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On Logistic Diffusion Equations with Nonlocal Effects

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1 Introduction

In this paper we discuss the following problem for logistic equations with diffusion and nonlocal effects:

$$
\text{(P)} \qquad \begin{cases} u_t = d\Delta u + u\left(a - bu - \displaystyle\int_\Omega k(x,y)u(y,t)dy\right) & \text{in } \ \Omega \times (0,\infty), \\ u = 0 & \text{on } \ \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \ \Omega, \end{cases}
$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, a, d are positive constants, b is a nonnegative constant, $k \in C(\overline{\Omega} \times \overline{\Omega})$ is a nonnegative function and u_0 is a nonnegative function. In (P), u denotes the population density of a certain species. Usually, the dynamics of the population density is governed by a logistic diffusion equation (without nonlocal terms). If $k \equiv 0$ in (P), it is well known that there exists a unique global solution u and that

$$
\lim_{t\to\infty} u(\cdot,t) = \begin{cases} 0 & \text{uniformly in } \ \Omega \ \text{ if } \ 0 < a \le d\lambda_1, \\ \theta & \text{uniformly in } \ \Omega \ \text{ if } \ a > d\lambda_1, \end{cases}
$$

where λ_1 is the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition and θ is a unique positive stationary solution (which exists if and only if $a > d\lambda_1$). However, it is sometimes reasonable to take account of nonlocal effects since each individual species interacts either visually or by chemical means in a real world. So we will discuss a logistic diffusion equation by adding a nonlocal reaction term as in (P).

Our main purpose is to investigate the difference or similarity between local problems and nonlocal problems for logistic diffusion equations. In particular, we are interested in the following points:

(a) Existence and uniqueness of bounded global solutions for (P),
(b) Asymptotic behavior of global solutions as $t \to \infty$,
(c) Structure of positive solutions for the corresponding stationary problem:

$$
\text{(SP)} \qquad \begin{cases} d\Delta u + u\left(a - bu - \displaystyle\int_\Omega k(x,y)u(y)dy\right) = 0 & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases}
$$

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For semilinear elliptic equations with nonlocal terms, there are a lo of works (see, e.g, [1], [2], [3], [6], [8]). In most papers, existence of positive solutions has been established with use of bifurcation theory or the Leray-Schauder degree theory. Here we will give a very elementary method to construct a positive stationary solution to (SP).

The contents of the present paper are as follows. In Section 2, we will show that (P) admits a unique global solution for any nonnegative initial data in a suitable class. Section 3 is devoted to the analysis of (SP). We will look for a positive solution of (SP) by a constructive manner. Finally, some remarks are given in section 4.

Notation. We denote by $L^p(\Omega)$ the space of measurable functions $u: \ \Omega \to R$ such that $|u(x)|^p$ is integrable over Ω with norm

$$\|u\|_p := \left\{ \int_\Omega |u(x)|^p \ dx \right\}^{1/p}.$$

For $p = 2$, we simply write $\|\cdot\|$ in place of $\|\cdot\|_2$. By $W^{k,p}(\Omega)$, we denote the Sobolev space of functions $u \to R$ such that u and its distributional derivatives up to order k belong to $L^p(\Omega)$. Its norm is defined by

$$\|u\|^p_{W^{k,p}} = \sum_{|\rho| \le k} \|D^\rho u\|^p_p,$$

where ρ denotes a multi-index for derivatives.

