Free Boundary Problems for Reaction-Diffusion Equations Arising in Ecology

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1 Introduction

This is a survey article on a simplified free boundary problem which was investigated in a series of works by Mimura, Yamada and Yotsutani [7, 8, 9]. We consider the situation that two species, which cannot coexist in the same region, are struggling each other at the boundaries of their territories to extend their own habitats. Assume that each species lives separately in a one-dimensional interval $[0, L]$: one species lives in $[0, s(t)]$ and the other species lives in $[s(t), L]$. Here $x = s(t)$ is a free boundary at which both species are competing. Denote by $u_1(t, x)$ (resp. $u_2(t, x)$) the population density of the species living in the region $0 \le x \le s(t)$ (resp. $s(t) \le x \le L$) at time t. We assume that the dynamical behaviors of u_1 and u_2 are governed by reaction-diffusion equations as follows:

$$\begin{cases} u_{1,t} = D_1 u_{1,xx} + u_1 f_1(u_1), & t > 0,\ 0 < x < s(t), \\ u_{2,t} = D_2 u_{2,xx} + u_2 f_2(u_2), & t > 0,\ s(t) < x < L, \end{cases} \tag{1.1}$$

where D_i $(i = 1, 2)$ are positive constants and f_i $(i = 1, 2)$ are locally Lipschitz continuous functions satisfying $f_i(\alpha_i) = 0$ and $f_i(u) < 0$ for $u > \alpha_i$ with some $\alpha_i > 0$ $(i = 1, 2)$. On the fixed boundaries $x = 0, L$ and at initial time $t = 0$ we impose the following conditions

$$\begin{cases} u_1(t, 0) = M_1, \quad u_2(t, L) = M_2, & t > 0, \\ s(0) = s_0, & \\ u_1(0, x) = u_{10}(x), & 0 < x < s_0, \\ u_2(0, x) = u_{20}(x), & s_0 < x < L, \end{cases} \tag{1.2}$$

where M_i $(i = 1, 2)$ are nonnegative constants, $0 < s_0 < L$ is a constant and u_{i0} $(i = 1, 2)$ are nonnegative functions. We also assume that the dynamics of the free boundary $x = s(t)$ is determined by the interaction between u_1 and u_2 in the following manner

$$\begin{cases} u_1(t, s(t)) = u_2(t, s(t)) = 0, & t > 0, \\ \dot{s}(t) = -\beta_1 u_{1,x}(t, s(t)) - \beta_2 u_{2,x}(t, s(t)), & t \in \{\tau > 0 :\ 0 < s(\tau) < 1\}, \end{cases} \tag{1.3}$$

where $\dot{s} = ds/dt$ and β_i $(i = 1, 2)$ are positive constants. Roughly speaking, the above condition (1.3) implies that the dynamics of the free boundary is controlled by balance of the population pressures of two species at the competing front.

Our purpose in the present article is to study the existence, uniqueness, regularity and asymptotic behaviors of solutions for (1.1)-(1.3). We replace the space variable x and the free boundary $s(t)$ by Lx and $Ls(t)$, respectively. Moreover, we also introduce a new unknown function

$$u(t,x) = \begin{cases} u_1(t,Lx)/\alpha_1 & \text{for } \ t \geq 0, \ \ 0 \leq x \leq s(t), \\ -u_2(t,Lx)/\alpha_2 & \text{for } \ t \geq 0, \ \ s(t) \leq x \leq 1. \end{cases}$$

Then (1.1) is rewritten as

$$u_t = d_1 u_{xx} + uf(u), \qquad t > 0, \quad 0 < x < s(t), \tag{1.4}$$

$$u_t = d_2 u_{xx} + ug(u), \qquad t > 0, \quad s(t) < x < 1, \tag{1.5}$$

where $d_i = D_i/L^2 \ \ (i = 1,2)$, $f(u) = f_1(\alpha_1 u)$ and $g(u) = f_2(-\alpha_2 u)$. The free boundary conditions (1.3) are rewritten as

$$u(t, s(t)) = 0, \qquad t > 0, \tag{1.6}$$

$$\dot{s}(t) = -\mu_1 u_x(t, s(t) - 0) + \mu_2 u_x(t, s(t) + 0), \qquad t \in \{\tau > 0 : \ 0 < s(\tau) < 1\}, \tag{1.7}$$

where $\mu_i = \alpha_i \beta_i / L^2$ $(i = 1,2)$. In (1.7), $u_x(t, s(t) - 0)$ (resp. $u_x(t, s(t) + 0)$) denotes the limit of $u_x(t,x)$ at $x = s(t)$ from the left (resp. right). These conditions are quite similar to the free boundary conditions in a two-phase Stefan problem. Finally. it is easy to rewrite (1.2) as

$$\begin{cases} u(t,0) = m_1, \quad u(t,1) = -m_2, & t > 0, \\ s(0) = \ell, & \\ u(0,x) = \phi(x), & 0 < x < 1, \end{cases} \tag{1.8}$$

where $m_i = M_i/\alpha_i$ $(i = 1,2), \ell = s_0/L$ and

$$\phi(x) = u_{0,1}(Lx)/\alpha_1 \quad \text{for } \ x \in [0, \ell] \quad \text{and} \quad \phi(x) = -u_{0,2}(Lx)/\alpha_2 \quad \text{for } \ x \in [\ell, 1].$$

So we will study the following problem for reaction-diffusion equations with Stefan-like free boundary conditions

$$(\mathrm{P}) \quad \begin{cases} u_t = d_1 u_{xx} + uf(u) & \text{for } \ (t,x) \in S^-, \\ u_t = d_2 u_{xx} + ug(u) & \text{for } \ (x,t) \in S^+, \\ u(t,0) = m_1, \quad u(t,1) = -m_2 & \text{for } \ t > 0, \\ u(t,s(t)) = 0 & \text{for } \ t > 0, \\ \dot{s}(t) = -\mu_1 u_x(t, s(t) - 0) + \mu_2 u_x(t, s(t) + 0) & \text{for } \ t \in \{\tau > 0 : \ 0 < s(\tau) < 1\}, \\ u(0,x) = \phi(x) & \text{for } \ 0 \leq x \leq 1, \\ s(0) = \ell, & \end{cases}$$

where S^- (resp. S^+) is an open subset of $Q := (0, \infty) \times (0,1)$ with $x < s(t)$ (resp $x > s(t)$. When $f \equiv 0$ and $g \equiv 0$, the free boundary problem (P) is quite similar to a two-phase Stefan problem for which there are a lot of contributions.

We can construct a local solution $(u(t,x), s(t))$ of (P) with use of Schauder's fixed point theorem. In order to extend the local solution to a time-global solution, we employ the maximum principle for parabolic differential equations and energy methods to derive some a priori estimates. Here it should be noted that, if $m_1 = 0$ (resp. $m_2 = 0$), then the free boundary $x = s(t)$ may touch the fixed boundary $x = 0$ (resp. $x = 1$) in a finite time. When such a phenomenon happens, we take acount of (1.6) and continue to solve (P) as the standard boundary value problem without free boundary condition (1.7). Then it is possible to find a unique global smooth solution of (P) when f, g and (ϕ, ℓ) satisfy appropriate conditions (see (A.1)-(A.5) in §2).

Our next purpose is to study large-time behaviors of global smooth solutions to (P). We will apply the theory of dynamical systems and introduce the notion of the ω-limit set associated with each solution orbit $\{(u(t,\cdot), s(t)); t \geq 0\}$. Since it is proved that (P) possesses the Lyapunov functional, the standard theory for dynamical systems allows us to show that every element (u^*, s^*) of the ω-,imit set satisfies the stationary problem corresponding to (P):

$$\text{(SP)} \qquad \begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, \quad u^* \geq 0 & \text{in} \quad (0, s^*), \\ d_2 u^*_{xx} + u^* g(u^*) = 0, \quad u^* \leq 0 & \text{in} \quad (s^*, 1), \\ u^*(0) = m_1, \quad u^*(s^*) = 0, \quad u^*(1) = -m_2, & \\ -\mu_1 u^*_x(s^* - 0) + \mu_2 u^*_x(s^* + 0) = 0 & \text{if} \ \ 0 < s^* < 1. \end{cases}$$

For the analysis of asymptotic properties for (P), the comparison principle is a very useful and powerful tool because the smooth solutions of (P) possess the order preserving property. For instance, if (ϕ, ℓ) is a subsolution for (SP), then it will be shown that the smooth solution $(u(t,\cdot), s(t))$ of (P) with initial data (ϕ, ℓ) is monotone increasing in t and converges as $t \to \infty$ to a minimal solution (u^*, s^*) of (SP) in the class $u^* \geq \phi$ and $s^* \geq \ell$ (Theorem 4.3).

In the study of (SP), we will assume (for the sake of simplicity) that $uf(u)$ and $-ug(-u)$ are nonlinear functions of logistic type. Then one can use the phase plane method and get complete information on the structure of solutions to (SP). When both m_1 and m_2 are positive, it will be proved that the set of solutions of (SP) consists of discrete elements. In such a situation, any solution of (P) converges to one of stationary solutions as $t \to \infty$. When $m_1 = m_2 = 0$, every pair of the form $(0, s^*)$ with $0 \leq s^* \leq 1$ satisfies (SP); such a pair is called a trivial stationary solution. So the analysis in case $m_1 = m_2 = 0$ becomes more complicate than the case $m_1 > 0$ and $m_2 > 0$. In order to investigate asymptotic behaviors of solutions to (P) as $t \to \infty$, the most important task is to construct suitable supersolutions and subsolutions. We will give several examples of super- and subsolutions to derive valuable information on stability and/or instability of every stationary solution.

As concluding remarks, we should note that our arguments are also valid for the free boundary problem with Dirichlet boundary conditions in (1.2) replaced by homogeneous Neumann boundary conditions

$$u_{1,x}(t, 0) = 0, \quad u_{2,x}(t, L) = 0, \quad t > 0.$$

In 2010, Du and Lin [2] proposed an interesting free boundary problem related with (1.1)-(1.3). Their problem models the invasion of a single species and it can be interpreted as a one-phase nonlinear Stefan problem. Set $u_1 = u, u_2 = 0, D_1 = d, L = \infty$ and $\beta_1 = \mu$ in (1.1) and (1.3); then

$$\begin{cases} u_t = du_{xx} + uf(u) & t > 0, \ 0 < x < s(t), \\ u(t, s(t)) = 0, & t > 0, \\ \dot{s}(t) = -\mu u_x(t, s(t)), & t > 0. \end{cases} \tag{1.9}$$

In [2], initial and boundary ($x = 0$) conditions are given by

$$\begin{cases} u_x(t, 0) = 0, & \\ s(0) = s_0, & \\ u(0, x) = u_0(x) & 0 \leq x \leq s_0, \end{cases} \tag{1.10}$$

where u_0 is a nonnegative function. When f is given by $f(u) = a - bu$ with positive a, b, Du and Lin obtained a remarkable result on spreading-vanishing dichotomy for (1.9) -(1.10); that is, every solution (u, s) of (1.9)-(1.10) satisfies one of the following asymptotic behaviors as $t \to \infty$:

- Vanishing; $\lim_{t\to\infty} s(t) \le (\pi/2)\sqrt{d/a}$ and $\lim_{t\to\infty} \|u(t)\|_{C([0,s(t)])} = 0$,
- Spreading; $\lim_{t\to\infty} s(t) = \infty$ and $\lim_{t\to\infty} u(t,\cdot) = \dfrac{a}{b}$ locally uniformly in $[0,\infty)$.

Moreover, it was also shown that, whenever the spreading happens, the free boundary satisfies

$$\lim_{t\to\infty} \frac{s(t)}{t} = c^*,$$

where c^* is a positive constant that is determined, independently of initial data, from a corresponding semi-wave problem (for more details see also the work of Du and Lou [3]). The above results attracted attentions of lots of researchers. The free boundary problem (1.9)-,(1.10) and related problems have been investigated quite intensively. So one can find a number of interesting results such as the classification of asymptotic behaviors of solutions, asymptotic estimates of free boundaries and asymptotic profiles of solutions. See, for instance, [2], [3] and [5].

2 Assumptions and global solutions

We will study (P), in place of (1.1)-(1.3), under the following assumptions:

(A.1) f is locally Lipschitz continuous in $[0,\infty)$ and satisfies

$$f(1) = 0 \quad \text{and} \quad f(u) < 0 \quad \text{for} \quad u > 1.$$

(A.2) g is locally Lipschitz continuous in $(-\infty, 0]$ and satisfies

$$g(-1) = 0 \quad \text{and} \quad g(u) < 0 \quad \text{for} \quad u < -1.$$

(A.3) $0 \le m_1 \le 1$ and $0 \le m_2 \le 1$.

(A.4) $0 < \ell < 1$.

