Evolution Equations and Their Applications to Partial Differential Equations

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1 Introduction

I would like to give a lecture about evolution equations and their applications to partial differential equations (PDEs). In this lecture, evolution equations are understood as abstract differential equations of the form

$$u_t + Au = 0 \tag{E}$$

in an infinite dimensional space X (usually, X will be a Banach space). Here A is a linear or nonlinear operator whose fundamental properties are usually determined by reflecting partial differential operators. Although the treatment of (E) is frequently carried out with the aid of functional analysis, it is desirable that abstract results for (E) inherit some of basic properties of original PDEs.

The theory of evolution equations has developed to study the dynamics of solutions for (E). The main purpose of this lecture is to explain (as far as I know) how the theory of evolution equations has been applied to nonlinear PDEs, what kind of contributions it has given in the related fields and what kind of limits it has for the better understanding of nonlinear PDEs. Topics are selected according to my interest; so that some important ones may not be discussed here. There are a lot of good monographs which deal with evolution equations and their applications to PDEs; say, Friedman [?], Henry [?], Pazy [?], Reed and Simon [?], and Tanabe [?]. As is seen from these monographs, the theory of abstract evolution equations is mostly applied to parabolic equations, while, until now (1990), there are not so many works where its applications to hyperbolic or Schrödinger equations are investigated. This is because, in the analysis of hyperbolic or Schrödinger equations, it is very difficult for the abstract theory to yield important information (as PDEs) other than the existence and uniqueness of solutions. Therefore, my lecture is also principally concentrated to parabolic equations.

2 Prototypes of evolution equations

As prototypes for linear evolution equations, I will give three classes of linear PDEs; diffusion equations, wave equations and Schrödinger equations.

Let $\Omega \subset \mathbb{R}^N (N \in \mathbb{N})$ be a bounded domain with smooth boundary $\partial\Omega$. Set $X = L^2(\Omega)$.

(I) Consider the following initial boundary value problem for diffusion equations

$$\begin{cases} u_t = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases}$$

If we define a positive self-adjoint operator A in X by

$$Au = -\Delta u \qquad \text{with domain} \quad D(A) = H^2(\Omega) \cap H^1_0(\Omega),$$

then the above problem is written as (E).

(II) We next consider the initial boundary value problem for wave equations

$$\begin{cases} u_{tt} = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases}$$

Define A as in (I). Since A is a positive self-adjoint operator in X, there exists a spectral decomposition $\{E(\lambda)\}_{\lambda > 0}$ associated with A. Therefore,

$$A^{1/2} = \int_0^\infty \lambda^{1/2} dE(\lambda).$$

Introduce a new unknown function $U = {}^t(v, w)$ by $v = u_t$ and $w = A^{1/2}u$; then the above wave equation is represented as $U_t + \mathcal{B}U = 0$, where $\mathcal{B}$ is an anti-self-adjoint operator in $X \times X$ defined by

$$\mathcal{B} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & 0 \end{pmatrix}.$$

(III) Finally consider the following problem for Schrödinger equations

$$\begin{cases} iu_t = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases}$$

With use of A defined in (I), set $C = -iA$. Then this problem is written as in the form of (E) with A replaced by C. It should be noted here that there are no differences between (II) and (III) from the view point of operator theory because C is also an anti-self-adjoint operator.

3 Abstract setting of semilinear parabolic equations

Typical semilinear parabolic equations are expressed as

$$u_t = \Delta u + F(u, \nabla u).$$

Such equations are regarded as abstract semilinear evolution equations of the form

$$u_t + Au = F(u), \quad t > 0, \tag{3.1}$$

in an appropriate function space X. In (**??**), A is a linear operator and F is a nonlinear operator.

Before studying (**??**), it is convenient to give some preliminary results on the Cauchy problem in a Banach space X with norm $\|\cdot\|$:

$$u_t + Au = 0, \quad t > 0, \quad \text{with} \quad u(0) = a, \tag{3.2}$$

where A is a closed linear operator with dense domain $D(A)$. Usually,it is assumed that A satisfies the following resolvent condition:

$$\|(\lambda + A)^{-1}\| \le \frac{C}{(1 + |\lambda|)^\beta} \quad \text{with} \quad 0 \le \beta \le 1 \tag{3.3}$$

for every $\lambda \in \Sigma_\omega := \{\lambda \in \mathbb{C};\ |\lambda| \geq c_0,\ |\arg \lambda| \leq \pi - \omega$ with $c_0 > 0$ and $\omega \in (0, \pi/2)\}$.

We will take the Laplace transform for (**??**): the Laplace transform $\hat{u}(\lambda)$ of $u(t)$ is defined by

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt.$$

Then it follows that $-a + \lambda \hat{u}(\lambda) + A\hat{u}(\lambda) = 0$; so that $\hat{u}(\lambda) = (\lambda + A)^{-1} a$. Therefore, the inverse Laplace transform of $\hat{u}(\lambda)$ formally leads to

$$u(t) = \frac{1}{2\pi i} \int_{c-i\omega}^{c+i\omega} e^{\lambda t} (\lambda + A)^{-1} a \ d\lambda$$

with some $c \in \mathbb{R}$. By Cauchy's integral theorem, this formal calculation is justified when $u(t)$ is defined by

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + A)^{-1} a \ d\lambda := e^{-tA} a, \tag{3.4}$$

where the integral is understood as Dunford integral and Γ is a contour in Σ_ω with $\arg \lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\pi/2, \pi - \omega]$.

We will indicate several examples associated with $2m$-th order elliptic differential operator $A(x, D)$ in Ω, which is supplemented by an appropriate class of normal boundary conditions (for details, see, e.g., Tanabe [**?**]).