2 Existence of global solutions

We will discuss (P) in the framework of $L^p(\Omega)$ with $p > 1$. Define a closed linear operator A in $L^p(\Omega)$ by

$$Au = -d\Delta u \quad \text{with domain} \quad D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega).$$

Then it is well known that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t \ge 0}$ in $L^p(\Omega)$ (see, e.g., [9, 11]). Our problem (P) can be written as

$$\begin{cases} u_t + Au = f(u, \ell(u)), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where

$$f(u, v) = u(a - bu - v) \qquad \text{with} \quad \ell(u) = \int_\Omega k(x, y)u(y)dy.$$

For (2.1) we can prove the following local existence theorem:

Theorem 2.1. *Let $p > \max\{1, N/2\}$. For any $u_0 \in L^p(\Omega)$, there exists a positive number T such that* (2.1) *has a unique solution u in the class*

$$u \in C([0, T]; L^p(\Omega)) \cap C((0, T]; W^{2,p}(\Omega)) \cap C^1((0, T]; L^p(\Omega)).$$

Proof. The proof is standard. The first procedure is to rewrite (2.1) in the form of integral equation

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s), \ell(u(s)))ds. \tag{2.2}$$

The second procedure is to apply Banach's fixed point theorem to (2.2) in order to show the existence and uniqueness of a local solution. For details, see [9] or [11]. □

In what follows we assume

$$u_0 \in L^\infty(\Omega) \tag{2.3}$$

and establish the global existence theorem.

Theorem 2.2. *Let $p > \max\{1, N/2\}$ and assume* (2.3).
(i) *If $b > 0$, then* (2.1) *has a unique solution u in the class*

$$u \in C([0,\infty); L^p(\Omega)) \cap C((0,\infty); W^{2,p}(\Omega)) \cap C^1((0,\infty); L^p(\Omega)).$$

Moreover, u satisfies

$$0 \le u(x,t) \le \max\left\{\|u_0\|_\infty, \frac{a}{b}\right\}$$

for all $(x,t) \in \Omega \times [0,\infty)$.
(ii) *If $b = 0$, then* (2.1) *has a unique solution u in the same class as* (i). *Moreover, if there exists a positive constant k_0 such that $k(x,y) \ge k_0$ for all $x, y \in \Omega$, then*

$$0 \le u(x,t) \le m$$

with a positive number m for all $(x,t) \in \Omega \times [0,\infty)$.

Proof. (i) Since $u_0 \ge 0$, it is easy to show by the maximum principle for parabolic equations (see [12]) that $u(\cdot,t) \ge 0$ as long as it exists. Therefore, u satisfies

$$u_t \le d\Delta u + u(a - bu) \qquad \text{in } \ \Omega \times [0,T),$$

where T is a maximal existence time. The comparison theorem for parabolic equations enables us to show that

$$u \le \max\left\{\|u_0\|_\infty, \frac{b}{a}\right\}$$

for $(x,t) \in \Omega \times [0,T)$. Hence we can conclude $T = \infty$ and obtain a required estimate.

(ii) We will show the uniform boundedness of the solution u in case $k \ge k_0$. Integrating the first equation of (P) leads to

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u(x,t)dx &= d\int_\Omega \Delta u(x,t) + a\int_\Omega u(x,t)dx - \int_\Omega u(x,t)\ell(u(t))dx \\
&= d\int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma + a\int_\Omega u(x,t)dx - \int_\Omega u(x,t)\left(\int_\Omega k(x,y)u(y,t)dy\right)dx \\
&< a\int_\Omega u(x,t)dx - k_0\left(\int_\Omega u(x,t)dx\right)^2.
\end{aligned} \tag{2.4}$$

Here we have used $\partial u/\partial n|_{\partial\Omega} < 0$ by the strong maximum principle (see [12]). Solving differential inequality (2.4) we get

$$\int_\Omega u(x,t)dx \le \max\left\{\|u_0\|_1, \frac{a}{k_0}\right\}. \tag{2.5}$$

Since $|\ell(u)| \le k_\infty \|u\|_1$ with $k_\infty = \sup\{k(x,y);\ x,y \in \Omega\}$, we see

$$\|f(u,\ell(u))\|_1 = \|u(a - \ell(u))\|_1 \le a\|u\|_1 + k_\infty \|u\|_1^2;$$

so that it follows from (2.5) that

$$\sup_{t\ge 0}\{\|f(u(t), \ell(u(t)))\|_1\} = m_1.$$

In order to derive uniform boundedness of $u(t)$, it is sufficient to use $L^p - L^q$ estimates for $\{e^{-tA}\}_{t\ge 0}$ with $p,q \in [1,\infty]$ and follow the arguments developed in the work of Rothe [13]. So we omit the rest of the proof.