(A.5) $\phi \in H^1(I)$ with $I := (0, 1)$ satisfies $\phi(0) = m_1, \phi(\ell) = 0, \phi(1) = -m_2$ and $(\ell - x)\phi(x) \ge 0$ for $x \in I$.

Case $m_1 > 0$ and $m_2 > 0$

We begin with the existence of a unique global solution for (P) in case $m_1 > 0$ and $m_2 > 0$.

Theorem 2.1. ([7, Theorem I]) *There exists a unique pair of functions $(u, s) \in C(\overline{Q}) \times C([0,\infty))$, $Q = [0,\infty) \times I$, with the following properties:*

(i) *s satisfies $s(0) = \ell, \dot{s} \in L^3(0,\infty) \cap L^\infty(\delta,\infty)$ for any $\delta > 0$ and*

$$b_1 \le s(t) \le b_2 \quad \textit{for all} \quad t \ge 0$$

with some $b_i \in (0, 1)$ $(i = 1, 2)$.

(ii) *u satisfies (1.8) and*

$$0 \le u \le M := \max\{1, \sup_{0\le x\le \ell} \phi(x)\} \quad \textit{in} \quad \overline{S^-},$$

$$0 \ge u \ge -N := \min\{-1, \inf_{\ell\le x\le 1} \phi(x)\} \quad \textit{in} \quad \overline{S^+}.$$

(iii) *Set* $u^+ = \max\{u, 0\}$ *and* $u^- = -\min\{u, 0\}$. *Then* $u^\pm \in C([0,\infty);\ H^1(I))$ *and*

$$\sup_{t\geq 0} \|u^+(t)\|_{H^1(I)} < \infty, \quad \sup_{t\geq 0} \|u^-(t)\|_{H^1(I)} < \infty.$$

Moreover,

$$\sup_{t\geq \delta} \|(u^+)_x(t)\|_{L^\infty(I)} < \infty, \quad \sup_{t\geq \delta} \|(u^-)_x(t)\|_{L^\infty(I)} < \infty$$

for any $\delta > 0$.

(iv) $u_t \in L^2(S^-) \cap L^2(S^+)$.

(v) $u_t, u_{xx} \in C(S^-) \cap C(S^+)$ *and* u *satisfies* (1.4)-(1.5).

(vi) *For any* $\delta > 0$, $u_x(t,x)$ *is Hölder continuous in* $(t,x) \in \{(\tau,y):\ \tau \geq \delta, 0 \leq y \leq s(t)\} \cup \{(\tau,y):\ \tau \geq \delta, s(t) \leq y \leq 1\}$ *and* $\dot{s}(t)$ *is Hölder continuous in* $t \in [\delta,\infty)$.

(vii) (u,s) *satisfies* (1.6) *and* (1.7).

Remark 2.1. *Theorem 2.1 assures the existence of a global solution* (u,s) *for (P). The existence of a local solution to (P) can be proved with use of Schauder's fixed point theorem and its extension to a global solution is assured by some a priori estimates. The uniqueness of a solution is a consequence of the comparison theorem for (P). Finally, the regularity of the solution is shown by using the theory of evolution equations and embedding theorems. For details, see [7].*

In what follows, we say that (u,s) is a *smooth solution* of (P) when it possesses the regularity properties given in Theorem 2.1.

Since the global existence result is established in case that both m_1 and m_2 are positive, our next task is to study large-time behaviors of smooth solutions. Let $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ denote the smooth solution of (P) with initial data (ϕ,ℓ) satisfying (A.4) and (A.5). We introduce the notion of the *ω-limit set* associated with the solution orbit $\{(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)):\ \ t \geq 0\}$ as follows:

$$\begin{aligned}\omega(\phi,\ell) := \{(u^*,s^*) \in H^1(I)\times\bar{I}:\ &\text{there exists a sequence } \ \{t_n\}\uparrow\infty \ \text{ such that} \\ &s(t_n;\phi,\ell)\to s^* \ \text{ and } \ u^\pm(t_n;\phi,\ell)\to(u^\pm)^* \ \text{ in } \ H^1(I) \ \text{ as } \ n\to\infty\}.\end{aligned} \tag{2.1}$$

We say that $\{(u(t_n;\phi,\ell), s(t_n;\phi,\ell))\}$ *converges to* (u^*,s^*) *in the sense of* Ω*-topology* if it has the convergence properties in (2.1).

By Theorem 2.1 $\dot{s}(\cdot;\phi,\ell) \in L^3(0,\infty)$ and that $\{u(t;\phi,\ell);\ t\geq 0\}$ is bounded in $H^1(I)$. Then Ascoli-Arzela's theorem implies that $\{u(t;\phi,\ell);\ t\geq 0\}$ is relatively compact in $C(\bar{I})$. Moreover, $t \mapsto s(t;\phi,\ell)$ is uniformly Hölder continuous with exponent $2/3$. Therefore, repeating the arguments of [7, §4 and §6] one can show the $\{u^\pm(t;\phi,\ell);\ t\geq 0\}$ is relatively compact in $H^1(I)$. These considerations show that $\omega(\phi,\ell)$ is a nonempty compact set in $H^1(I)\times\bar{I}$.

The structure of the ω-imit set is given by the following theorem.

Theorem 2.2. ([7, Theorem II]) *Assume* $m_1, m_2 \in (0,1]$ *and let* $\omega(\phi,\ell)$ *be the* ω*-limit set associated with the smooth solution* $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ *of* (P).

(i) $\omega(\phi,\ell)$ *is a nonempty, connected and compact set in* $H^1(I)\times\bar{I}$.

(ii) $\omega(\phi,\ell)$ *is positively invariant: if* $(u^*,s^*)\in\omega(\phi\,\ell)$, *then* $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) \in \omega(\phi,\ell)$ *for every* $t\geq 0$.

(iii) *If* $(u^*,s^*)\in\omega(\phi,\ell)$, *then it satisfies* (SP) *with* $0 < s^* < 1$.

In Theorem 2.2, (SP) is called a *stationary problem* associated with (P) and any pair (u^*, s^*) satisfying (SP) is called a *stationary solution* for (P). Theorem 2.2 helps us to get useful information on asymptotic behaviors of smooth solutions to (P) as $t \to \infty$. For instance, if it is proved that stationary solutions are isolated, then $\{(u(t;\phi,\ell), s(t;\phi,\ell))\}$ converges to one of stationary solutions (in the sense of Ω-topology) as $t \to \infty$. So it becomes very important to investigate the structure of the set of stationary solutions.

Case $m_1 = 0$ or $m_2 = 0$

When $m_1 = 0$ or $m_2 = 0$, there is possibility that the free boundary $x = s(t)$ hits one of the fixed boundaries $x = 0, 1$ in a finite time; say, $s(T^*) = 1$ with some $T^* \in (0, \infty)$ Then one can see from the strong maximum principle (see the monograph of Protter and Weinberger [10]) that the free boundary never leaves the fixed boundary $x = 1$ after $t = T^*$.

We will give an existence result for (P) in a typical case $m_1 = m_2 = 0$. We use the following notation:

$$S^-_{\delta,T} = \{(t,x) \in Q : \ \delta < t < T \ \text{ and } \ 0 < x < s(t)\},$$
$$S^+_{\delta,T} = \{(t,x) \in Q : \ \delta < t < T \ \text{ and } \ s(t) < x < 1\}.$$

Theorem 2.3. ([9, Theorems 2.1 and 2.2]) *Assume $m_1 = m_2 = 0$. Under assumptions* (A.1)-(A.5)*, there exists $T^* \in (0, \infty]$ such that*(P) *admits a unique solution $(u, s) \in C([0,T^*] \times \bar{I}) \times C([0,T^*])$ with the following properties:*

(i) *(u, s) satisfies* (1.8).

(ii) $\dot{s} \in L^3(0,T^*)$, $\ 0 < s(t) < 1$ *for* $t \in [0,T^*)$. *If $T^* < \infty$, then $s(T^*) = 0$ or 1.*

(iii) *(u, s) satisfies*

$$0 \le u \le M := \max\{1, \sup_{0\le x\le \ell} \phi(x)\} \qquad \text{in } \overline{S^-_{0,T^*}},$$
$$0 \ge u \ge -N := \min\{-1, \inf_{\ell\le x\le 1} \phi(x)\} \ \text{ in } \overline{S^+_{0,T^*}}.$$

(iv) $u^\pm \in C([0,T^*);\ H^1_0(I))$. *If $T^* < \infty$, then $u^\pm \in C([0,T^*];\ H^1_0(I))$.*

(v) $(u^+)_x \in L^\infty(S^-_{\delta,T^*})$, $(u^-)_x \in L^\infty(S^+_{\delta,T^*})$ *and $\dot{s} \in L^\infty(\delta,T^*)$ for any $\delta \in (0,T^*)$.*

(vi) $u_t \in L^2(S^-_{0,T^*}) \cap L^2(S^+_{0,T^*})$.

(vii) $u_t, u_{xx} \in C(S^-_{0,T^*}) \cap C(S^+_{0,T^*})$ *and (u, s) satisfies* (1.4)-(1.5).

(viii) *For any $\delta, \delta' \in (0,T^*)$, u_x is Hölder continuous with respect to (t,x)in $\{(\tau,y) \in \overline{S^-_{\delta,T^*}} : s(\tau) \ge \delta'\}$ and $\{(\tau,y) \in \overline{S^+_{\delta,T^*}} : s(\tau) \le 1-\delta'\}$ and $\dot{s}$ is Hölder continuous with respect to $t \in [\delta,T^*]$.*

(ix) *$(u.s)$ satisfies* (1.6) *and* (1.7).

As is stated before, the free boundary $x = s(t)$ may reach one of fixed boundaries at a finite time $t = T^*$, but it never leaves the fixed boundary after $t = T^*$. If we intend to continue to solve (P) for $t \ge T^*$, it would be reasonable to study the problem by setting $s(t) \equiv 0$ or $s(t) \equiv 1$ for $t \ge T^*$ in view of (1.6). If $s(T^*) = 1$ (resp. $s(T^*) = 0$), we understand to solve the standard initial boundary value problem with $s(t) \equiv 1$ (resp. $s(t) \equiv 0$) for $t \ge T^*$ and initial data $u(T^*)$

at initial time T^*. In this situation, we have to neglect the Stefan condition (1.7). Our global existence result is given in the following manner.

Theorem 2.4. ([9, Theorem 3.1]) *Assume $m_1 = m_2 = 0$. Under assumptions* (A.1)-(A.5), *there exists a unique solution $(u, s) \in C(\bar{Q}) \times C[0, \infty)$ with the following properties:*

(i) *(u, s) satisfies* (1.8).

(ii) *$\dot{s} \in L^3(0, \infty)$ and s satisfies one of the following properties; $0 < s(t) < 1$ for all $t > 0$, $s(t) = 0$ for all $t \geq T^*$ or $s(t) = 1$ for all $t \geq T^*$ with some $T^* \in (0, \infty)$.*

(iii) *(u, s) satisfies*

$$0 \leq u \leq M := \max\{1, \sup_{0\leq x\leq \ell} \phi(x)\} \qquad in \quad \overline{S^-},$$
$$0 \geq u \geq -N := \min\{-1, \inf_{\ell\leq x\leq 1} \phi(x)\} \quad in \quad \overline{S^+}.$$

(iv) *$u^{\pm} \in C([0, \infty);\ H_0^1(I))$.*

(v) *$(u^+)_x \in L^\infty(S^-_{\delta,\infty})$, $(u^-)_x \in L^\infty(S^+_{\delta,\infty})$ and $\dot{s} \in L^\infty(\delta, \infty)$ for any $\delta > 0$.*

(vi) *$u_t \in L^2(S^-) \cap L^2(S^+)$.*

(vii) *$u_t, u_{xx} \in C(S^-) \cap C(S^+)$ and (u, s) satisfies* (1.4)-(1.5).

(viii) *For any $\delta > 0$ and $\delta' > 0$, u_x is Hölder continuous with respect to (t, x) in $\{(\tau, y) \in \overline{S^-_{\delta,\infty}} :\ s(\tau) \geq \delta'\}$ and $\{(\tau, y) \in \overline{S^+_{\delta,\infty}} :\ s(\tau) \leq 1 - \delta'\}$ and $\dot{s}$ is Hölder continuous with respect to t in $\overline{\{\tau \geq \delta;\ 0 < s(\tau) < 1\}}$.*

(ix) *(u, s) satisfies* (1.6) *and* (1.7).

Finally we will give complete information on the structure of the ω limit set.