Cace (A). Set $X = L^p(\Omega)$ with $p > 1$ and define $(Au)(x) = A(x, D)u(x)$ for $u \in D(A) = \{v \in W^{2m,p}(\Omega);\ v$ satisfies suitable boundary conditions$\}$. Then it is well known that A becomes a closed linear operator satisfying (**??**) with $\beta = 1$ (see Tanabe [**?**]). Moreover, it can be shown that $\{e^{-tA}\}$ is an analytic semigroup.

Case (B). Set $X = C(\Omega)$ and define A as in Case (A) with

$$D(A) = \{v \in \bigcap_{q>N} W^{m,q}(\Omega);\ A(x, D)v \in C(\Omega) \text{and } v \text{ satisfies boundary conditions}\}.$$

By Stewart [**?**] it is shown that A satisfies (**??**) with $\beta = 1$ and that $\{e^{-tA}\}$ becomes an infinitely differentiable semigroup although $D(A)$ is not dense in X.

Case (C). Set $X = C^k(\Omega)$ ad define A as in Case (A) with domain

$$D(A) = \{v \in C^{2m+k}(\Omega);\ v \text{ satisfies boundary conditions}\}.$$

It follows from the results of Von Wahl [**?**] that (**??**) holds true with $\beta = 1 - k/(2m)$. In this case, $\{e^{-tA}\}$ also becomes an infinitely differentiable semigroup.

Further information can be found in the works of Yagi [**?**, **?**, **?**].

4 Local existence theory of semilinear evolution equations

As one of important contributions of abstract theory to nonlinear PDEs, we will discuss the local existence theory for (**??**). Suppose that A is a densely defined closed linear operator in a Banach space X, which satisfies resolvent condition (**??**) with $\beta = 1$. Moreover, let the spectrum $\sigma(A)$ of A satisfy $\mathrm{Re}\ \sigma(A) > 0$ for the sake of simplicity.

The essential idea to construct a local solution to (**??**) is almost the same as for ordinary differential equations (ODEs). The first step is to rewrite (**??**) as an integral equation of the form

$$u(t) = e^{-tA} a + \int_0^t e^{-(t-s)A} F(u(s)) ds. \tag{4.1}$$

If $u \to F(u)$ is locally Lipschitz continuous in X, it is quite easy to construct a local solution of (**??**) by an iteration procedure or a fixed point theorem for strictly contraction mapping. However, the local Lipschitz continuity of $F(u)$ is too severe to apply this local existence result to semilinear diffusion equations.

Fractional powers of A play a crucial role in the application of the abstract theory to a very wide class of semilinear parabolic equations. For every $\alpha > 0$, define

$$A^{-\alpha} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\alpha}(\lambda + A)^{-1} d\lambda$$

and A^α is defined by the inverse of $A^{-\alpha}$. In applications, $D(A^\alpha)$ is important rather than A^α. Its complete characterization is established by Triebel [**?**].

Suppose that $A:\ L^p(\Omega) \to L^p(\Omega), p > 1$, is a realization of $2m$-order elliptic differential operator with a suitable normal boundary condition. It is well known that the following inclusion relations hold:

$$\begin{cases} D(A^\alpha) \subset W^{k,q}(\Omega) & \text{if} \quad k - \frac{N}{q} \le 2m\alpha - \frac{N}{p},\ q > p, \\ D(A^\alpha) \subset C^\mu(\Omega) & \text{if} \quad 0 \le \mu < 2m\alpha - \frac{N}{p}, \end{cases} \tag{4.2}$$

(see, e.g., Henry [**?**] or Triebel [**?**]). Here we note that each inclusion mapping in (**??**) is continuous. The continuity of nonlinear operator $F(u)$ can be described in terms of fractional powers of A; say,

$$\|F(u) - F(v)| \le L(\|A^\alpha u\|, \|A^\alpha v\|)\|A^\alpha(u - v)\| \quad \text{for all} \quad u, v \in D(A^\alpha)$$

with some $\alpha \in (0, 1)$, where $L(U, V)$ is a continuous function of U and V. Since the analytic semigroup $\{e^{-tA}\}$ satisfies

$$\|A^\alpha e^{-tA} u\| \le C_\alpha t^{-\alpha}\|u\| \qquad \text{for all} \quad u \in X \quad \text{and} \quad t > 0 \tag{4.3}$$

with a positive constant C_α, an iteration procedure will lead us to a local existence reslut for (**??**). The first analytic semigroup approach which has produced interesting contributions to semilinear parabolic equations will originate in a pioneer work of Fujita and Kato [**?**, **?**]. In their works, the Navier-Stokes equations were studied in the framework of L^2 theory with use of fractional powers of the Stokes operator. See also Giga and Miyakawa [**?**] or Hishida [**?**], where the semigroup approach is carried out in L^p framework.

In what follows, we assume that F satisfies $F(0) = 0$ and

$$\|F(u) - F(v)\| \le C(\|A^\gamma u\| + \|A^\gamma v\|)^{m-1}\|A^\gamma(u - v)\| \qquad \text{for all} \quad u, v \in D(A^\gamma) \tag{4.4}$$

with $C > 0, m > 1$ and $0 \le \gamma < 1$. Then the following existence result holds true for (**??**). The proof can be found in the work of Hoshino and Yamada [**?**].

Theorem 4.1. *Let m, γ and θ satisfy*

$$\begin{cases} \dfrac{m\gamma - 1}{m - 1} \le \theta < 1 & \text{if} \quad m\gamma > 1, \\ 0 < \theta < 1 & \text{if} \quad m\gamma = 1, \\ 0 \le \theta & \text{if} \quad m\gamma < 1. \end{cases} \tag{4.5}$$

Then for every $a \in D(A^\theta)$ there exists a positive number T such that (**??**) *has a unique solution $u \in C([0, T];\ D(A^\theta))$ with the following properties:*

(i) *$t^{(\gamma-\theta)^+} u \in C((0, T];\ D(A^\gamma)$ and $t^{(\gamma-\theta)^+}\|A^\gamma u(t)\| \le M_\gamma\|A^\theta a\|$ fot $0 \le t \le T$ with some $M_\gamma > 0$.*

(ii) *$\|A^\gamma u(t)\| = o(t^{-(\gamma-\theta)})$ as $t \to 0$ if $\gamma > \theta$.*

Theorem **??** assures the solvability of integral equation (**??**). Further regularity properties can be obtained in the following theorem (see [**?**]).