□

3 Stationary positive solutions

In this section we will study (SP) associated with (P). In particular, we are interested in positive stationary solutions and look for them in the case

$$k(x, y) = p(x)q(y), \tag{3.1}$$

where $p, q(\not\equiv 0)$ are nonnegative continuous functions in $\overline{\Omega}$. So our problem is written as follows:

$$\begin{cases} d\Delta u + u\left(a - bu - p(x)\displaystyle\int_\Omega q(y)u(y)dy\right) = 0 & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ u > 0 & \text{in } \ \Omega, \end{cases} \tag{3.2}$$

where a, d are positive constants, b is a nonnegative number. Lots of authors (e.g., [1], [2], [3]. [6], [8]) have discussed the existence of positive solutions for semilinear elliptic equations with nonlocal terms by means of bifurcation theory, the Leray-Schauder degree theory and monotone methods. Among them, Corréa, Delgado and Suárez [2] have studied (3.2) in case $b = 0$ and obtained an interesting result.

Theorem 3.1. ([2]) *Assume that* $\Omega_0 := \mathrm{Int}\{x \in \Omega; p(x) = 0\}$ *is connected. Then* (3.2) *has a unique positive solution* u *if and only if*

$$\begin{cases} a \in (\lambda_{1,\Omega}, \infty) & \textit{in case } \ \Omega_0 = \emptyset, \\ a \in (\lambda_{1,\Omega}, \lambda_{1,\Omega_0}) & \textit{in case } \ \Omega_0 \neq \emptyset. \end{cases}$$

Here $\lambda_{1,D}$ *stands for the principal eigenvalue of the following eigenvalue problem*

$$-\Delta u = \lambda u \quad \textit{in } \ D \quad \textit{with } \ u = 0 \quad \textit{on } \ \partial D.$$

We will briefly explain the idea of the proof of Theorem 3.1. Let u be a positive solution of (3.2) with $b = 0$. If we put

$$\alpha = \int_\Omega q(x)u(x)dx, \tag{3.3}$$

we can rewrite (3.2) in the following form

$$\begin{cases} -d\Delta u + \alpha p(x)u = au & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ u > 0 & \text{in } \ \Omega. \end{cases} \tag{3.4}$$

Since u is a positive definite function, a must be identical with the principal eigenvalue of the following eigenvalue problem

$$-d\Delta u + \alpha p(x)u = \lambda u \quad \text{in } \ \Omega \quad \text{and } \ u = 0 \quad \text{on } \ \partial\Omega. \tag{3.5}$$

If we denote by $\lambda_1(\alpha p)$ the principal eigenvalue of (3.5), we have only to find α satisfying $\lambda_1(\alpha p) = a$.

It is well known that $\lambda_1(\alpha p)$ can be expressed by the following variational characterization

$$\lambda_1(\alpha p) = \inf\left\{d\int_\Omega |\nabla u|^2 dx + \alpha\int_\Omega p(x)u^2 dx; u \in H_0^1(\Omega), \|u\|_2 = 1\right\}. \tag{3.6}$$

It should be noted that $\lambda_1(\alpha p)$ has the following properties:

Lemma 3.1. *Let $p(\not\equiv 0)$ be a nonnegative continuous function in $\overline{\Omega}$ and assume that Ω_0 is connected. Then the following properties hold true.*
(i) *The mapping $\alpha \to \lambda_1(\alpha p)$ is continuous and strictly increasing for $\alpha \geq 0$.*
(ii) $\lim_{\alpha\to 0} \lambda_1(\alpha p) = \lambda_1(0) = \lambda_{1,\Omega}$.
(iii) $\lim_{\alpha\to\infty} = \begin{cases} \infty & \textit{in case } \ \Omega_0 = \emptyset, \\ \lambda_{1,\Omega_0} & \textit{in case } \ \Omega_0 \neq \emptyset. \end{cases}$