Theorem 2.5. ([9, Theorem 6.2]) *Under the assumption of $m_1 = m_2 = 0$ and let $\omega(\phi, \ell)$ be the ω-limit set associated with the smooth solution $(u(t; \phi, \ell), s(t; \phi, \ell))$ of* (P). *Then the following properties hold true.*

(i) *$\omega(\phi, \ell)$ is a nonempty, connected and compact set in $H_0^1(I) \times \bar{I}$.*

(ii) *$\omega(\phi, \ell)$ is positively invariant: if $(u^*, s^*) \in \omega(\phi\, \ell)$, then $(u(t, \cdot; \phi, \ell), s(t; \phi, \ell)) \in \omega(\phi, \ell)$ for every $t \geq 0$.*

(iii) *If $(u^*, s^*) \in \omega(\phi, \ell)$, then it satisfies*

$$(\mathrm{SP}-0) \qquad \begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, \quad u^* \geq 0 & in \quad (0, s^*), \\ d_2 u^*_{xx} + u^* g(u^*) = 0, \quad u^* \leq 0 & in \quad (s^*, 1), \\ u^*(0) = u^*(s^*) = u^*(1) = 0, & \\ -\mu_1 u^*_x(s^* - 0) + \mu_2 u^*_x(s^* + 0) = 0 & if \quad 0 < s^* < 1. \end{cases}$$

3 Basic estimates

In this section we will provide basic estimates for every smooth solution of (P). For $s \in I = (0, 1)$ and $u \in H^1(I)$ satisfying $u(s) = 0$, we introduce the following functional

$$E(u, s) = \frac{\mu_1^2}{2} \int_0^s u_x^2\ dx + \frac{\mu_2^2}{2} \int_s^1 u_x^2\ dx - \frac{\mu_1^2}{d_1} \int_0^1 F(u)\ dx - \frac{\mu_2^2}{d_2} \int_s^1 G(u) dx, \tag{3.1}$$

where

$$F(u) = \int_0^u vf(v)dv \quad \text{and} \quad G(u) = \int_0^u vg(v)dv.$$

Proposition 3.1. *Assume $m_1, m_2 \in (0,1]$ and let (u,s) be the smooth solution of* (P). *Then it holds that*

$$E(u(t),s(t)) + \frac{\mu_1^2}{d_1}\int_0^t\int_0^{s(\tau)} u_t^2(\tau,x)dxd\tau + \frac{\mu_2^2}{d_2}\int_0^t\int_{s(\tau)}^1 u_t^2(\tau,x)dxd\tau$$
$$+\frac{1}{2}\int_0^t |\dot{s}(\tau)|^3 d\tau \le E(\phi,\ell) \qquad \textit{for all} \ \ t \ge 0.$$

Proof. We begin with the following identity for the smooth solution $(u(t,x),s(t))$ of (P):

$$\frac{d}{dt}\int_0^{s(t)} u_x^2\,dx = u_x^2(t,s(t)-0)\dot{s}(t) + 2u_x(t,s(t)-0)u_t(t,s(t)) - 2\int_0^{s(t)} u_{xx}u_t\,dx. \tag{3.2}$$

It follows from the free boundary condition that

$$J(t,\Delta t) := u(t+\Delta t, s(t+\Delta t)) - u(t,s(t)) = 0.$$

If $s(t+\Delta t) \ge s(t)$, then

$$\begin{aligned} J(t,\Delta t) &= \{u(t+\Delta t, s(t+\Delta t) - u(t+\Delta t, s(t))\} + \{u(t+\Delta t, s(t)) - u(t,s(t))\} \\ &= u_x(t+\Delta t, s(t) + \theta(s(t+\Delta t)-s(t)))(s(t+\Delta t)-s(t)) + u_t(t+\theta'\Delta t, s(t))\Delta t, \end{aligned}$$

with some $\theta,\theta' \in [0,1]$. Recalling the Hölder continuity of u_x with respect to (t,x) up to the boundary (see [7, (4.27)]) we divide the above relation by Δt and let $\Delta t \to 0$. Then, by virtue of $J(t,\Delta t) = 0$,

$$u_x(t,s(t)-0)\dot{s}(t) + u_t(t,s(t)) = 0. \tag{3.3}$$

On the other hand, if $s(t+\Delta t) \le s(t)$, then

$$\begin{aligned} J(t,\Delta t) &= \{u(t+\Delta t, s(t+\Delta t)) - u(t, s(t+\Delta t))\} + \{u(t, s(t+\Delta t)) - u(t,s(t))\} \\ &= u_t(t+\theta\Delta t, s(t+\Delta t))\Delta t + u_x(t, s(t) + \theta'(s(t+\Delta t)-s(t)))(s(t+\Delta t)-s(t)) \end{aligned}$$

with some $\theta,\theta' \in [0,1]$. Dividing the above relation by Δt and letting $\Delta \to 0$ we se from $J(t,\Delta t) = 0$ that

$$u_t(t,s(t)) + u_x(t,s(t)-0)\dot{s}(t) = 0. \tag{3.4}$$

Therefore, it follows from (3.2), (3.3) and (3.4) that

$$\frac{d}{dt}\int_0^{s(t)} u_x^2(t,x)\,dx = -u_x^2(t,s(t)-0)\dot{s}(t) - 2\int_0^{s(t)} u_{xx}(t,x)u_t(t,x)\,dx. \tag{3.5}$$

Similarly, one can derive

$$\frac{d}{dt}\int_{s(t)}^1 u_x^2(t,x)\,dx = u_x^2(t,s(t)+0)\dot{s}(t) - 2\int_{s(t)}^1 u_{xx}(t,x)u_t(t,x)\,dx. \tag{3.6}$$

Since $F(u(t,s(t)) = G(u(t,s(t)) = 0$, it is easy to see

$$\frac{d}{dt}\int_0^{s(t)} F(u(t,x))dx = \int_0^{s(t)} (uf(u))(t,x)u_t(t,x)dx, \tag{3.7}$$

$$\frac{d}{dt}\int_{s(t)}^1 G(u(t,x))dx = \int_{s(t)}^1 (ug(u))(t,x)u_t(t,x)dx. \tag{3.8}$$

Then it follows from (3.5)-(3.8) that

$$\begin{aligned}\frac{d}{dt}E(u(t),s(t)) &= -\frac{\mu_1^2}{d_1}\int_0^{s(t)}(d_1u_{xx}+uf(u))u_t\ dx - \frac{\mu_2^2}{d_2}\int_{s(t)}^1(d_2u_{xx}+ug(u))u_t\ dx \\ &\quad + \frac{\dot{s}(t)}{2}\{-\mu_1^2u_x^2(t,s(t)-0)+\mu_2^2u_x^2(t,s(t)+0)\} \\ &= -\frac{\mu_1^2}{d_1}\int_0^{s(t)}u_t^2\ dx - \frac{\mu_2^2}{d_2}\int_{s(t)}^1 u_t^2\ dx \\ &\quad + \frac{\dot{s}(t)}{2}\{-\mu_1u_x(t,s(t)-0)+\mu_2u_x(t,s(t)+0)\} \\ &\quad\quad \times\{\mu_1u_x(t,s(t)-0)+\mu_2u_x(t,s(t)+0)\}\end{aligned} \tag{3.9}$$

Since $u \geq 0$ in S^- and $u \leq 0$ in S^+, the strong maximum principle gives

$$u_x(t,s(t)-0) < 0 \quad \text{and} \quad u_x(t,s(t)+0) < 0$$

(see [10]). Hence it follows from $\dot{s}(t) = -\mu_1u_x(t,s(t)-0)+\mu_2u_x(t,s(t)+0)$ that

$$|\dot{s}(t)| \leq -\mu_1u_x(t,s(t)-0)-\mu_2u_x(t,s(t)+0).$$

In view of the above relations, we obtain from (3.9)

$$\frac{d}{dt}E(u(t),s(t)) \leq -\frac{\mu_1^2}{d_1}\int_0^{s(t)}u_t^2\ dx - \frac{\mu_2^2}{d_2}\int_{s(t)}^1 u_t^2\ dx - \frac{1}{2}|\dot{s}(t)|^3. \tag{3.10}$$

The integration of (3.10) with respect to t yields the assertion. □

Remark 3.1. *The estimate (ii) of Theorem 2.1 is a consequence of the maximum principle. So Proposition 3.1 together with (ii) yields some other estimates in Theorem 2.1;* $\dot{s} \in L^3(0,\infty)$, $u_t \in L^2(S^-)\cap L^2(S^+)$ *and* $u^\pm \in L^\infty(0,\infty;H^1(I))$.

Proof of Theorem 2.2

As an application of Proposition 3.1 we will give a sketch of the proof of Theorem 2.2.

Let (ϕ,ℓ) satisfy (A.4), (A.5) and denote by $(u(t,\cdot;\phi,\ell),s(t;\phi,\ell))$ a smooth solution of (P) with initial data (ϕ,ℓ). By Theorem 2.1 $\{u(t,\cdot;\phi,\ell):\ t\geq 0\}$ is uniformly bounded in $C([0,\infty)\times\bar{I})$ and $b \leq s(t;\phi,\ell) \leq 1-b$ with some $b\in(0,1)$. So Proposition 3.1 gives

$$\sup_{t\geq 0}\|u^+(t;\phi,\ell)\|_{H^1(I)} < \infty \quad \text{and} \quad \sup_{t\geq 0}\|u^-(t;\phi,\ell)\|_{H^1(I)} < \infty \tag{3.11}$$

and

$$\int_0^\infty |\dot{s}(t;\phi,\ell)|^3\ dt < \infty. \tag{3.12}$$

Ascoli-Arzela's theorem together with (3.11) assues that $\{u(t;\phi,\ell):\ t\geq 0\}$ is relatively compact in $C(\bar{I})$. Moreover, (3.12) implies that $t\mapsto s(t;\phi,\ell)$ uniformly Hölder continuous with exponent 2/3. Then repeating the arguments of [7, §4] and making use of the embedding theorems in the theory of evolution equations (see [4]) we can prove that $\{u^\pm(t;\phi,\ell):\ t\geq 0\}$ is relatively compact in $H^1(I)$ (see also [9, Lemma 6.1]). Hence we see that $\omega(\phi,\ell)$ is nonempty.

In order to complete the proof of (i) and (ii) of Theorem 2.2, we recall that $(\phi,\ell)\mapsto(u(t;\phi,\ell),s(t,\phi,\ell))$ is continuous in $H^(I)\times[0,1]$ for every $t\geq 0$ (see [7, Theorem 6.5] or [16]).

Therefore, by the standard method in the theory of dynamical systems, it is possible to show that $\omega(\phi, \ell)$ is compact, connected and positively invariant in $H^1(I) \times \bar{I}$.

We will next prove (iii). By (3.10), $t \mapsto E(u(t; \phi, \ell), s(t; \phi, \ell))$ is strictly decreasing and bounded from below; so that there exists

$$\lim_{t \to \infty} E(u(t; \phi, \ell), s(t; \phi, \ell)) = E_\infty. \tag{3.13}$$

Now take any $(u^*, s^*) \in \omega(\phi, \ell)$. Since there exists a sequence $\{t_n\} \uparrow \infty$ such that $\lim_{n \to \infty} (u(t_n; \phi, \ell), s(t_n; \phi, \ell)) = (u^*, s^*)$ in the sense of Ω-topology, it follows from the definition of $E(u, s)$ that

$$\lim_{n \to \infty} E(u(t_n; \phi, \ell), s(t_n; \phi, \ell)) = E(u^*.s^*). \tag{3.14}$$

Then we see from (3.13) and (3.14) that any $(u^*, s^*) \in \omega(\phi, \ell)$ satisfies

$$E(u^*, s^{\text{`}}) = E_\infty. \tag{3.15}$$

Owing to the result of (ii), $\omega(\phi.\ell)$ is positively invariant, which implies that, if $(u^*, s^*) \in \omega(\phi, \ell)$, then

$$(u(t; u^*, s^*), s(t; u^*, s^*)) \in \omega(\phi, \ell) \qquad \text{for every} \quad t \geq 0.$$

Hence it follows from (3.15)

$$E(u(t; u^*, s^*), s(t; u^*, s^*)) = E_\infty \qquad \text{for every} \quad t \geq 0; \tag{3.16}$$

so that

$$\frac{d}{dt} E(u(t; u^*, s^*), s(t; u^*, s^*)) = 0 \qquad \text{for} \quad t \geq 0.$$

Therefore, we see from (3.10) that

$$u_t(t, \cdot; u^*, s^*) = 0 \quad \text{and} \qquad \dot{s}(t; u^*, s^*) = 0.$$

So it follows that $u(t, \cdot; u^*, s^*) \equiv u^*$ and $s(t; u^*, s^*) \equiv s^*$. This fact implies that (u^*, s^*) satisfies (SP).

We next give another important estimate in case $m_1 = m_2 = 0$.