Theorem 4.2. *Let u be the solution in Theorem* **??**. *Then the following properties hold true:*

(i) *For every $\alpha \in [0,1]$, $t^{(\alpha-\theta)^+}u \in C((0,T]; D(A^\alpha)$ and $t^{(\alpha-\theta)^+}\|A^\alpha u(t)\| \le M_\alpha \|A^\theta a\|$ for $0 < t \le T$ with a positive number M_α.*

(ii) *For every $\alpha \in [0,1)$, $t^{1+(\alpha-\theta)^+}u_t \in C((0,T]; D(A^\alpha)$ and $t^{1+(\alpha-\theta)^+}\|A^\alpha u_t(t)\| \le M^*_\alpha \|A^\theta a\|$ for $0 < t \le T$ with a positive number M^*_α.*

(iii) *u satisfies* (**??**).

As an application of Theorems **??** and **??**, we consider the following initial boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$:

$$\begin{cases} u_t = \Delta u + |u|^{m-1}u & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in} \quad \Omega, \end{cases} \tag{4.6}$$

where $m > 1$ and a is a measurable function defined in Ω. Set $X = L^p(\Omega)$ with $p > 1$ and define $A_p u = -\Delta u$ with $D(A_p) = \{u \in W^{2,p}(\Omega);\ u = 0 \ \text{ on } \ \partial\Omega\}$. Since A_p satisfies (**??**) (with $m = 1$), it is easy to see that, if

$$p > \max\{1, (m-1)N/2m\},$$

then $F(u) =$ satisfies (**??**) with $\gamma = (m-1)N/2mp$. Simple applications of Theorems **??** and **??** assure that, for every $a \in D(A^\theta)$ with

$$\begin{cases} \theta \ge 0 & \text{if} \quad p > \frac{(m-1)N}{2}, \\ \theta > 0 & \text{if} \quad p = \frac{(m-1)N}{2}, \\ \theta \ge \frac{N}{2p} - \frac{1}{m-1} & \text{if} \quad \frac{(m-1)N}{2} > p, \end{cases} \tag{4.7}$$

then (**??**) has a unique local solution u in the framework of $L^p(\Omega)$ space.

It is also possible to derive from Theorems **??** and **??** much more information; say, asymptotic behavior or singular behavior of $u(t)$ as $\to 0$ in various function spaces.

Take any $\tau \in (0,T]$. Since $u(\tau) \in D(A_p^\alpha)$ for every $\alpha \in [0,1]$, we can regard $u(\tau) \in D(A_{p'}^{\theta'})$ with $p' > p$ and $\theta' < \theta$. If we choose p', θ' snf $\gamma' := (m-1)/2p'm$ such that they satisfy (**??**), then $u(t)$ for $\tau \le t \le T$ may be regarded as a solution in $L^{p'}(\Omega)$. Then the uniqueness and regularity results yield $u(t) \in D(A_{p'}^\beta)$ for $\beta \in [0,1]$ and $t \in [\tau, T]$. Repeating this procedure and making use of (**??**), we arrive at the following result on the asymptotic behavior of u in various Sobolev spaces near $t = 0$.

Proposition 4.1. *Assume* (**??**). *Then there exists a positive number T such that* (**??**) *has a unique solution u jn $[0,T]$, which satisfies for every $q \ge p$*

$$\max_{|\mu|\le 2, |\nu|\le 1} \left\| \sup_{t\in(0,T]} t^{\rho-\theta+|\mu|/2}\|D_x^\mu u(t)\|_{L^q}, \sup_{t\in(0,T]} t^{\rho-\theta+1+|\nu|/2}\|D_x^\nu u_t(t)\|_{L^q} \right\} \le C\|A_p^\theta a\|_{L^p}$$

with $\rho = \dfrac{N}{2mp} - \dfrac{N}{2mq}$, where C is apositive number independent of a.

This proposition is concerned with smoothing effect of parabolic equations. As its corollary, making use of Sobolev's imbedding theorem, we can show that $u(x,t), D_x u(x,t)$ and $u_t(x,t)$ are Hölder continuous with respect to $x \in \Omega$ for each $t \in (0,T]$. These facts imply that, even if initial function a is not smooth, (**??**) has a classical solution. Details can be found in the work of Hoshino and Yamada [**?**]. See also the related work of Weissler, where the solvability of (**??**) is also studied in terms of the corresponding integral equation.

5 global existence theory for semilinear parabolic equations

A. Abstract global existence

Consider the Cauchy problem for abstract evolution equations of the form (**??**) when F satisfies (**??**). We assume that A satisfies, in place of (**??**),

$$\|A^{\alpha}e^{-tA}u\| \le C(\alpha)t^{-\alpha}e^{-\omega_0 t}\|u\| \quad \text{for all} \quad u \in X \quad \text{and} \quad t > 0 \tag{5.1}$$

with some $C(\alpha) > 0$ and $\omega_0 > 0$. The global existence theorem reads as follows.

Theorem 5.1. *Let m, γ and θ satisfy (**??**). Then there exists a positive constant δ_0 such that, for every $a \in D(A^{\theta})$ with $\|A^{\theta}a\| \le \delta_0$, (**??**) admits a unique global solution u such that $u(t)$ decays exponentially to zero as $t \to \infty$ (in a suitable topology).*

For the proof, see Hoshino and Yamada [**?**]. It is easy to apply Theorem **??** to initial boundary value problem (**??**) to derive the global existence of solutions. See Weissler [**?**], where $L^p - L^q$ estimates for $\{e^{-tA}\}$ are used in place (**??**). See also Lions [**?**].