Proof. Assertions (i) and (ii) come from (3.6). For the proof of (iii), see López-Gómez [10]. □

In order to find a positive solution u of (3.2), it is sufficient to look for α^* satisfying $\lambda_1(\alpha^* p) = a$ for given a. Then u can be obtained as $u = c\varphi$ with positive constant c, where φ is a positive eigenfunction of (3.5) corresponding to $\lambda_1(\alpha^* p)$. In view of (3.3), positive constant c can be determined from

$$\alpha^* = c\int_\Omega q(x)\varphi(x)dx.$$

Therefore, it is easy to prove Theorem 3.1 if we use Lemma 3.1.

We now discuss the existence of positive solutions of (3.2) in case $b > 0$. Let u be a positive solution of (3.2). If we define α by (3.3), then the first equation of (3.2) can be written as

$$\begin{cases} -d\Delta u + \alpha p(x)u = u(a - bu) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{3.7}$$

Our strategy is to look for a positive solution $\theta(x : \alpha p)$ for (3.7) for each $\alpha \geq 0$ and determine α from

$$\alpha = \int_\Omega q(x)\theta(x; \alpha p)dx. \tag{3.8}$$

In place of (3.7) we will study the existence of positive solutions for the following auxiliary problem:

$$\begin{cases} -d\Delta u + m(x)u = u(a - bu) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ u > 0 & \text{in } \ \Omega, \end{cases} \tag{3.9}$$

where a, b, d are positive constants and $m : \overline{\Omega} \to R$ is a nonnegative continuous function. We have the following result.

Proposition 3.1. *Let m be a nonnegative continuous function in $\overline{\Omega}$. Then* (3.9) *has a unique positive solution $\theta(x; m)$ if and only if $a > \lambda_1(m)$. Moreover, if $m_1 \geq m_2$ $(m_1 \not\equiv m_2)$, then $\theta(x; m_2) > \theta(x; m_1)$ for $x \in \Omega$.*

Proof. Since $\lambda_1(m)$ is the principal eigenvalue, one can choose a positive eigenfunction $\varphi(x; m)$ corresponding to $\lambda_1(m)$ such that

$$\max_{x\in\Omega} \varphi(x; m) = 1 \quad \text{and} \quad \varphi(x; m) > 0 \quad \text{in} \ \ \Omega.$$

If we set $u^*(x) = c_1$ with positive constant c_1 satisfying $c_1 \geq a/b$, then we see that u^* is a supersolution of (3.9). We next take

$$v_*(x) = \varepsilon\varphi(x; m) \quad \text{with positive constant} \ \ \varepsilon.$$

Then

$$-d\Delta v_* + v_*(m(x) - a + bv_*) = \epsilon\varphi(x;m)(\lambda_1(m) - a + b\varepsilon\varphi(x;m)).$$

Hence, if $a > \lambda_1(m)$, one can take a sufficiently small $\varepsilon > 0$ such that $b\varepsilon \le a - \lambda_1(m)$. In this case,

$$-d\Delta v_* + v_*(m(x) - a + bv_*) \le 0;$$

that is, v_* is a subsolution of (3.9). Thus we can construct a supersolution u^* and a subsolution v_* satisfying $u^* \ge v_*$. Hence it follows from the result of Sattinger [14] that (3.9) has a positive solution.

The proofs of the necessity part and the uniqueness of positive solutions are standard; so we omit them.