Proposition 3.2. *In addition to*(A.1)-(A.5)*, assume* $m_1 = m_2 = 0$*. Let* $(u, s) \in C([0, T] \times \bar{I}) \times C([0, T])$ *be a smooth solution of* (P) *satisfying*(i)-(iv), (vi)-(ix) *of Theorem 2.3 with* T^* *replaced by* T*. Then*

$$(u^+)_x \in L^\infty(S^-_{\delta,T}), \quad (u^-)_x \in L^\infty(S^+_{\delta,T}) \quad \textit{and} \quad \dot{s} \in L^\infty(\delta, T)$$

for any $\delta \in (0, T)$*.*

Proof. First we assume

$$\phi_x \in L^\infty(I) \tag{3.17}$$

in addition to (A.5). Applying Moser's technique we will show

$$u_x \in L^\infty(S^-_{0,T}) \cap L^\infty(S^+_{0,T}), \tag{3.18}$$

in other words,

$$(u^+)_x \in L^\infty(S^-_{0,T}) \quad \text{and} \quad (u^-)_x \in L^\infty(S^+_{0,T}).$$

(For Moser's technique, see the works of [1] or [11].)

Let p be any positive integer. Then

$$\begin{aligned}\frac{d}{dt}\int_0^{s(t)} u_x^{2p}\,dx &= u_x^{2p}(t,s(t)-0)\dot{s}(t) + 2p\int_0^{s(t)} u_x^{2p-1}u_{xt}\,dx \\ &= u_x^{2p}(t,s(t)-0)\dot{s}(t) + 2pu_x^{2p-1}(t,s(t)-0)u_t(t,s(t)) \\ &\quad - 2p(2p-1)\int_0^{s(t)} u_x^{2p-2}u_{xx}u_t\,dx.\end{aligned}$$

By virtue of (3.3) and (3.4) , the above relation yields

$$\begin{aligned}\frac{d}{dt}\int_0^{s(t)} u_x^{2p}\,dx &= -(2p-1)u_x^{2p}(t,s(t)-0)\dot{s}(t) \\ &\quad -2p(2p-1)\int_0^{s(t)} u_x^{2p-2}u_{xx}(d_1u_{xx}+uf(u))\,dx.\end{aligned} \tag{3.19}$$

Here it should be noted that

$$\int_0^{s(t)} u_x^{2p-2}u_{xx}^2\,dx = \frac{1}{p^2}\int_0^{s(t)} |(u_x^p)_x|^2\,dx,$$

$$-\int_0^{s(t)} u_x^{2p-2}u_{xx}uf(u)\,dx = -\frac{1}{2p-1}\int_0^{s(t)} (u_x^{2p-1})_x uf(u)\,dx = \frac{1}{2p-1}\int_0^{s(t)} u_x^{2p}\tilde{f}(u)\,dx,$$

where $\tilde{f}(u) = \frac{d}{du}uf(u)$. Hence it follows from (3.19) that

$$\begin{aligned}&\frac{d}{dt}\int_0^{s(t)} (\mu_1u_x)^{2p}dx + \frac{2(2p-1)d_1}{p}\int_0^{s(t)} |(\mu_1^pu_x^p)_x|^2\,dx \\ &\quad + (2p-1)(\mu_1u_x)^{2p}(t,s(t)-0)\dot{s}(t) = 2p\int_0^{s(t)} (\mu_1u_x)^{2p}\tilde{f}(u)\,dx.\end{aligned} \tag{3.20}$$

Similarly, once can deduce

$$\begin{aligned}&\frac{d}{dt}\int_{s(t)}^1 (\mu_2u_x)^{2p}dx + \frac{2(2p-1)d_2}{p}\int_{s(t)}^1 |(\mu_2^pu_x^p)_x|^2\,dx \\ &\quad - (2p-1)(\mu_2u_x)^{2p}(t,s(t)+0)\dot{s}(t) = 2p\int_{s(t)}^1 (\mu_2u_x)^{2p}\tilde{g}(u)\,dx,\end{aligned} \tag{3.21}$$

where $\tilde{g}(u) = \frac{d}{du}ug(u)$. By (iii) of Theorem 2.3, $-N \le u(t,x) \le M$ in $[0,T]\times\bar{I}$; so that

$$\sup_{(t,x)\in S_{0,T}^-} |\tilde{f}(u(t,x))| \le M_1^* \quad \text{and} \quad \sup_{(t,x)\in S_{0,T}^+} |\tilde{g}(u(t,x))| \le M_2^* \tag{3.22}$$

with positive constants M_1^* and M_2^* independent of T. Therefore, it follows from (3.20), (3.21) and (3.22) that

$$
\begin{aligned}
\frac{d}{dt}\left\{\int_0^{s(t)}(\mu_1 u_x)^{2p}dx+\int_{s(t)}^1(\mu_2 u_x)^{2p}dx\right\}& \\
+\frac{2(2p-1)}{p}\left\{d_1\int_0^{s(t)}|(\mu_1^p u_x^p)_x|^2\,dx+d_2\int_{s(t)}^1|(\mu_2^p u_x^p)_x|^2\,dx\right\}& \\
+(2p-1)\{(\mu_1 u_x)^{2p}(t,s(t)-0)-(\mu_2 u_x)^{2p}(t,s(t)+0)\}\dot{s}(t)& \\
\le 2p\left\{M_1^*\int_0^{s(t)}(\mu_1 u_x)^{2p}dx+M_2^*\int_{s(t)}^1(\mu_2 u_x)^{2p}dx\right\}&.
\end{aligned}
\tag{3.23}
$$

Since $u_x(t,s(t)\pm 0)<0$ by the strong maximum principle and p is a positive integer, we have

$$
\begin{aligned}
&\{(\mu_1 u_x)^{2p}(t,s(t)-0)-(\mu_2 u_x)^{2p}(t,s(t)+0)\}\dot{s}(t)\\
&=\left\{(-\mu_1 u_x(t,s(t)-0))^{2p-1}+\cdots+(-\mu_2 u_x(t,s(t)+0))^{2p-1}\right\}\\
&\qquad\qquad\times\left\{-\mu_1 u_x(t,s(t)-0)+\mu_2 u_x(t,s(t)+0)\right\}\dot{s}(t)\\
&=\left\{(-\mu_1 u_x(t,s(t)-0))^{2p-1}+\cdots+(-\mu_2 u_x(t,s(t)+0))^{2p-1}\right\}\dot{s}(t)^2\\
&\ge\left\{(-\mu_1 u_x(t,s(t)-0))^{2p-1}+(-\mu_2 u_x(t,s(t)+0))^{2p-1}\right\}\dot{s}(t)^2.
\end{aligned}
$$

Observe that

$$
|\dot{s}(t)|\le -\mu_1 u_x(t,s(t)-0)-\mu_2 u_x(t,s(t)+0)
$$

and

$$
a^{2p-1}+b^{2p-1}\ge 2^{2-2p}(a+b)^{2p-1}\quad\text{for all}\quad a,b>0;
$$

so that the above relations yield

$$
\{(\mu_1 u_x)^{2p}(t,s(t)-0)-(\mu_2 u_x)^{2p}(t,s(t)+0)\}\dot{s}(t)\ge 2^{2-2p}|\dot{s}(t)|^{2p+1}. \tag{3.24}
$$

Therefore, the following estimate can be obtained from (3.23) and (3.24) ;

$$
\begin{aligned}
\frac{d}{dt}\left\{\int_0^{s(t)}(\mu_1 u_x)^{2p}dx+\int_{s(t)}^1(\mu_2 u_x)^{2p}dx\right\}& \\
+2d_0\left\{\int_0^{s(t)}|(\mu_1^p u_x^p)_x|^2\,dx+\int_{s(t)}^1|(\mu_2^p u_x^p)_x|^2\,dx\right\}+\frac{2p-1}{2^{2p-2}}|\dot{s}(t)|^{2p+1}& \\
\le 2pM^*\left\{\int_0^{s(t)}(\mu_1 u_x)^{2p}dx+\int_{s(t)}^1(\mu_2 u_x)^{2p}dx\right\}&,
\end{aligned}
\tag{3.25}
$$

where $d_0=\min\{d_1,d_2\}$ and $M^*=\max\{M_1^*,M_2^*\}$.

We now recall the Gagliardo-Nirenberg inequality in the following form:

$$
\|v\|_{L^2(\alpha,\beta)}\le C\|v_x\|_{L^2(\alpha,\beta)}^{1/3}\|v\|_{L^1(\alpha,\beta)}^{2/3}\quad\text{for all}\quad v\in H^1(\alpha,\beta), \tag{3.26}
$$

where C is a positive number independent of α and β. Applying Young's inequality to (3.26) we see that for any $\varepsilon>0$

$$
\|v\|_{L^2(\alpha,\beta)}^2\le\frac{\varepsilon}{3}\|v_x\|_{L^2(\alpha,\beta)}^2+\frac{2C^3}{3\varepsilon^{1/2}}\|v\|_{L^1(\alpha,\beta)}^2. \tag{3.27}
$$

We set $v = u_x^p$ and $(\alpha, \beta) = (0, s(t))$ or $(s(t), 1)$ in (3.27); then

$$\|v\|^2_{L^2(\alpha,\beta)} = \|u_x\|^{2p}_{L^{2p}(\alpha,\beta)}, \quad \|v\|^2_{L^1(\alpha,\beta)} = \|u_x\|^{2p}_{L^p(\alpha,\beta)} \quad \text{and} \quad \|v_x\|^2_{L^2(\alpha,\beta)} = \|(u_x^p)_x\|^2_{L^2(\alpha,\beta)}.$$

Therefore, (3.27) assures that, for any $\varepsilon > 0$,

$$\|(u_x^p)_x\|^2_{L^2(\alpha,\beta)} \geq \frac{3}{\varepsilon}\|u_x\|^{2p}_{L^{2p}(\alpha,\beta)} - \frac{2C^3}{\varepsilon^{3/2}}\|u_x\|^{2p}_{L^p(\alpha,\beta)}. \tag{3.28}$$

Substitution of (3.28) into (3.25) gives

$$\begin{aligned} &\frac{d}{dt}\left\{\int_0^{s(t)} (\mu_1 u_x)^{2p} dx + \int_{s(t)}^1 (\mu_2 u_x)^{2p} dx\right\} \\ &\quad + \left(\frac{6d_0}{\varepsilon} - 2pM^*\right)\left(\int_0^{s(t)} (\mu_1 u_x)^{2p} dx + \int_{s(t)}^1 (\mu_2 u_x)^{2p} dx\right) + \frac{2p-1}{2^{2p-2}}|\dot{s}(t)|^{2p+1} \\ &\leq \frac{2C^3 d_0}{3\varepsilon^{3/2}}\left\{\left(\int_0^{s(t)} (\mu_1 u_x)^p \, dx\right)^2 + \left(\int_{s(t)}^1 (\mu_2 u_x)^p \, dx\right)^2\right\}. \end{aligned} \tag{3.29}$$

If we take $p = 2^k$ $(k = 1, 2, 3, \cdots)$ and set

$$X_k(t) = \int_0^{s(t)} (\mu_1 u_x(t,x))^{2^k} dx + \int_{s(t)}^1 (\mu_2 u_x(t,x))^{2^k} dx,$$

then it follows from (3.29) that

$$\frac{d}{dt}X_{k+1}(t) + \left(\frac{6d_0}{\varepsilon} - 2^{k+1}M^*\right)X_{k+1}(t) \leq \frac{2C^3 d_0}{3\varepsilon^{3/2}}X_k(t)^2.$$

Here $\varepsilon > 0$ is arbitrary. Taking $\varepsilon = \dfrac{3d_0}{2^{k+1}M^*}$ in the above inequality we are lead to

$$\frac{d}{dt}X_{k+1}(t) + 2^{k+1}M^* X_{k+1}(t) \leq 2^{3(k+1)/2}L^* X_k(t)^2 \tag{3.30}$$

with $L^* = \dfrac{2C^3(M^*)^{3/2}}{9\sqrt{3d_0}}$. Solving differential inequality (3.30) we obtain

$$X_{k+1}(t) \leq \max\left\{X_{k+1}(0), 2^{k/2}C^*\left(\sup_{0\leq t\leq T} X_k(t)\right)^2\right\} \quad \text{for all} \quad t \in [0,T] \tag{3.31}$$

with $C^* = \dfrac{\sqrt{2}L^*}{M^*} = \dfrac{2\sqrt{2M^*}C^3}{9\sqrt{3d_0}}$. By (A.6)

$$X_k(0) = \int_0^\ell (\mu_1\phi_x)^{2^k} dx + \int_\ell^1 (\mu_2\phi_x)^{2^k} dx \leq K^{2^k}, \tag{3.32}$$

where $K = \mu\|\phi_x\|_{L^\infty(I)}$ with $\mu = \max\{\mu_1, \mu_2\}$. Moreover, Proposition 3.1 gives

$$X_1(t) \leq K_1 \quad \text{for all} \quad 0 \leq t \leq T, \tag{3.33}$$

where K_1 is a positive constant depending only on $\|\phi\|_{H^1(I)}$. From (3.31), (3.32) and (3.33) we have

$$X_2(t) \leq \max\{X_2(0), \sqrt{2}C^*(\sup_{0\leq t\leq T} X_1(t))^2\} \leq \max\{K^4, \sqrt{2}C^*K_1^2\}$$

for $t \in [0,T]$. Here we may assume

$$K^4 \leq \sqrt{2}C^*K_1^2, \quad 2C^* \geq 1$$

without loss of generality. Indeed, it suffices to make $M^* = \max\{M_1^*, M_2^*\}$ large so that the above conditions are satisfied (see (3.22)). Then solving (3.31) by iteration one can deduce

$$\sup_{0\leq t\leq T} X_{k+1}(t) \leq 2^{a_k}(C^*)^{b_k}K_1^{c_k} \tag{3.34}$$

with

$$a_k = \frac{1}{2}\sum_{i=0}^{k-1}(k-i)2^i = 2^k - \frac{k}{2} - 1,$$

$$b_k = \sum_{i=0}^{k-1} 2^i = 2^k - 1 \quad \text{and} \quad c_k = 2^k.$$

Hence

$$\big(\sup_{0\leq t\leq T} X_{k+1}(t)\big)^{1/2^{k+1}} \leq 2^{a_k/2^{k+1}}(C^*)^{b_k/2^{k+1}}(K_1)^{c_k/2^{k+1}}.$$

Letting $k \to \infty$ in the above relation we are led to

$$\mu_1 \sup_{0\leq t\leq T} \|u_x(t)\|_{L^\infty(0,s(t))} + \mu_2 \sup_{0\leq t\leq T} \|u_x(t)\|_{L^\infty(s(t),1)} \leq \sqrt{2C^*K_1}. \tag{3.35}$$

In view of $u = u^+ - u^=$ with $u^+ = u|_{[0,s(t)]}$ and $u^- = -u|_{[s(t),1]}$, we have shown

$$(u^+)_x \in L^\infty(S_{0,T}^-) \quad \text{and} \quad (u^-)_x \in L^\infty(S_{0,T}^+)$$

when ϕ satisfies (A.6). It is easy to prove $\dot{s} \in L^\infty(0,T)$ from the Stefan condition (1.7).