B. Use of a priori estimates

A priori estimates are often combined with the local existence theorem in order to assure the global existence of solutions for (**??**). Their derivation has, in a sense, a very technical feature. According to special forms of nonlinear terms, one can use, for instance, the comparison principle for parabolic equations, Moser's technique (see, e.g., Alikakos [**?**]) or feedback arguments based on $L^p - L^q$ estimates for $\{e^{-tA}\}$ (see, e.g., Rothe [**?**, **?**]).

We will give a simple example:

$$\begin{cases} u_t = \Delta u + cu(1-u^2) & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in} \quad \Omega, \end{cases} \tag{5.2}$$

where c is a positive constant. Take $p > \max\{1, N\}$. Theorems **??** and **??** imply the existence of a local solution u to (**??**) for any $a \in L^p(\Omega)$. Therefore, it is sufficient to get an a priori estimate of $\|u(t)\|_{L^p}$ with some $p > \max\{1, N\}$ to extend $u(t)$ globally over $[0, \infty)$. For the sake of simplicity, assume $a \in L^{\infty}(\Omega)$. As a comparison function, we take the solution of the following ODE

$$v_t = cv(1-v^2), \quad v(0) = \|a\|_{L^{\infty}}.$$

Then it is easy to derive the following estimate by the comparison theorem

$$\|u(t)\|_{L^{\infty}} \le v(t) \le \max\{1, \|a\|_{L^{\infty}}\},$$

which assures the global existence of the solution to (**??**).

6 Stability of steady-state solutions

After the existence of global solutions to (**??**) is established, it becomes important to study their asymptotic behaviors as $t \to \infty$. In this connection, the stability or instability of steady-states is very closely related with the asymptotic analysis.

Let u^* be a steady state solution for (**??**); that is, u^* satisfies

$$Au^* - F(u^*) = 0. \tag{6.1}$$

The stability of u^* is decided by the asymptotic behaviors of solutions of (**??**) when initial data are near u^* with respect to the topology of a suitable Banach space Y (sometimes stronger than X). Let $u(t; a)$ be the solution of (**??**) with $u(0) = a$. We say that u^* is stable in Y if, for any neighborhood U of u^* in Y, there exists a neighborhood V of u^* in Y such that $u(t; a) \in U$ for all $t \geq 0$ whenever $a \in V$. If this property does not hold, it is said that u^* is unstable (in Y). Moreover, a stable steady state u^* is called an asymptotically stable-state if $u(t; a) \to u^*$ in Y as $t \to \infty$ for every $a \in V$.

Set $v = u - u^*$; then (**??**) is rewritten as

$$v_t + Av - dF(u^*)v = G(v), \tag{6.2}$$

where $dF(u^*)$ denotes the Fréchet derivative of F at u^* and $G(u)$ is, roughly speaking, a higher-order term. The linearized principle is based on the idea that the asymptotic behaviors of solutions of (**??**) will be quite similar to those of solutions to

$$w_t + Bw = 0 \qquad \text{with } \; B := A - dF(u^*),$$

when the former solutions remain near zero. The (linearized) stability or instability of u^* can be stated as follows; for details, see the monograph of Henry [**?**].

Theorem 6.1. *Let B satisfy* (**??**) *with $\beta = 1$ and let G satisfy $\|G(v)\|_Y = o(\|v\|_Y)$ with respect to suitable Banach space Y.*

(i) *If $\sigma(B)$ lies in the set $\{\lambda \in \mathbb{C};\ Re\ \lambda \geq c\}$ with some $c > 0$, then u^* is asymptotically stable in Y.*

(ii) *If $\sigma(B) \bigcap \{\lambda \in \mathbb{C};\ Re \lambda < 0\}$ is nonempty, then u^* is unstable in Y.*

The stability theorem has a lot of modifications; see for instance the works of Kielhöfer [**?**, **?**], where semilinear parabolic equations are discussed. Theorem **??** is also applicable to the stability analysis of steady-states.

We will apply Theorem **??** to the stationary problem for (**??**). Clearly, $u = 0$ is a steady-state solution. The linearized operator for (**??**) at 0 os given by

$$B = -\Delta - c \qquad \text{with zero Dirichlet boundary condition.}$$

Denote by λ_1 the least eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Theorem **??** assures the asymptotic stability of $u = 0$ in case $c < \lambda_1$. More precisely, we can show that, if $\|a\|_p$ is sufficiently small for $p > \max\{1, N\}$ and $c < \lambda_1$, then the solution u of (**??**) satisfies

$$\lim_{t\to\infty} \|u(t)\|_q = 0 \quad \text{expenentially for every } \; q \geq p.$$

For $c > \lambda_1$,$u = 0$ loses its stability. Are there any stable steady-states for (**??**) in case $c > \lambda_1$? It can be shown by the comparison method that, for every $c > \lambda_1$, the stationary problem associated with (**??**) has a unique positive (resp. negative) solution ϕ (resp.$-\phi$) (see, e.g., Sattinger [**?**]). We will study the stability of ϕ. The linearized operator at ϕ is

$$B = -\Delta - c + 3c\phi^2 \qquad \text{with zero Dirichlet boundary condition.}$$

Since B is self-adjoint in $L^2(\Omega)$, all eigenvalues of B are real. Let μ_1 be the least eigenvalue of B and denote by ψ the corresponding eigenfunction. Since ψ does not change the sign, ψ is allowed to be positive. Multiply $B\psi = \mu_1\psi$ by ϕ and integrate the resulting expression over Ω; then

$$\begin{aligned}\mu_1 \int_\Omega \phi\psi \, dx &= -\int_\Omega \phi\Delta\psi \, dx + \int_\Omega \phi(-c + 3c\phi^2)\psi \, dx \\ &= -\int_\Omega \Delta\phi \, \psi \, dx + \int_\Omega (-c + 3c\phi^2)\phi\psi \, dx \\ &= 2c \int_\Omega \phi^3\psi \, dx > 0.\end{aligned}$$

Since μ_1 is positive from the above relations, $\sigma(B)$ lies in the right half plane. Thus the asymptotic stability of the positive solution ϕ follows from Theorem **??**. The stability of $-\phi$ can be shown in the same way.

