rangle) | \xi \rangle,$$

and hence

$$\begin{aligned} & P^{A(0)}(x) P^{A(0)}(y) | \psi \rangle | \xi \rangle \\ &= P^{A(0)}(x) (P^A(y) | \psi \rangle) | \xi \rangle \\ &= P^{M(\tau)}(x) (P^A(y) | \psi \rangle) | \xi \rangle \\ &= P^{M(\tau)}(x) P^{A(0)}(y) | \psi \rangle | \xi \rangle \\ &= P^{M(\tau)}(x) P^{M(\tau)}(y) | \psi \rangle | \xi \rangle \\ &= \delta_{x,y} P^{M(\tau)}(x) | \psi \rangle | \xi \rangle \\ &= \delta_{x,y} P^{A(0)}(x) | \psi \rangle | \xi \rangle. \end{aligned}$$

Thus, we have

$$P^A(x) P^A(y) | \psi \rangle = \delta_{x,y} P^A(x) | \psi \rangle \quad (\text{D2})$$

for all  $|\psi\rangle$ , so that  $A$  is an observable.  $\square$

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