Finally we will show the assertion for general ϕ. The regularity properties of smooth solutions show that the solution (u,s) of (P) satisfies

$$u_x(t,\cdot) \in L^\infty(0,s(t)) \cap L^\infty(s(t),1) \quad \text{for any} \quad t \in (0,T].$$

Then take any $\delta \in (0,T]$. Repeating the preceding arguments with ϕ and $[0,T]$ replaced by $u(\delta,\cdot)$ and $[\delta,T]$ one can prove (3.35). This fact completes the proof. □

Proof of Theorem 2.3 Applying Propositions 3.1 and 3.2 we will give the proof of Theorem 2.3.

Since $0 < \ell < 1$, it is possible to apply the existence result of local smooth solutions of (P) even if $m_1 = m_2 = 0$ (see [7, Theorems 3.1 and 4.3]). So there exists a positive number T such that (P) possesses a smooth solution (u,s) in $[0,T]$. Here a smooth solution (u,s) of (P) in $[0,T]$ means that (u,s) has properties (i)-(iv) and (vi)-(ix) of Theorem 2.3 with T^* replaced by T. We set

$$T^* = \sup\{T > 0 : \text{ (P) has a smooth solution } (u,s) \text{ in } [0,T]\}.$$

Hereafter we assume $T^* < \infty$ because there is nothing left to prove if $T^* = \infty$. Note that $0 < s(t) < 1$ for $0 \le t < T^*$; so that Proposition 3.1 is still valid for all $t \in (0, T^*)$. Hence we see $\dot{s} \in L^3(0, T^*)$, which assures that $\lim_{t\to T^*} s(t)$ exists. So define

$$s(T^*) = \lim_{t\to T^*} s(t).$$

We will show

$$s(T^*) = 0 \quad \text{or} \quad S(T^*) = 1 \tag{3.36}$$

by contradiction. Assume $0 < s(T^*) < 1$. Repeating the arguments in [7, §4] we can rewrite the initial boundary value problem in S^-_{0,T^*} (resp. S^+_{0,T^*}) as the initial value problem for an appropriate semilinear evolution equation in $[0, T^*]$. With use of the regularity theory of evolution equations and embedding theorems we can prove that the limits of $u^-(t)$ and $u^+(t)$ as $t \to T^*$ exist with respect to $H^1_0(I)$-norm. So define $u(T^*) = u^+(T^*) - u^-(T^*)$ with

$$u^\pm(T^*) = \lim_{t\to T^*} u^\pm(t) \quad \text{in} \quad H^1_0(I).$$

Since $0 < s(T^*) < 1$, we can study (P) for $t \ge T^*$ with (ϕ, ℓ) replaced by $(u(T^*), s(T^*))$ and, therefore, prove the existence of a smooth solution in $[T^*, T^* + \tau^*]$ with some $\tau^* > 0$. This fact allows us to show that (P) has a smooth solution in $[0, T^* + \tau^*]$, which contradicts to the definition of T^*. Thus we have shown (3.36).

For the sake of simplicity, assume $s(T^*) = 1$. Then one can rewrite the initial boundary value problem in S^-_{0,T^*} as the initial boundary value problem in an appropriate cylindrical domain which has been discussed in [7]. So the parabolic regularity theory implies that u is smooth with respect to $(t, x) \in S^-_{0,T^*}$ up to $t = T^*$. In particular, we see

$$u^+ \in C([0, T^\cdot]; H^1_0(I)).$$

In order to show $u^- \in C([0, T^*]; H^1_0(I))$, we will use Proposition 3.2. It should be noted that the proof of this proposition allows us to derive

$$\sup_{\delta \le t < T^*} \|u^-_x(t)\|_{L^\infty(I)} = C_\delta$$

with a positive number C_δ depending only on δ. Therefore,

$$\int_{s(t)}^1 (u^-)_x(t,x)^2 dx \le \|u^-_x(t)\|^2_{L^\infty}(1 - s(t)) \le C^2_\delta(1 - s(t)) \to 0 \quad \text{as} \quad t \to T^*.$$

This fact assures $u^- \in C([0, T^*]; H^1_0(I))$; so that $u \in C([0, T^*]; H^1_0(I))$.

Finally it is easy to get other regularity properties of (u, s) from Propositions 3.1 and 3.2.

Proof of Theorem 2.5

We will give an alternative proof of Theorem 2.5 (in particular, (iii) of Theorem 2.5), which is different from that of Theorem 2.2. For $(\phi.\ell)$ satisfying (A.4) and (A.5), let (u, s) be the smooth solution obtained in Theorem 2.4.

Step 1. We will collect some basic estimates for (u, s). Recall that Proposition 3.1 gives us

$$\sup_{t\ge 0} \|(u^+)_x(t)\|_{L^2(I)} \le C_1, \qquad \sup_{t\ge 0} \|(u^-_x(t)\|_{L^2(I)} \le C_1, \tag{3.37}$$

$$\int_0^\infty |\dot{s}(t)|^3 dt \le C_2, \tag{3.38}$$

$$\int_0^\infty \|(u^+)_t(t)\|^2_{L^2(I)} dt \le C_3, \qquad \int_0^\infty \|(u^-)_t(t)\|^2_{L^2(I)} dt \le C_3, \tag{3.39}$$

where C_i $(i = 1, 2, 3)$ are positive constants depending only on (ϕ, ℓ). It also follows from Proposition 3.2 that, for any $\delta > 0$,

$$\sup_{t\geq\delta} \|(u^+)_x(t)\|_{L^\infty(I)} \leq C_4(\delta), \qquad \sup_{t\geq\delta} \|(u^-)_x(t)\|_{L^\infty(I)} \leq C_4(\delta) \tag{3.40}$$

with a positive constant $C_4(\delta)$.

Step 2. We will give another important estimates of solutions for parabolic boundary value problems with moving boundaries.

We first assume

$$\delta^* \leq s(t) \leq 1 - \delta^* \quad \text{for} \quad t \geq 0 \tag{3.41}$$

with some $\delta^* > 0$. Our strategy is to rewrite the initial boundary value problem in S^- by putting

$$v(t, y) = u(t, x) \quad \text{with} \quad x = s(t)y \in [0, s(t)].$$

Then v satisfies the following problem with fixed boundaries $y = 0, 1$:

$$\begin{cases} v_t = a(t)v_{yy} + k(t, y)v_y + vf(v), & t > 0, \quad 0 < y < 1, \\ v(t, 0) = v(t, 1) = 0, & t > 0, \end{cases} \tag{3.42}$$

where

$$a(t) = \frac{d_1}{s(t)^2} \quad \text{and} \quad k(t, y) = \frac{\dot{s}(t)y}{s(t)}.$$

Define a closed linear operator A_p with $1 \leq p < \infty$ by

$$A_p v = -v_{yy} \qquad \text{with domain} \ \ D(A_p) = W_0^{1,p}(I) \cap W^{2,p}(I).$$

We also define $A_p(t) = a(t)A_p$ and handle (3.42) in the framework of an evolution equation.

In what follows, we take any $\tau \geq 1 + \delta$ wth $\delta > 0$ and fix any $T > 0$. We study (3.42) in the following form

$$v_t + A_p(t)v = h(t, y), \quad \tau - 1 \leq t \leq \tau + T, \tag{3.43}$$

where

$$h(t, y) = k(t, y)v_y(t, y) + v(t, y)f(v(t, y)).$$

Since

$$0 \leq v(t, y) \leq M \quad \text{for} \quad (t, y) \in [0, \infty) \times [0, 1]$$

by Theorem 2.4, it follows from (3.40) that

$$\|h(t)\|_{L^p(I)} \leq M_1(1 + |\dot{s}(t)|), \tag{3.44}$$

where M_1 is a positive number independent of τ and T. Here we observe that $\{A_p(t)\}_{\tau-1\leq t\leq\tau+T}$ generates an evolution operator $\{U_p(t, \sigma)\}_{\tau-1\leq\sigma\leq t\leq\tau+T}$ and that $\{U_p(t, \sigma)\}$ satisfies basic estimates independent of τ and T (see [7, §4]). By (3.43), v is expressed as

$$v(t) = U_p(t, \tau - 1)v(\tau - 1) + \int_{\tau-1}^{t} U_p(t, \sigma)h(\sigma)d\sigma. \qquad \tau - 1 \leq t \leq \tau + T.$$

On account of (3.44) one can apply the arguments in [7, §4] to the above expression and derive

$$\|v_y\|_{C^{\rho,2\rho}([\tau,\tau+T]\times\bar{I})} \leq M_2 \tag{3.45}$$

with some $\rho \in (0,1)$, where M_2 is a positive constant independent of τ and T. In particular, we note that

$$u_x(t, s(t)-0) = \frac{v_y(t,1)}{s(t)} \in C^\rho([\tau, \tau+T]). \tag{3.46}$$

It is also possible to rewrite the boundary value problem in S^+ by setting

$$u(t,x) = w(t,z) \quad \text{with} \quad z = \frac{x - s(t)}{1 - s(t)}.$$

Then w satisfies

$$\begin{cases} w_t = \tilde{a}(t) w_{zz} + \tilde{k}(t,z) w_z + w g(w), & t > 0, 0 < z < 1, \\ w(t,0) = w(t,1) = 0, & t > 0, \end{cases}$$

where

$$\tilde{a}(t) = \frac{d_2}{(1-s(t))^2} \quad \text{and} \quad \tilde{k}(t,z) = \frac{\dot{s}(t)(1-z)}{1-s(t)}.$$

Therefore, we repeat the preceding arguments and derive the following estimate in the same way as (3.45):

$$\|w_z\|_{C^{\rho,2\rho}([\tau,\tau+T]\times \bar{I})} \leq M_3 \tag{3.47}$$

with some $\rho \in (0,1)$ and $M_3 > 0$ independently of τ and T. Then (3.47) implies

$$u_x(t, s(t)+0) = \frac{w_z(t,0)}{1-s(t)} \in C^\rho([\tau, \tau+T]). \tag{3.48}$$

By virtue of (3.46) and (3.48), the Stefan free boundary condition (1.7) yields

$$\dot{s} \in C^\rho([\tau, \tau+T]).$$

Step 3. We are now ready to study the structure of $\omega(\phi, \ell)$. Take any $(u^*, \ell^*) \in \omega(\phi, \ell)$. Then there exists $\{t_n\} \uparrow$ such that

$$s(t_n) \to \ell^*, \qquad u^\pm(t_n, \cdot) \to (u^*)^\pm \quad \text{in} \quad H_0^1(I) \quad \text{as} \quad n \to \infty. \tag{3.49}$$

Note

$$\begin{aligned} u(t,x) &= u^+(t,x) && \text{for} \;\; t > 0,\; 0 < x < s(t), \\ u(t,x) &= -u^-(t,x) && \text{for} \;\; t > 0,\; s(t) < x < 1. \end{aligned}$$

For $t \geq 0$, we set

$$s_n(t) = s(t + t_n).$$

First we assume

$$0 < \ell^* < 1.$$

Then there exists $\delta^* \in (0, 1/4)$ such that

$$2\delta^* < s_n(0) = s(t_n) < 1 - 2\delta^* \qquad \text{for} \quad n \geq 1.$$