Finally, we will prove the order preserving property. Let $m_1 \ge m_2$; then $\theta(x;m_2)$ is a supersolution of (3.9) with $m = m_1$. Therefore

$$\theta(x;m_2) \ge \theta(x;m_1) \quad \text{in} \ \ \Omega.$$

Moreover, if we set $w(x) = \theta(x;m_2) - \theta(x;m_1)$, then w satisfies

$$\begin{cases} -d\Delta w + m_2 w + w\{b(\theta(x;m_1) + \theta(x;m_2)) - a\} \ge 0 & \text{in} \ \ \Omega, \\ w = 0 & \text{on} \ \ \partial\Omega. \end{cases}$$

Therefore, one can apply the strong maximum principle ([12]) to conclude $w > 0$ in Ω. □

We are ready to study (3.2) in case $b > 0$. It follows from Proposition 3.1 that (3.7) has a unique solution $\theta(x;\alpha p)$ if and only if

$$a > \lambda_1(\alpha p). \tag{3.10}$$

Here we should recall basic properties of $\lambda_1(\alpha p)$ as a function of α (see Lemma 3.1).

In what follows, assume

$$a > d\lambda_{1,\Omega}. \tag{3.11}$$

Then it is possible to find a unique $\bar{\alpha} > 0$ satisfying $a = \lambda_1(\bar{\alpha}p)$ in case $\Omega_0 = \emptyset$. In case $\Omega_0 \ne \emptyset$, if we additionally assume $a < d\lambda_{1,\Omega_0}$; then it is also possible to find $\bar{\alpha}$ which satisfies the same property as above. When a satisfies $a \ge d\lambda_{1,\Omega_0}$ in case $\Omega_0 \ne \emptyset$, we set $\bar{\alpha} = \infty$. Then we see that (3.10) is equivalent to

$$0 \le \alpha < \bar{\alpha} \tag{3.12}$$

and that, if α satisfies (3.12), then (3.7) has a unique positive solution $\theta(x;\alpha p)$.

Lemma 3.2. *The mapping $\alpha \to \theta(x;\alpha p)$ is of class C^1 from $[0,\bar{\alpha})$ to $C(\overline{\Omega})$ and strictly decreasing. Moreover, it satisfies the following properties:*

(i) $\lim_{\alpha\to 0} \theta(\cdot;\alpha p) = \theta_0$ *uniformly in Ω, where θ_0 is a unique positive solution of*

$$d\Delta\theta + \theta(a - b\theta) = 0 \quad in \ \ \Omega \quad and \ \ \theta = 0 \quad on \ \ \partial\Omega.$$

(ii) $$\lim_{\alpha\to\bar{\alpha}} \theta(\cdot;\alpha p) = \begin{cases} 0 & \textit{uniformly in} \ \ \Omega \quad \textit{if} \ \ \bar{\alpha} < \infty, \\ \theta_\infty & \textit{uniformly in} \ \ \Omega \quad \textit{if} \ \ \bar{\alpha} = \infty. \end{cases}$$

Here θ_∞ is a function satisfying $\theta_\infty \equiv 0$ in $\Omega \setminus \Omega_0$ and

$$\begin{cases} d\Delta\theta_\infty + \theta_\infty(a - b\theta_\infty) = 0 & in \ \ \Omega_0, \\ \theta_\infty = 0 & on \ \ \partial\Omega_0, \\ \theta_\infty > 0 & in \ \ \Omega_0. \end{cases}$$

Before giving the proof of Lemma 3.2 we will prove the solvability of (3.2).

Theorem 3.2. *Let $a > d\lambda_{1,\Omega}$. Then* (3.2) *has a unique positive solution u^*.*

Remark 3.1. *It is easy to show that* (3.2) *has no positive solution for $a \le d\lambda_{1,\Omega}$.*

Proof. Since $\theta(x; \alpha p)$ is a positive solution of (3.7) for $0 \le \alpha < \bar{\alpha}$, we see, in view of (3.3), that $\theta(x; \alpha p)$ is a positive solution of (3.2) if and only if α satisfies (3.8). Denote the right-hand side of (3.8) by $F(\alpha)$. It follows from Lemma 3.2 that $F(\alpha)$ is strictly decreasing for $\alpha \in [0, \bar{\alpha}]$ and satisfies

$$F(0) = \int_\Omega q(x)\theta_0(x)dx > 0$$

and

$$\begin{cases} F(\bar{\alpha}) = 0 & \text{in case } \ \bar{\alpha} < \infty, \\ \lim\limits_{\alpha\to\infty} F(\alpha) = \displaystyle\int_{\Omega_0} q(x)\theta_\infty(x)dx & \text{in case } \ \bar{\alpha} = \infty. \end{cases}$$