As above, we have discussed the local stability properties of 0 and $\pm\phi$. Moreover, we can derive their "global" stability property. Define the following functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx - \frac{c}{2}\int_\Omega u^2(x)dx + \frac{c}{4}\int_\Omega u^4(x)dx.$$

Let $u(t;a)$ be the solution of (**??**). Then

$$\begin{aligned}\frac{d}{dt}E(u(t;a)) &= \int_\Omega \nabla u(t;a) \cdot \nabla u_t(t;a)dx - c\int_\Omega u_t(t;a)(u(t;a) - u^3(t;a))dx \\ &= -\int_\Omega u_t(t;a)\{\Delta u(t;a) + cu(t;a)(1 - u^2(t;a))\}dx \\ &= -\int_\Omega u_t(t;a)^2 \, dx \le 0,\end{aligned}$$

which means that $E(u(t;a))$ is decreasing with respect to t. This fact, together with the theory of dynamical systems (see, for instance, [**?**]), allows us to conclude that

$$\lim_{t\to\infty} u(t;a) = u^* \quad \text{uniformly in } \Omega \tag{6.3}$$

and

$$\lim_{t\to\infty} E(u(t;a)) = E(u^*), \tag{6.4}$$

where u^* is a steady-state for (**??**), that is, u^* satisfies

$$\Delta v + cv(1 - v^2) = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{6.5}$$

If $c \le \lambda_1$, one can show that $u^* \equiv 0$ is the only solution of (**??**); so that it follows from (**??**) that all solutions of (**??**) converge to 0 uniformly in Ω. In this sense, $u^* \equiv 0$ is a global attractor for (**??**) and, furthermore, it is a global minimizer of $E(u)$.

In case $c > \lambda_1$, take nonnegative $a(\not\equiv 0)$. The strong maximum principle yields $u(t;a) > 0$ in Ω for $t > 0$ (see [**?**]). Let φ_1 be a positive eigenfunction of $-\Delta$ corresponding to λ_1. For fixed $t_0 > 0$, ake a sufficiently small $\varepsilon > 0$ satisfying $u(t_0;a) > \varepsilon\varphi_1$ in Ω. Then the comparison theorem for parabolic equations assures

$$\max\{1, \|a\|_{L^\infty}\} \ge u(t;a) \ge u(t - t_0; \varepsilon\varphi_1) \ge \varepsilon\varphi_1 \qquad \text{in } \Omega \quad \text{and for } \ t \ge t_0. \tag{6.6}$$

Here it should be noted that

$$\begin{aligned}E(\varepsilon\varphi_1) &= \frac{1}{2}(\lambda_1 - c)\varepsilon^2 \int_\Omega \varphi_1^2 \, dx + \frac{c}{4}\varepsilon^4 \int_\Omega \varphi_1^4 dx \\ &= \frac{\varepsilon^2}{4}\left\{c\varepsilon^2 \int_\Omega \varphi_1^4 dx - 2(c - \lambda_1)\int_\Omega \varphi_1^2 dx\right\} < 0\end{aligned}$$

provided that $\varepsilon > 0$ is sufficiently small. Therefore, it follows from (**??**) and (**??**) that

$$\lim_{t\to\infty} u(t;\varepsilon\varphi_1) = \varphi^* \quad \text{uniformly in } \Omega \qquad \text{and} \qquad \lim_{t\to\infty} E(u(t;\varepsilon\varphi_1)) = E(\varphi^*) < E(\varepsilon\varphi_1) < 0,$$

where φ^* is a nonnegative and nontrivial solution of (**??**). Hence the maximum principle for elliptic equations enables us to see that φ^* is positive in Ω (the positivity of φ^* also follows from (**??**) by letting $t \to \infty$). Here we use the fact that a positive solution of (**??**) is unique; so that $\varphi^* = \phi$. This fact together with (**??**) and (**??**) implies that, if $a \geq 0$ and $a \not\equiv 0$, then $u(t;a)$ satisfies

$$\lim_{t\to\infty} u(t;a) = \phi \qquad \text{uniformly in } \Omega.$$

Thus one can derive the global attractivity of ϕ in case $a \geq 0$ ($\not\equiv 0$). It is also possible to show that ϕ is a global minimizer of E in case $c > \lambda_1$.

For further information, see the monographs of Hale [**?**] or Henry [**?**], where the one dimensional case is studied in detail as the Chafee-Infante problem.

Remark 6.1. *Generally, some special tools are required to look for solutions of* (**??**)*; say, fixed point theorems, implicit function theorem, comparison method, variational method, degree theory, etc. When one relies on an operator theoretical approach, there is a very useful tool based on the bifurcation theory. This theory has been developed by Crandall and Rabinowitz* (*see [**?**, **?**] and the references therein*) *and the procedure to determine the stability of bifurcating solutions as well as their construction has been established. Therefore, this theory has been successfully applied to nonlinear PDEs in various fields.*

7 Analysis of semilinear parabolic systems

Following the lines stated in Sections 5 and 6, we can also study semilinear parabolic systems although the construction of steady-states becomes more complicate. As a typical example, we consider the following prey-predator system with diffusion

$$\begin{cases} u_t = \Delta u + au(1 - u - cv) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + bv(1 + du - v) & \text{in } \Omega \times (0,\infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases} \tag{7.1}$$

where $a, b, c.d$ are positive constants and u_0, v_0 are nonnegative measurable functions defined in Ω. In (**??**) u denotes the population density of a prey species and v denotes the population density of a predator species. This system has been investigated by lots of authors as a model to describe the population dynamics between two species.