Since $t \mapsto s_n(t)$ is uniformly continuous by (3.38), there exists a positive number $T > 0$ such that

$$\delta^* < s_n(t) < 1 - \delta^* \qquad \text{for all} \;\; n \geq 1 \quad \text{and} \;\; 0 \leq t \leq T. \tag{3.50}$$

Moreover, (3.38) also implies that $\{s_n(t)\}_{n=1}^{\infty}$ is equi-continuous for $t \in [0, T]$. Then Ascoli-Arzela's theorem allows us to see that $\{s_n(t)\}_{n=1}^{\infty}$ is relatively compact in $C([0, T])$. Choosing a suitable subsequence $\{s_{n'}\}$ we can prove that

$$\lim_{n' \to \infty} s_{n'}(t) = S^*(t) \qquad \text{uniformly for } \ 0 \le t \le T \tag{3.51}$$

with some $S^* \in C([0, T])$. It also follows from (3.38) that

$$\int_0^T |\dot{s}_n(t)|^3 dt = \int_{t_n}^{t_n+T} |\dot{s}(t)|^3 dt \le \int_{t_n}^{\infty} |\dot{s}(t)|^3 dt \to 0 \quad \text{as} \ \ n \to \infty. \tag{3.52}$$

Here note the following identity:

$$s_n(t) = s_n(0) + \int_0^t \dot{s}_n(\sigma) d\sigma \qquad \text{for} \ \ 0 \le t \le T.$$

Setting $n = n'$ and letting $n' \to \infty$ in this identity one can prove from (3.49), (3.51) and (3.52) that

$$S^*(t) = \ell^* \quad \text{for all} \ \ 0 \le t \le T.$$

Hence we have the following convergence in place of (3.51):

$$\lim_{n \to \infty} s_n(t) = S^*(t) \equiv \ell^* \quad \text{uniformly for} \ \ t \in [0, T]. \tag{3.53}$$

Set

$$u_n(t, x) = u(t + t_n, x) \qquad \text{and} \quad v_n(t, y) = v(t + t_n, y).$$

It follows from (3.39) that

$$\int_0^T \|(u_n^+)_t(t)\|_{L^2(I)}^2 dt = \int_{t_n}^{t_n+T} \|(u^+)_t(t)\|_{L^2(I)}^2 dt \le \int_{t_n}^{\infty} \|(u^+)_t(t)\|_{L^2(I)}^2 dt \to 0 \tag{3.54}$$

as $n \to \infty$. Since

$$(u_n^+)_t = (v_n)_t - \frac{\dot{s}_n(t) y}{s_n(t)} (v_n)_y = (v_n)_t - \frac{\dot{s}_n(t) x}{s_n(t)^3} (u_n^+)_x \quad \text{for} \ \ 0 < x < s_n(t),$$

(3.54) together with (3.40) and (3.52) implies

$$\int_0^T \|(v_n)_t(t)\|_{L2(I)}^2 dt \to 0 \quad \text{as} \ \ n \to \infty. \tag{3.55}$$

Moreover, it follows from (3.45) that

$$\|(v_n)_y\|_{C^{\rho, 2\rho}([0,T] \times \bar{I})} \le M_2 \qquad \text{for all} \ \ n \ge 1.$$

This fact assures that both $\{v_n\}$ snf $\{(v_n)_y\}$ are relatively compact in $C([0, T] \times \bar{I})$. Therefore, choosing a subsequence if necessary, we may conclude that

$$\begin{cases} v_n \to V & \text{in} \ \ C([0, T] \times \bar{I}), \\ (v_n)_y \to V_y & \text{in} \ \ C([0, T] \times \bar{I}) \end{cases} \tag{3.56}$$

with some $V \in\in C^{0,1}([0,T]\times \bar{I})$. Here it should be noted that v_n satisfies

$$\begin{cases} (v_n)_t = a_n(t)(v_n)_{yy} + k_n(t,y)(v_n)_y + v_n f(v_n), & 0<t<T,\ 0<y<1, \\ v_n(t,0) = v_n(t,1) = 0, & 0<t<T, \\ v_n(0,y) = u^+(t_n, s(t_n)y), & 0<y<1, \end{cases} \tag{3.57}$$

where

$$a_n(t) = \frac{d_1}{s_n(t)^2} \quad \text{and} \quad k_n(t,y) = \frac{\dot{s}_n(t)y}{s_n(t)}.$$

In view of (3.53), (3.55), (3.56) and (3.57) we see that $\{(v_n)_{yy}\}$ is convergent in $L^2((0,T)\times I)$; so

$$(v_n)_{yy} \to V_{yy} \quad \text{in} \ \ L^2((0,T)\times I) \quad \text{as} \ \ n\to\infty. \tag{3.58}$$

Therefore, on account of these convergence properties, we see that V satisfies

$$V_t = 0 = a^* V_{yy} + V f(V) \qquad 0<t<T,\ 0<y<1, \tag{3.59}$$

with $a^* = d_1/(\ell^*)^2$. Since

$$\begin{aligned} |u^+(t_n, s(t_n)y) - u^*(\ell^* y)| &\le |u^+(t_n, s(t_n)y) - u^+(t_n, \ell^* y)| + |u^+(t_n,\ell^* y) - u^*(\ell^* y)| \\ &\le \|(u^+)_x\|_{L^\infty(I)} |s(t_n) - \ell^*| + |u^+(t_n,\ell^* y) - u^*(\ell^* y)| \to 0 \end{aligned}$$

for $0\le y\le 1$ as $n\to\infty$, it is seen from (3.56) and (3.59) that $V(t,y)$ is independent of t and

$$V(t,y) \equiv V(0,y) = u^*(\ell^* y) \quad \text{for} \ \ 0\le t\le T \ \ \text{and} \ \ 0\le y\le 1. \tag{3.60}$$

Thus (3.59) and (3.60) show

$$d_1(u^*)_{xx} + u^* f(u^*) = 0 \quad \text{for} \ \ 0\le x\le \ell^* \tag{3.61}$$

with $u^*(0) = u^*(\ell^*) = 0$. Moreover, note

$$\lim_{n\to\infty} (u_n)_x(t, s_n(t)-0) = \lim_{n\to\infty} \frac{(v_n)_y(t,1)}{s(t_n)} = \frac{V_y(t,1)}{\ell^*} = (u^*)_x(\ell^*-0) \tag{3.62}$$

for $0\le t\le T$. We have used (3.60) in the last equality.

One can use essentially the same considerations for $(u^-)(t,x)$. So it is possible to prove that

$$\begin{cases} d_2(u^*)_{xx} + u^* g(u^*) = 0, & \ell^* < x < 1, \\ u^*(\ell^*) = u(1) = 0 \end{cases} \tag{3.63}$$

and

$$\lim_{n\to\infty} (u_n)(t, s_n(t)+0) = (u^*)_x(\ell^+ + 0) \quad \text{for} \ \ t\in[0,T]. \tag{3.64}$$

Therefore, it follows from(3.52), (3.62) and (3.64) that for a.e. $t\in[0,T]$

$$\begin{aligned} 0 = \lim_{n\to\infty} \dot{s}_n(t) &= -\mu_1 \lim_{n\to\infty} (u_n)_x(t, s_n(t)-0) + \mu_2 \lim_{n\to\infty} (u_n)_x(t, s_n(t)+0) \\ &= -\mu_1 (u^*)_x(\ell^*-0) + \mu_2 (u^*)_x(\ell^*+0). \end{aligned}$$

Thus we have shown that (u^*, ℓ^*) satisfies (SP-0) in case $0<\ell^*<1$.

If $\ell^*=1$, the one can repeat the preceding arguments and derive the conclusion from (3.61). Since the analysis in case $\ell^*=0$ is essentially the same, we omit it.

4 Comparison principle

We will explain the comparison principle to study (P). For each $(u, s) \in C(\bar{Q}) \times C(\bar{I})$ with $Q = (0, \infty) \times I$ and $I = (0, 1)$, we say that (u, s) has *property* (R) when it satisfies the following conditions:

(i) $u_x \in C(\overline{S^-_{\delta,\infty}}) \cap C(\overline{S^+_{\delta,\infty}})$ for any $\delta > 0$,
(ii) $u_t, u_{xx} \in C(S^-) \cap C(S^+)$,
(iii) $s \in C^1((0, \infty))$.

Definition 4.1. *Let $(u, s) \in C(\bar{Q}) \times C(\bar{I})$ possess the property (R). Then (u, s) is called a* **supersolution** *of (P) with initial data (ϕ, ℓ) if it satisfies the following:*

$$\begin{cases} u_t \geq d_1 u_{xx} + uf(u) & \text{for } (t, x) \in S^-, \\ u_t \geq d_2 u_{xx} + ug(u), \qquad u \leq 0 & \text{for } (t, x) \in S^+, \\ u(t, 0) \geq m_1, \quad u(t, 1) \geq -m_2 & \text{for } t > 0, \\ u(t, s(t)) = 0 & \text{for } t > 0, \\ \dot{s}(t) \geq -\mu_1 u_x(t, s(t) - 0) + \mu_2 u_x(t, s(t) + 0) & \text{for } t \in \{\tau > 0;\ 0 < s(\tau) < 1\}, \\ u(0, x) = \phi(x) & \text{for } 0 \leq x \leq 1, \\ s(0) = \ell. \end{cases}$$

On the other hand, if (u, s) satisfies the above relations with "$\geq$" replaced by "$\leq$" and "$u \leq 0$ in S^+" replaced by "$u \geq 0$ in S^-", then it is called a **subsolution** *of (P) with initial data (ϕ, ℓ).*

Remark 4.1. *Let (u, s) be a suersolution of* (P). *Applying the strong maximum priciple to u in S^- we see that $u > 0$ in S^-. Similarly, it (u,s) is a subsolution of* (P), *then $u < 0$ in S^+.*

Theorem 4.1. *Let $(\phi_i, \ell_i)(i = 1, 2)$ with $\phi_1 \not\equiv \phi_2$ satisfy* (A.4) *and* (A.5). *Assume that (u_1, s_1) (resp. (u_2, s_2)) is a supersolution (resp. subsolution) of* (P) *with initial data (ϕ_1, ℓ_1) (resp. (ϕ_2, ℓ_2)). If $\phi_1 \geq \phi_2$ in I and $\ell_1 > \ell_2$, then*

$$u_1(t, x) > u_2(t, x) \qquad \textit{for} \quad (t, x) \in Q$$

and

$$s_1(t) > s_2(t) \quad \textit{for} \quad \textit{as long as} \quad s_1(t) > 0 \quad \textit{or} \quad s_2(t) < 1.$$

Proof. First we will show by way of contradiction

$$s_1(t) > s_2(t) \qquad \text{for} \quad t \geq 0 \tag{4.1}$$

as long as either $s_1(t)$ or $s_2(t)$ is distant from the fixed boundaries. Assume that there exists $T^* > 0$ such that

$$s_1(T^*) = s_2(T^*) \in (0, 1) \quad \text{and} \quad s_1(t) > s_2(t) \quad \text{for} \quad 0 \leq t < T^*.$$

Then

$$\dot{s}_1(T^*) \leq \dot{s}_2(T^*). \tag{4.2}$$

Since

$$u_1(t, 0) \geq m_1 \geq u_2(t, 0) \quad \text{and} \quad u_1(t, s_2(t)) > 0 = u_1(t, s_2(t)) \quad \text{for} \quad 0 < t < T^*$$

by virtue of Remark 4.1, the comparison principle for parabolic equations assures

$$u_1(t,x) > u_2(t,x) \quad \text{for} \;\; 0 < t \le T^*, \quad 0 < x < s_2(t)$$

(see, for instance, the monograph of Smoller [13]). In view of $u_1(T^*, s_1(T^*)) = u_2(T^*, s_2(T^*)) = 0$, Hopf's boundary lemma implies

$$u_{1,x}(T^*, s_1(T^*) - 0) < u_{2,x}(T^*, s_2(T^*) - 0) \tag{4.3}$$

Similarly, one can also prove

$$u_1(t,x) > u_2(t,x) \quad \text{for} \;\; 0 < t \le T^*, \quad s_1(t) < x < 1.$$

Applying Hopf's boundary lemma again we see

$$u_{1,x}(T^*, s_1(T^*) + 0) > u_{2,x}(T^*, s_2(T^*) + 0). \tag{4.4}$$

Then it follows from Definition 4.1 together with (4.3) and (4.4) that

$$\begin{aligned} \dot{s}_1(T^*) &\ge -\mu_1 u_{1,x}(T^*, s_1(T^*) - 0) + \mu_2 u_{1,x}(T^*, s_1(T^*) + 0) \\ &> -\mu_1 u_{2,x}(T^*, s_2(T^*) - 0) + \mu_2 u_{2,x}(T^*, s_2(T^*) + 0) \ge \dot{s}_2(T^*), \end{aligned}$$

which contradicts to (4.2). Thus we have shown (4.1).