Therefore, it is easy to find a unique α^* satisfying $\alpha^* = F(\alpha^*)$ in both cases $\bar{\alpha} < \infty$ and $\bar{\alpha} = \infty$. Clearly, $\theta(x; \alpha^* p)$ becomes a unique positive solution of (3.2). □

Proof of Lemma 3.2. Observe that $\theta(x; \alpha p)$ satisfies

$$-d\Delta\theta(x; \alpha p) + \alpha p(x)\theta(x; \alpha p) + \theta(x; \alpha p)(b\theta(x; \alpha p) - a) = 0 \quad \text{in } \ \Omega$$

with $\theta(x; \alpha p) = 0$ on $\partial\Omega$. Differentiation of the above equation with respect to α leads us to

$$-d\Delta w + \alpha p(x)w + (2b\theta(x; \alpha p) - a)w = -p(x)\theta(x; \alpha p) \quad \text{in } \ \Omega \quad \text{and} \quad w = 0 \quad \text{on } \ \partial\Omega$$

with $w(x) = (\partial/\partial\alpha)\theta(x; \alpha p)$. We should recall that $-d\Delta + \alpha p(x) + 2b\theta(x; \alpha p) - a$ is an invertible and order-preserving operator from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to $L^p(\Omega)$ (see, e.g., [15, Lemma 1.1]). Therefore, the implicit function theorem assures to show

$$\frac{\partial\theta(\alpha p)}{\partial\alpha} = -\{-d\Delta + \alpha p(x) + 2\theta(x; \alpha p) - a\}^{-1}(p\theta(\alpha p)) < 0 \quad \text{in } \ \Omega.$$

Thus $\alpha \to \theta(x; \alpha p)$ is strictly decreasing.

It is easy to see $\theta(0) = \theta_0$ and $\theta(\bar{\alpha}p) = 0$ in case $\bar{\alpha} < \infty$.

It remains to study $\lim_{\alpha\to\infty} \theta(\alpha p)$ in case $\bar{\alpha} = \infty$. Since $\theta(\alpha p)$ is positive and strictly decreasing with respect to α, there exists a nonnegative function θ_∞ such that

$$\lim_{\alpha\to\infty} \theta(\alpha p) = \theta_\infty \quad \text{pointwise in } \ \Omega. \tag{3.13}$$

Take any $\varphi \in C_0^\infty(\Omega)$; then it holds that

$$-d\int_\Omega \theta(x; \alpha p)\Delta\varphi dx + \alpha\int_\Omega p(x)\theta(x; \alpha p)\varphi dx = \int_\Omega \theta(x; \alpha p)(a - b\theta(x; \alpha p))dx. \tag{3.14}$$

Since $p(x) = 0$ in Ω_0, we see from (3.14) that

$$\int_{\Omega\setminus\Omega_0} p(x)\theta(x; \alpha p)\varphi dx = \frac{1}{\alpha}\left\{d\int_\Omega \theta(x; \alpha p)\Delta\varphi dx + \int_\Omega \theta(x; \alpha p)(a - b\theta(x; \alpha p))dx\right\}. \tag{3.15}$$

Making use of the uniform boundedness of $\theta(x; \alpha p)$ for $\alpha \ge 0$ and letting $\alpha \to \infty$ in (3.15) one can derive

$$\int_{\Omega\setminus\Omega_0} p(x)\theta_\infty(x)\varphi dx = 0 \quad \text{for any } \ \varphi \in C_0^\infty(\Omega).$$

Therefore, $\theta_\infty(x) = 0$ for $x \in \Omega \setminus \Omega_0$.