Take $X = L^p(\Omega) \times L^p(\Omega)$ with $p > \max\{1, N/2\}$ and define norm $\|\cdot\|$ of X by $\|U\| = \|u\|_{L^p} + \|v\|_{L^p}$ for $U = {}^t(u,v) \in X$. We define a closed linear operator A from domain $D(A) \subset X$ to X by

$$AU = \begin{pmatrix} -\Delta - a & 0 \\ 0 & -\Delta - b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{for } U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A) = \{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\}^2.$$

Then one can show that A satisfies the resolvent condition (**??**) with $\beta = 1$; so that it generates an analytic semigroup $\{e^{-tA}\}$. We rewrite (**??**) in the form of (**??**) with

$$F(U) = \begin{pmatrix} -au(u + cv) \\ bv(du - v) \end{pmatrix} \qquad \text{for } U = \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

Since $F(U)$ satisfies the Lipschitz condition (**??**) with $m = 2$ and $\gamma = N/4p$, it is possible to apply Theorems **??** and **??** to (**??**). For any nonnegative $(u_0, v_0) \in X$, there exists a unique (local) solution $U = (u, v) \in C([0,T];\ X) \cap C^1((0,T];\ X) \cap C((0,T];\ W^{2,p}(\Omega) \times W^{2,p}(\Omega))$ of (**??**) with some $T > 0$. Moreover, by the maximum principle for parabolic equations, the nonnegativity of (u_0, v_0) assures the $u(\cdot, t) \geq 0$ and $v(\cdot, t) \geq 0$ as long as the solution exists.

In what follows, assuming $(u_0, v_0) \in \{L^\infty(\Omega)\}^2$ for the sake of simplicity, we will derive an a priori estimate for $U(t) = (u(t), v(t))$, which allow us to extend the local solution $U(t)$ in$[0, T]$ to the whole interval $[0, \infty)$. Since u and v are nonnegative, u satisfies

$$u_t = \Delta u + au(1 - u - cv) \leq \Delta u + au(1 - u),$$

the comparison theorem yields the following estimate

$$0 \leq u(x,t) \leq \max\{1, \|u_0\|_{L^\infty}\} =: m_1 \qquad \text{for} \ \ (x,t) \in \Omega \times (0, \infty). \tag{7.2}$$

Hence v satisfies

$$v_t = \Delta v + bv(1 + du - v) \leq \Delta v + bv(1 + dm_1 - v).$$

Applying the comparison theorem again we get

$$0 \leq v(x,t) \leq \max\{1 + dm_1, \|v_0\|_{L^\infty}\} =: m_2 \qquad \text{for} \ \ (x,t) \in \Omega \times (0, \infty). \tag{7.3}$$

Owing to (**??**) and (**??**) it is easy to see that (**??**) has a bounded global solution $U = (u, v) \in C([0, \infty);\ X) \cap C^1((0, \infty);\ X)$.

Our nest task is to study asymptotic behaviors of solutions to (**??**) as $t \to \infty$. As a preparatory problem, we will consider the following logistic diffusion equation:

$$\begin{cases} w_t = \Delta w + aw(1 - w) & \text{in} \ \ \Omega \times (0, \infty), \\ w = 0 & \text{on} \ \ \partial\Omega \times (0, \infty), \\ w(\cdot, 0) = w_0 \geq 0 (\not\equiv 0) & \text{in} \ \ \Omega, \end{cases} \tag{7.4}$$

where a is a positive constant and w_0 is $L^\infty(\Omega)$ function. The steady-state problem associated with (**??**) is given by

$$\Delta w + aw(1 - w) = 0 \quad \text{in} \ \ \Omega \ \ \text{and} \ \ w = 0 \quad \text{on} \ \ \partial\Omega. \tag{7.5}$$

We will look for a nonnegative solution of (**??**). It ia well known that (**??**) has no nontrivial solutions in case $a \leq \lambda_1$, while it has a unique positive solution θ_a in case $a > \lambda_1$. Moreover it is also known that for any $w_0 \geq 0 (\not\equiv 0)$, every solution of (**??**) satisfies

$$\lim_{t \to \infty} w(t) = \theta_a \quad \text{uniformly in} \ \ \Omega.$$

After these preparations, we will study the steady-state problem associated with the prey-predator model with diffusion:

$$\begin{cases} \Delta u + au(1 - u - cv) = 0, & u \geq 0, & \text{in} \ \ \Omega, \\ \Delta v + bv(1 + du - v) = 0, & v \geq 0, & \text{in} \ \ \Omega, \\ u = v = 0 & & \text{on} \ \ \partial\Omega \end{cases} \tag{7.6}$$

In addition to the trivial solution $(u, v) = (0, 0)$, the steady-state problem (**??**) possesses the following two semitrivial solutions

$$\begin{aligned} (u, v) &= (\theta_a, 0) \quad \text{if} \ \ a > \lambda_1, \\ (u, v) &= (0, \theta_b) \quad \text{if} \ \ b > \lambda_1. \end{aligned}$$

The stability of these solutions can be obtained by the linearization principle (Theorem **??**).