We now note that

$$u_1(t, s_2(t)) > 0 = u_2(t, s_2(t)) \quad \text{as long as} \;\; 0 < s_2(t) < 1$$

and

$$u_1(t, s_1(t)) = 0 \ge u_2(t, s_1(t)) \quad \text{as long as} \;\; 0 < s_1(t) < 1.$$

It follows from the comparison principle for parabolic equations that

$$u_1(t,x) > u_2(t,x) \qquad \text{for} \;\; (t,x) \in \overline{S_2^-} \cup \overline{S_1^+},$$

where $S_2^- = \{(t,x) \in Q : \; t > 0, \; 0 < x < s_2(t)\}$ and $S_1^+ = \{(t,x) \in Q : \; t > 0, \; s_1(t) < x < 1\}$. For $(t,x) \in S_2^+ \cap S_1^-$ with $S_2^+ = Q \setminus \overline{S_2^-}$ and $S_1^- = Q \setminus \overline{S_1^+}$, it holds that

$$u_1(t,x) > 0 \ge u_2(t,x).$$

Thus we complete the proof of $u_1 > u_2$ in Q. □

In order to give another comparison result which is stronger than Theorem 4.1 we will prepare the following lemma which implies the continuous dependence of solutions upon initial data under a certain condition.

Lemma 4.1. *Assume that* $(\phi_i, \ell_i), i = 1, 2$, *satisfy* (A.4), (A.5) *and*

$$\phi_1 \ge \phi_2 \;\; (\phi_1 \not\equiv \phi_2) \quad \text{in} \;\; I \quad \text{and} \quad \ell_1 > \ell_2.$$

For each $i = 1, 2$, *let* (u_i, s_i) *be a smooth solution of* (P) *with initial data* (ϕ_i, ℓ_i). *Then there exist positive constants* C_1 *and* C_2 *such that*

$$0 \le (s_1(t) - s_2(t)) + \int_0^1 (u_1(t,x) - u_2(t,x))dx \le C_1 \left\{ (\ell_1 - \ell_2) + \int_0^1 (\phi_1(x) - \phi_2(x))dx \right\} e^{C_2 t}$$

for all $t \ge 0$.

Proof. Let (u, s) be any smooth solution of (P). The following identity holds true:

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{\mu_1}{d_1}\int_0^{s(t)} u\ dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 u\ dx + s(t))\right\}\\ &= -\mu_1 u_x(t,0) + \mu_2 u_x(t,1) + \frac{\mu_1}{d_1}\int_0^{s(t)} uf(u)\ dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 ug(u)\ dx.\end{aligned} \tag{4.5}$$

Put $(u, s) = (u_i, s_i), i = 1, 2$, in (4.5) and subtract the resulting expressions; then

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{\mu_1}{d_1}\left(\int_0^{s_1(t)} u_1 dx - \int_0^{s_2(t)} u_2 dx\right) + \frac{\mu_2}{d_2}\left(\int_{s_1(t)}^1 u_1 dx - \int_{s_2(t)}^1 u_2 dx\right) + (s_1 - s_2)(t)\right\}\\ &= -\mu_1(u_{1,x}(t,0) - u_{2,x}(t,0)) + \mu_2(u_{1,x}(t,1) - u_{2,x}(t,1))\\ &\quad + \frac{\mu_1}{d_1}\left(\int_0^{s_1(t)} u_1 f(u_1)dx - \int_0^{s_2(t)} u_2 f(u_2)dx\right)\\ &\quad\quad + \frac{\mu_2}{d_2}\left(\int_{s_1(t)}^1 u_1 g(u_1)dx - \int_{s_2(t)}^1 u_2 g(u_2)dx\right).\end{aligned} \tag{4.6}$$

By Theorem 4.1

$$u_1(t, x) > u_2(t, x), \quad (t, x) \in Q \quad \text{and} \quad s_1(t) \geq s_2(t), \quad t > 0: \tag{4.7}$$

so that

$$u_{1,x}(t, 0) - u_{2,x}(t, 0) > 0 \qquad \text{and} \qquad u_{1,x}(t, 1) - u_{2,x}(t, 1) < 0. \tag{4.8}$$

We now define the following functional:

$$\begin{aligned}U(t) :=&\frac{\mu_1}{d_1}\left(\int_0^{s_1(t)} u_1 dx - \int_0^{s_2(t)} u_2 dx\right) + \frac{\mu_2}{d_2}\left(\int_{s_1(t)}^1 u_1 dx - \int_{s_2(t)}^1 u_2 dx\right) + s_1(t) - s_2(t)\\ \geq&\frac{\mu_1}{d_1}\int_0^{s_2(t)} (u_1 - u_2)(t,x)dx + \frac{\mu_2}{d_2}\int_{s_1(t)}^1 (u_1 - u_2)(t,x)dx + (s_1(t) - s_2(t)) > 0.\end{aligned}$$

Here we have used (4.7), $u_1 \geq 0$ in S_1^- and $u_2 \leq 0$ in S_2^+. By (A.1) and (4.7)

$$\left|\int_0^{s_1(t)} u_1 f(u_1)dx - \int_0^{s_2(t)} u_2 f(u_2)dx\right| \leq L_1\int_0^{s_2(t)} (u_1 - u_2)dx + L_2(s_1(t) - s_2(t)) \tag{4.9}$$

with some positive numbers L_1 and L_2. Similarly, by (A.2) and (4.7)

$$\left|\int_{s_1(t)}^1 u_1 g(u_1)dx - \int_{s_2(t)}^1 u_2 g(u_2)dx\right| \leq L_3\int_{s_1(t)}^1 (u_1 - u_2)dx + L_4(s_1(t) - s_2(t)) \tag{4.10}$$

with some positive numbers L_3 and L_4. Owing to (4.8), (4.9) and (4.10), the integration of (4.6) with respect to t gives us

$$U(t) \leq U(0) + L_5\int_0^t U(\tau)d\tau, \quad t \geq 0$$

with a positive constant L_5. Hence Gronwall's inequality implies

$$U(t) \leq U(0)e^{L_5 t}, \quad t \geq 0. \tag{4.11}$$

Since

$$U(t) \geq \min\left\{\frac{\mu_1}{d_1}, \frac{\mu_2}{d_2}\right\} \int_0^1 (u_1 - u_2)(x,t)dx + (s_1 - s_2)(t)$$

and

$$U(0) \leq \max\left\{\frac{\mu_1}{d_1}, \frac{\mu_2}{d_2}\right\} \int_0^1 (\phi_1 - \phi_2)(x)dx + (\ell_1 - \ell_2),$$

the assertion follows from (4.11). □

Theorem 4.2. *In additions to the assumptions of Theorem 4.1, assume that either* (u_1, s_1) *or* (u_2, s_2) *is a smooth solution of* (P). *If* $\phi_1 \geq \phi_2$ *in* I *and* $\ell_1 \geq \ell_2$*, then*

$$u_1(t,x) \geq u_2(t,x), \quad (t,x) \in Q \quad \textit{and} \quad s_1(t) \geq s_2(t), \quad t \geq 0.$$

Moreover, if $\phi_1 \not\equiv \phi_2$*, then*

$$u_1(t,x) > u_2(t,x), \quad \textit{for} \quad (t,x) \in Q$$

and

$$s_1(t) > s_2(t) \quad \textit{for} \quad t > 0 \quad \textit{as long as} \;\; s_1(t) > 0 \;\; \textit{or} \;\; s_2(t) < 1.$$

Proof. We will prove this theorem when (u_2, s_2) is a smooth solution. Choose an increasing sequence $\{\psi_n, \hat{\ell}_n\}$ such that

$$\psi_n \leq \psi_{n+1} \leq \phi_2 \leq \phi_1 \qquad \text{and} \quad \hat{\ell}_n < \hat{\ell}_{n+1} < \ell_2 \leq \ell_1 \quad \text{for} \;\; n \geq 1$$

and

$$\lim_{n\to\infty} (\psi_n)^{\pm} = (\phi_2)^{\pm} \quad \text{in} \quad H^1(I) \quad \text{and} \quad \lim_{n\to\infty} \hat{\ell}_n = \ell_2.$$

Let (v_n, h_n) be a smooth solution of (P) with initial data $(\psi_n, \hat{\ell}_n)$. Since $\psi_n \leq \phi_1$ in I and $\hat{\ell}_n < \ell_1$, Theorem 4.1 implies

$$u_1(x,t) > v_n(x,t), \quad (x,t) \in Q \quad \text{and} \quad s_1(t) \geq h_n(t), \quad t > 0, \tag{4.12}$$

for all $n \geq 1$. On the other hand, Lemma 4.1 implies

$$\lim_{n\to\infty} h_n(t) = s_2(t) \quad \text{locally uniformly in} \;\; t \geq 0.$$

Hence letting $n \to \infty$ in (4.12) leads us to

$$s_1(t) \geq s_2(t), \quad t \geq 0.$$

Snce $u_1(0,t) \geq m_1 = u_2(t,0), u_1(t, s_1(t)) = 0 \geq u_2(t, s_1(t))$ and $u_1(0,\cdot) = \phi_1 \geq \phi_2 = u_2(0,\cdot)$, it follows from the comparison theorem for parabolic equations that $u_1(t,x) \geq u_2(t,x)$ for $(t,x) \in S_1^-$. Similarly, it is also possible to show $u_1(t,x) \geq u_2(t,x)$ for $(t,x) \in S_1^+$.

Finally we note that the last assertion is a consequence the strong maximum principle. □

As an application of the comparison principle we will give the following result which provides us important information on the dynamical behaviors of smooth solutions of (P).

Theorem 4.3. *Let $(\phi, \ell) \in C([0,1]) \times (0,1)$ possess the following properties: $\phi \in C^2((0,\ell)) \cap C^2((\ell,1))$ satisfies*

$$\begin{cases} d_1\phi_{xx} + \phi f(\phi) \geq 0, & \phi \geq 0 \qquad \text{in } (0,\ell), \\ d_2\phi_{xx} + \phi g(\phi) \geq 0, & \phi \leq 0 \qquad \text{in } (\ell,1), \\ \phi(0) \leq m_1, \quad \phi(\ell) = 0, & \phi(1) \leq -m_2, \\ 0 \leq -\mu_1 \lim_{x\to\ell-0} \phi_x(x) + \mu_2 \lim_{x\to\ell+0} \phi_x(x). \end{cases}$$

Then the smooth solution $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ of (P) *with initial data (ϕ,ℓ) fulfills the following properties:*

(i) *$u_t(t,\cdot;\phi,\ell) \geq 0$ in Q and $\dot{s}(t) \geq 0$ for $t \geq 0$. In particular, if (ϕ,ℓ) is not a solution of* (SP)*, then $t \mapsto u(t,\cdot;\phi,\ell)$ is strictly increasing in $(0,\infty)$ and $t \mapsto s(t;\phi,\ell)$ is strictly increasing as long as $\ell \leq s(t;\phi,\ell) < 1$.*

(ii) *$\lim_{t\to\infty} u^{\pm}(t;\phi,\ell) = (u^*)^{\pm}$ in $H^1(I)$ and $\lim_{t\to\infty} s(t;\phi,\ell) = s^*$, where (u^*, s^*) is a minimal solution of* (SP) *in the class satisfying $u^* \geq \phi$ in I and $s^* \geq \ell$.*

Proof. Observe that $(u_2(t,x), s_2(t)) := (\phi(x), \ell)$ is a subsolution of (P) with initial data (ϕ,ℓ). Therefore, Theorem 4.2 implies

$$u(\tau,\cdot;\phi,\ell) \geq \phi \quad \text{in } I \quad \text{and} \quad s(\tau;\phi,\ell) \geq \ell \tag{4.13}$$

for all $\tau \geq 0$. Here it should be noted

$$(u(t,\cdot;u(\tau;\phi,\ell), s(\tau;\phi,\ell)), s(t;u(\tau;\phi,\ell), s(\tau;\phi,\ell))) = (u(t+\tau;\phi,\ell), s(t+\tau;\phi,\ell))$$

for any $t > 0$ and $\tau > 0$ by the uniqueness of solutions to (P). Hence it follows from (4.13) and Theorem 4.2 that

$$u(t+\tau,\cdot;\phi,\ell) \geq u(t,\cdot;\phi,\ell) \quad \text{in } I \quad \text{and} \quad s(t+\tau;\phi,\ell) \geq s(t;\phi,\ell) \tag{4.14}$$

for any $t > 0$ and $\tau > 0$. This fact implies $u_t(t,\cdot;\phi,\ell) \geq 0$ in I and $\dot{s}(t) \geq 0$ fot $t > 0$.