We next take any $\varphi \in C_0^\infty(\Omega_0)$ and define $\tilde{\varphi} \in C_0^\infty(\Omega)$ by $\tilde{\varphi}(x) = \varphi(x)$ if $x \in \Omega_0$ and $\tilde{\varphi}(x) = 0$ if $x \in \Omega \setminus \Omega_0$. Setting $\varphi = \tilde{\varphi}$ in (3.14) leads to

$$-d \int_{\Omega_0} \theta(x; \alpha p) \Delta \varphi dx = \int_{\Omega_0} \theta(x; \alpha p)(a - b\theta(x; \alpha p) \varphi dx.$$

Letting $\alpha \to \infty$ in the above identity we get

$$-d \int_{\Omega_0} \theta_\infty \Delta \varphi dx = \int_{\Omega_0} \theta_\infty (a - b\theta_\infty) \varphi dx;$$

which implies

$$\begin{cases} -d\Delta \theta_\infty = \theta_\infty (a - b\theta_\infty) & \text{in } \Omega, \\ \theta_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

It should be noted by elliptic regularity theory that θ_∞ becomes continuous in Ω. Therefore, one can conclude from Dini's theorem that the convergence in (3.13) is uniform. Thus the proof is complete.

4 Concluding remarks

4.1 Stability of stationary solution

In the previous section, we have shown in Theorem 3.2 that (3.2) has a unique positive solution u^*. Then it is a very important problem to study the stability of u^*. The spectral problem for the linearized operator around $u = u^*$ is given by

$$\begin{cases} -d\Delta v + a_1(x) v + p(x) u^*(x) \int_\Omega q(y) v(y) dy = \sigma v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$a_1(x) = 2bu^*(x) - a + p(x) \int_\Omega q(y) u^*(y) dy.$$

The above linearized operator is not self-adjoint; so that the spectral problem may have complex eigenvalues. Moreover, we do not know if the Krein-Rutman theorem holds for (4.1) or not. So it is difficult to get satisfactory information on the spectrum for (4.1). (Note that Theorem 2.1 in [2] is not applicable to (4.1).)

In general, it is a delicate and difficult problem to study the eigenvalues for the operator with nonlocal terms, see, e.g., [4], [5, 6, 7].

Finally, it should be noted that, if a is regarded as a bifurcation parameter in (3.2), then the local bifurcation theory assures the existence and uniqueness of bifurcating positive solutions of (3.2) if a $(> d\lambda_{1,\Omega})$ is very close to $d\lambda_{1,\Omega}$. We can also show that such bifurcating positive solutions are asymptotically stable when a is very close to $d\lambda_{1,\Omega}$. So we have a conjecture that u^* is asymptotically stable for every $a > d\lambda_{1,\Omega}$.

4.2 Positive solutions for general case

Our method of analysis is applicable for more general class of equations with diffusion and nonlocal effects:

$$u_t = d\Delta u + u(f(u) - p(x) \int_\Omega q(y) g(u(y,t)) dy),$$

where $f(u)$ is a deceasing and locally Lipschitz continuous function such that $f(0) > 0$ and $g(v)$ is an increasing, positive and locally Lipschitz continuous function for $v > 0$.

In Section 3, we have discussed the stationary problem in a case when k has a special form (3.1). Taking account of nonlocal effects it is also important to study the stationary problem in case k has the following form

$$k(x, y) = \rho(x - y),$$

where ρ is nonnegative and continuous function. For this problem, we can also apply the bifurcation theory by regarding a as a bifurcation parameter. So it is also possible to show that, for each $a > d\lambda_{1,\Omega}$

$$\begin{cases} d\Delta u + u\left(a - bu - \displaystyle\int_\Omega \rho(x-y)u(y)dy\right) = 0 & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases}$$

has at least one positive solution. We will discuss this fact elsewhere.

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