Theorem 7.1. (i) *Thr trivial solution* $(u,v)=(0,0)$ *is asymptotically stable if* $a<\lambda_1$ *and* $b<\lambda_1$*, while it is unstable if* $a>\lambda_1$ *or* $b>\lambda_1$.
(ii) $(u,v)=(\theta_a,0)$ *is asymptotically stable if* $\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-b\int_\Omega(1+d\theta_a)\varphi^2dx\}>0$, *while it is unstable if* $\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-b\int_\Omega(1+d\theta_a)\varphi^2dx\}<0$.
(iii) $(u,v)=(0,\theta_b)$ *is asymptotically stable if* $\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-a\int_\Omega(1-c\theta_b)\varphi^2dx\}>0$, *while it is unstable if* $\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-a\int_\Omega(1-c\theta_b)\varphi^2dx\}<0$.

Remark 7.1. *We can also study the global asymptotic stability of the trivial and semitrivial solutions in Theorem* **??**. *For instance, take* $(\theta_a,0)$. *Applying comparison theorems to reaction diffusion equations for* u *and* v *separately, one can prove the global asymptotic stability of* $(\theta_a,0)$ *provided that* (a,b) *satisfies* $\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-b\int_\Omega(1+d\theta_a)\varphi^2dx\}>0$. *That is, under this condition, every solution* (u,v) *of* (**??**) *satisfies*

$$\lim_{t\to\infty}(u(t),v(t))=(\theta_a,0)\quad \textit{niformly in}\ \ \Omega.$$

Similarly, it is also possible to prove the global asymptotic stability of $(0,\theta_b)$ *when* $(a.b)$ *satisfies* $\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-a\int_\Omega(1-c\theta_b)\varphi^2dx\}>0$.

Theorem **??** implies that the trivial and semitrivial solutions are unstable when (a,b) satisfies

$$\begin{aligned}&\mathrm{Inf}_{\varphi\in \mathrm{H}_0^1(\Omega)}\{\|\nabla\varphi\|^2-\mathrm{b}\int_\Omega(1+\mathrm{d}\theta_\mathrm{a})\varphi^2\mathrm{dx}\}<0 \qquad \text{and}\\ &\mathrm{Inf}_{\varphi\in H_0^1(\Omega)}\{\|\nabla\varphi\|^2-a\int_\Omega(1-c\theta_b)\varphi^2dx\}<0.\end{aligned} \tag{7.7}$$

The most significant and interesting task is to answer the following questions:

- What will happen for solutions of (**??**) if (a,b) satisfies (**??**)?
- Does the steady-state problem (**??**) possess a positive solution if (a,b) satisfies (**??**)?

In particular, a positive solution corresponds to the coexistence of two species; so that it is a very important solution from an ecological point of view as well as a mathematical point. As to the existence of a positive solution of (**??**) we have the following result (see Yamada [**?**]).

Theorem 7.2. *Steady-state problem* (**??**) *has a positive solution if and only if* (a,b) *satisfies* (**??**).

We can prove this theorem with use of local and global bifurcation theory (see Crandall and Rabinowitz [**?**, **?**]) or degree theory (see Dancer [**?**]).

8 Periodic solutions and orbital stability

A. Hopf bifurcation for PDEs

Consider the following reaction-diffusion system:

$$\begin{cases}u_t=\Delta u+au+bv+f(u,v) & \text{in}\ \ \Omega\times(0,\infty),\\ v_t=\Delta v-cu+g(u,v) & \text{in}\ \ \Omega\times(0,\infty),\\ u=v=0 & \text{on}\ \ \partial\Omega\times(0,\infty),\end{cases} \tag{8.1}$$

where a,b and c are positive constants and $f(u,v),g(u,v)$ are higher-order nonlinear functions. We assume $bc>\lambda_1$ (we denote by $\{\lambda_n\}_{n=1}^\infty$ the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition). As in Sections 6 and 7, take the linearized operator A for (**??**) at $(u,v)=(0,0)$; then its eigenvalues are given by $\{\lambda_n-(a\pm\sqrt{a^2-4bc})/2\}_{n=1}^\infty$. We will regard a as a parameter.

For $0 < a < 2\lambda_1$, all eigenvalues of A have positive real parts; so that Theorem **??** assures the asymptotic stability of $(u,v) = (0,0)$. When a becomes larger that n $2\lambda_1$, a pair of two eigenvalues $\lambda_1 - (a \pm \sqrt{a^2-4bc})/2$ crosses the imaginary axis in the complex plane. In this situation, bifurcation of periodic solutions (not another steady state) occurs from the trivial state $(0,0)$ (see Crandall and Rabinowitz [**?**]).

In this section, we will explain how to study the stability of periodic solutions for (**??**). An important tool is the Poincaré map associated with linear evolution equations of the form

$$u_t + C(t)u = 0, \quad u(0) = u_0, \tag{8.2}$$

where $C(t)$ is a $T-$periodic linear operator. Let $u(t;u_0)$ be the solution of (**??**). When $U(t)$ is defined by $U(t)u_0 = u(t;u_0)$, $U(T)$ is called the Poincaré map for (**??**). If η satisfies $e^{\eta T} \in \sigma(U(T))$, it is called a Floquet exponent. Cleary, η is a Floquet exponent if and only if

$$v_t + C(t)v = -\eta v$$

has a nontrivial T-periodic solution. Therefore, if all Floquet exponents have negative real parts, then every solution of (**??**) decays exponentially to zero as $t \to \infty$.

We now return to the case when (**??**) has T-periodic solution $p(t)$. The linearization of (**??**) around $u = p(t)$ leads to

$$w_t + Aw = dF(p(t))w. \tag{8.3}$$

Here we should observe that zero is one of Floquet exponents for (**??**). Indeed, since $p(t)$ satisfies (**??**), $w = dp/dt$ satisfies (**??**); so that zero is a Floquet exponent for (**??**). This fact makes the stability analysis more involved.

A stability result for any periodic solution $p(t)$ is stated in terms of the orbital stability for $\gamma := \{p(t);\ 0 \le t \le T\}$ in a suitable topology Y (usually stronger than X).

Theorem 8.1. *Suppose that zero is a simple Floquet exponent and that real parts of other exponents are negative. Then there exists a positive constant δ such that, if* $\mathrm{dist}\{a,\gamma\} := \mathrm{Inf}_{0\le t\le T}\{\|a-p(t)\|_Y\} \le \delta$, *every solution u of* (**??**) *with $u(0) = a$ satisfies*

$$\|u(t) - p(t-\theta)\|_Y \to 0 \qquad \textit{exponentially as } \ t \to \infty$$

with some θ depending on a.

Making use of this theorem, Crandall and Rabinowitz have established the existence and orbital stability of bifurcating periodic solutions in the framework of functional analysis (see [**?**, **?**] and the references therein).

B. Hopf bifurcation for functional differential equations

As is shown for (**??**), Hopf bifurcations often take place for reaction diffusion systems. However, they also occur for scalar parabolic equations with time delays. Consider

$$\begin{cases} u_t = \Delta u + u(a - bu - \int_{-\infty}^t k(t-s)u(s)ds) & \text{in } \ \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \ \partial\Omega \times (0,\infty), \\ u(\cdot,\tau) = \phi(\cdot,\tau) \ge 0 & \text{in } \ \Omega \times (-\infty,0], \end{cases} \tag{8.4}$$

where a and b are positive constants and k is s a nonnegative continuous function with

$$\alpha := \int_0^\infty k(t)dt < \infty.$$

If $k \equiv 0$, the asymptotic behavior for (**??**) is completely known similarly as is discussed for (**??**) in Section 7. The presence of k strongly influences the asymptotic behavior of (**??**).

We set $u^* = a/(b+\alpha)$. The linearization of (**??**) at u^* leads to the following "characteristic problem"

$$\begin{cases} (\lambda - \Delta + bu^* + \hat{k}(\lambda)u^*)w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.5}$$

where $\hat{k}(\lambda)$ is the Laplace transformation of $k(t)$. In [**?**], it is provedt hat, if (**??**) has no nontrivial solutions λ with $\mathrm{Re}\,\lambda \geq 0$, then u^* is asymptotically stable in an appropriate topology.

As a special example, we take $k(t) = \rho\alpha e^{-\rho t}$. Since

$$\hat{k}(\lambda) = \frac{\rho\alpha}{\lambda + \rho},$$

(**??**) has no nontrivial solutions for λ with $\mathrm{Re}\,\lambda \geq 0$; so that u^* is asymptotically stable. The asymptotic behavior of (**??**) is precisely studied by Iida [**?**] n this seminar. See also [**?**].

As a next example, taking $k(t) = \alpha\rho^2 te^{-\rho t}$, we regard α as a bifurcation parameter (other numbers are fixed). It is possible to prove that for $0 < \alpha < \alpha_0$ with some α_0, (**??**) has no characteristic values λ with $\mathrm{Re}\,\lambda \geq 0$; while, for $\alpha > \alpha_0$, a pair of characteristic values λ ($\mathrm{Re}\,\lambda \geq 0$) with imaginary parts appears. This fact allows us to expect Hopf bifurcations. Indeed, Yamada and Niikura [**?**] have established the existence of periodic solutions by using the bifurcation theory in such a situation.

9 Some topics

In this section I will briefly state some topics which are related to evolution equations, but are not discussed in previous sections,

A. Inertial manifolds

When we study dynamical systems in infinite dimensional spaces, the theory of invariant manifolds (say, stable manifolds, unstable manifolds, center manifolds etc.) has been proved to be very important. In particular, global attractors have been studied to describe the dynamics of dissipative evolution equations. Some authors have discussed specific problems to estimate the Hausdorff or fractal dimensions of the corresponding global attractors in terms of physical parameters (see, for instance, the monographs of Constantin, Foias, Nicolaenko and Temam [**?**] and Temam [**?**]). In the course of the study, it has turned out that these global attractors are often embedded in exponentially attractive invariant manifolds with finite dimensions, which are called inertial manifolds. This fact allows us to believe that the asymptotic behaviors of solutions of infinite dimensional dynamical systems are strongly governed by those of finite dimensional dynamical systems.

Since an inertial manifold is finite dimensional and invariant, the restriction of the original system to this manifold gives a system of ODEs. Moreover, the exponential attractivity of the manifold assures that every solution of the original system converges to a solution of the ODE on the manifold as $t \to \infty$. On account of these properties, the inertial manifold has the possibility that enables us to go deeply in the study of asymptotic behavior of solutions for dissipative PDEs. For details, see the works of Chow and Lu [**?**], Foias, Sell and Temam [**?**] and Mallet-Paret and Sell [**?**].

B. Quasilinear evolution equations

Many authors have studied the existence, uniqueness and some other properties of solutions to abstract evolution equations of the form

$$u_t + A(u)u = F(u)$$

in a Banach space X, where, for each u in a subset of X, $A(u)$ is a linear operator generating a C_0-semigroup or an analytic semigroup. Among them, I would like to refer to the work of Amann [**?**] and Yagi [**?**] as interesting results with emphasis on applications to quasilinear parabolic equations.

C. Application of nonlinear semigroup theory

Nonlinear semigroup theory was first introduced by Kōmura [**?**, **?**] and it has been applied to many PDEs such as degenerate parabolic equations, parabolic variational inequalities, first-order quasilinear equations of conservation law, etc. In particular, it seems to me that this theory has given great contributions to the analysis of porous medium equations and related ones (see, e.g., Benilan, Brezis and Crandall [**?**]).

It is difficult to give a complete review on the nonlinear semigroup theory and its applications. So I would like to end this lecture by referring to some works of Barbu [**?**], Brezis [**?**], Crandall [**?**] and Evans [**?**] on applications of nonlinear semigroup theory to nonlinear PDEs.

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