In particular, if (ϕ,ℓ) is not a stationary solution, then Theorem 4.2 enables us to see that the former inequality of (4.14) is valid with " $\geq$" replaced by "$>$". This fact shows the strictly increasing property of $u(t,\cdot,\phi,\ell)$ with respect to t. The same reasoning shows the strictly increasing property of $s(t;\phi,\ell)$ as long as $\ell \leq s(t;\phi,\ell) < 1$.

(ii) Since both $u(t;\phi,\ell)$ and $s(t;\phi,\ell)$ are uniformly bounded in $t \geq 0$, the result of (i) implies that the following limit exists;

$$\lim_{t\to\infty} (u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (u^*, s^*) \tag{4.15}$$

with some $(u^*, s^*) \in L^\infty(I) \times \bar{I}$. If we recall Theorems 2.2 and 2.5, we see that (u^*, s^*) is a solution of (SP) and that the convergence in (4.15) holds in the sense of Ω-topology.

It remains to show that (u^*, s^*) is minimal in the class of stationary solutions $(\tilde{u}, \tilde{s})$ satisfying $\tilde{u} \geq \phi$ and $\tilde{s} \geq \ell$. Take any stationary solution $(\tilde{u}, \tilde{s})$ in the above class. The comparison principle (The orem 4.2) assures

$$\tilde{u} \geq u(t,\cdot;\phi,\ell) \quad \text{in } I \quad \text{and} \quad \tilde{s} \geq s(t;\phi,\ell) \tag{4.16}$$

for all $t \geq 0$. Letting $t \to \infty$ in (4.16) and using (4.15) we get

$$\tilde{u} \geq u^* \quad \text{in } I \text{ and} \quad \tilde{s} \geq s^*.$$

Thus we complete the proof. □

Remark 4.2. *Assume that (ϕ, ℓ) satisfies the opposite relations in Theorem 4.3:*

$$\begin{cases} d_1\phi_{xx} + \phi f(\phi) \le 0, & \phi \ge 0 & \text{in } (0,\ell), \\ d_2\phi_{xx} + \phi g(\phi) \le 0, & \phi \le 0 & \text{in } (\ell,1), \\ \phi(0) \ge m_1, \quad \phi(\ell) = 0, \quad \phi(1) \ge -m_2, \\ 0 \ge -\mu_1 \lim_{x\to\ell-0} \phi_x(x) + \mu_2 \lim_{x\to\ell+0} \phi_x(x). \end{cases}$$

Then analogous conclusions as Theorem 4.3 hold true with obvious changes.

Corollary 4.1. *Let (ϕ, ℓ) satisfy the assumptions of Theorem 4.3 and let (ϕ_1, ℓ_1) satisfy, in addition to*(A.4) *and* (A.5)*,*

$$\phi \le \phi_1 \le u^* \quad \text{in } I \quad \text{and} \quad \ell \le \ell_1 \le s^*,$$

where (u^*, s^*) is a stationary solution given in Theorem 4.2. Then

$$\lim_{t\to\infty} (u(t,\cdot;\phi_1,\ell_1), s(t;\phi_1,\ell_1)) = (u^*, s^*) \quad \text{in the sense of } \Omega\text{-topology.}$$

Proof. By Theorem 4.2

$$u(t,\cdot;\phi,\ell) \le u(t,\cdot;\phi_1,\ell_1) \le u^* \quad \text{in } I \quad \text{and} \quad s(t;\phi,\ell) \le s(t;\phi_1,\ell_1) \le s^*$$

for all $t \ge 0$. Hence letting $t \to \infty$ in the above relations one can get the conclusion with use of Theorem 4.3. □

In what follows, we will concentrate ourselves on the study of (P) with $m_1 = m_2 = 0$. In this case, the free boundary may arrive at one of the fixed boundaries in a finite time. We will investigate when this phenomenon happens for a smooth solution of (P). For instance, if we assume $s(T^*) = 1$ with some $T^* > 0$, then $s^* = 1$ and (SP-0) becomes

$$\begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, & u^* \ge 0 \quad \text{in } (0,1), \\ u^*(0) = u^*(1) = 0. \end{cases} \tag{4.17}$$

By the monotone method for elliptic boundary value problems (see Sattinger [12]) , it is possible to show that (4.17) has a minimal positive solution u_1^* provided that $f(0) > d_1\pi^2$. We will give a sufficient condition on (ϕ, ℓ) such that the smooth solution (u, s) of (P) with initial data (ϕ, ℓ) satisfies

$$s(T^*) = 1 \quad \text{at a finite time } T^*$$

and

$$\lim_{t\to\infty} u(t) = u_1^* \quad \text{in } H_0^1(I).$$

Proposition 4.1. *In addition to* (A.1) *and* (A.2)*, assume that f and g satisfy $f(0) > \pi^2 d_1, g(0) > 0$ and that g is nondecreasing near $u = 0$. Let $\ell^* \in (0,1)$ satisfy*

$$\ell^* > \max\left\{\pi\sqrt{\frac{d_1}{f(0)}}, 1 - \pi\sqrt{\frac{d_2}{g(0)}}\right\} \tag{4.18}$$

and define ϕ^* by

$$\phi^*(x) = \begin{cases} \varepsilon_1 \sin \dfrac{\pi x}{\ell^*} & \text{for } 0 \le x \le \ell^*, \\ -\varepsilon_2 \sin \dfrac{\pi(1-x)}{1-\ell^*} & \text{for } \ell^* \le x \le 1, \end{cases} \tag{4.19}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are sufficiently small numbers such that $\varepsilon_1\mu_1/\ell^* \geq \varepsilon_2\mu_2/(1-\ell^*)$. Suppose that (ϕ,ℓ) satisfies $\phi^* \leq \phi \leq u_1^*$ in I and $\ell^* \leq \ell$. Then the smooth solution $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ of (P) *with initial data (ϕ,ℓ) satisfies*

$$\lim_{t\to\infty}(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (u_1^*,1) \qquad \textit{in the sense of } \Omega\textit{-topology}, \tag{4.20}$$

where u_1^* is a minimal positive solution of (4.17)*. Moreover, there exists a positive number T^* such that*

$$s(T^*;\phi,\ell) = 1 \quad \textit{and} \quad \ell^* \leq s(t;\phi,\ell) < 1 \quad \textit{for} \ \ 0 \leq t < T^*. \tag{4.21}$$

Proof. We will show this proposition by dividing its proof into several steps.

Step 1. In this step we will verify that (ϕ^*,ℓ^*) fulfills the assumptions of Theorem 4.3 provided that ε_1 and ε_2 satisfy suitable conditions. Note that

$$d_1\phi^*_{xx} + \phi^* f(\phi^*) = \varepsilon_1 \sin\frac{\pi x}{\ell^*}\left\{-\left(\frac{\pi}{\ell^*}\right)^2 d_1 + f\left(\varepsilon_1\sin\frac{\pi x}{\ell^*}\right)\right\} \geq 0, \quad \phi^* \geq 0 \tag{4.22}$$

for $0 \leq x \leq \ell^*$ if $\min_{0\leq u\leq\varepsilon_1} f(u) \geq (\pi/\ell^*)^2 d_1$. Since $\ell^* > \pi\sqrt{d_1/f(0)}$ by (4.18), we see that (4.22) holds true if $\varepsilon_1 > 0$ is sufficiently small. Moreover, by (4.19)

$$d_2\phi^*_{xx} + \phi^* g(\phi^*) = \varepsilon_2 \sin\frac{\pi(1-x)}{1-\ell^*}\left\{\left(\frac{\pi}{1-\ell^*}\right)^2 d_2 - g\left(-\varepsilon_2\sin\frac{\pi(1-x)}{1-\ell^*}\right)\right\}.$$

By (4.18), $1-\ell^* < \pi\sqrt{d_2/g(0)}$ and $g(u)$ is nondecresing near $u=0$. Therefore, $(\pi/(1-\ell^*))^2 d_2 \geq \max_{-\varepsilon_2\leq u\leq 0} g(u) = g(0)$ if $\varepsilon_2 > 0$ is sufficiently small. Hence for such ε_2

$$d_2\phi^*_{xx} + \phi^* g(\phi^*) \geq 0, \qquad \phi^* \leq 0 \quad \text{for} \ \ \ell^* \leq x \leq 1. \tag{4.23}$$

Finally,

$$-\mu_1\phi^*_x(\ell^*-0) + \mu_2\phi^*_x(\ell+0) = \pi\left(\frac{\mu_1\varepsilon_1}{\ell^*} - \frac{\mu_2\varepsilon_2}{1-\ell^*}\right) \geq 0 \tag{4.24}$$

from the assumption on ε_i $(i=1,2)$. It follows from (4.19) that $\phi^*(0) = \phi^*(\ell^*) = \phi^*(1) = 0$. Thus we have verified that (ϕ^*,ℓ^*) fulfills the assumptions of Theorem 4.3.

Step 2. We will show (4.20). From the result of Step 1, Theorem 4.3 implies

$$\lim_{t\to\infty}(u(t,\cdot;\phi^*,\ell^*), s(t;\phi^*,\ell^*)) = (u^*,s^*) \quad \text{in the sense of } \Omega\text{- topology} \tag{4.25}$$

with a suitable solution (u^*,s^*) of (SP-0) satisfying $u^* \geq \phi^*$ in I and $s^* \geq \ell^*$. We will show $s^* = 1$ by contradiction. Assume $0 < s^* < 1$. Then u^* satisfies

$$\begin{cases} d_2 u^*_{xx} + u^* g(u^*) = 0, \qquad u^* \leq 0 \qquad \text{for} \ \ s^* \leq x \leq 1, \\ u^*(s^*) = u^*(1) = 0. \end{cases}$$

Taking $L^2(s^*,1)$-inner product of the above first equation with u^* we see

$$d_2\int_{s^*}^1 (u^*_x)^2 dx = \int_{s^*}^1 (u^*)^2 g(u^*) dx \leq g(0)\int_{s^*}^1 (u^*)^2 dx. \tag{4.26}$$

because $|u^*(x)| \leq \varepsilon_2$ with small $\varepsilon_2 > 0$ for $\ell^* \leq x \leq 1$ and $g(u)$ is nondecreasing near $u=0$.

Here we consider the following eigenvalue problem

$$\begin{cases} -d_2 w_{xx} = \lambda w \quad \text{in} \ \ (s^*, 1), \\ w(s^*) = w(1) = 0. \end{cases} \tag{4.27}$$

Note that the least eigenvalue of (4.27) is $d_2\pi^2/(1-s^*)^2$. Therefore, the variational characterization of the least eigenvalue provides us

$$d_2 \int_{s^*}^1 (u_x^*)^2 dx \geq \frac{d_2\pi^2}{(1-s^*)^2} \int_{s^*}^1 (u^*)^2 dx.$$

This fact together with (4.26) gives

$$\left(\frac{d_2\pi^2}{(1-s^*)^2} - g(0) \right) \int_{s^*}^1 (u^*)^2 dx \leq 0. \tag{4.28}$$

Recall that (4.18) implies

$$\frac{d_2\pi^2}{(1-s^*)^2} - g(0) \geq \frac{d_2\pi^2}{(1-\ell^*)^2} - g(0) > 0.$$

Hence we get $u^* \equiv 0$ in $[s^*, 1]$ from (4.28).

On the other hand, u^* satisfies

$$\begin{cases} d_1 u_{xx}^* + u^* f(u^*) = 0, \qquad u^* \geq 0 \qquad \text{in} \ \ (0, s^*), \\ u^*(0) = u^*(s^*) = 0. \end{cases} \tag{4.29}$$

Recall here that $u^* \geq \phi^*$ in $(0, \ell^*)$. This fact implies $u^* > 0$ in $(0, s^*)$ because it satisfies (4.29). Therefore, $u_x^*(s^* - 0) < 0$; so that

$$0 = -\mu_1 u_x^*(s^* - 0) + \mu_2 u_x^*(s^* + 0) = -\mu_1 u_x^*(s^* - 0) > 0.$$

Since this is a contradiction, it must be $s^* = 1$. Thus we have proved that u^* is a solution of (4.17).

Finally, by virtue of Theorem 4.3, it should be noted that u^* is identical with the minimal positive solution v_1^* of (4.17).

Step3. Since we have shown (4.20) for (ϕ^*, ℓ^*), we next consider the case when (ϕ, ℓ) satisfies $\phi^* \leq \phi \leq u_1^*$ in I and $\ell^* \leq \ell$. It is seen from the comparison principle (Theorem 4.2 that

$$u(t, \cdot; \phi^*, \ell^*) \leq u(t, \cdot; \phi, \ell) \leq u_1^* \quad \text{in} \ \ I \quad \text{and} \quad s(t; \phi^*, \ell^*) \leq s(t; \phi, \ell)$$

for all $t \geq 0$. Then letting $t \to \infty$ in the above relations we obtain (4.20) for (ϕ, ℓ).

Step 4. We will show (4.21) by contradiction. Assume

$$(\ell^* \leq) s(t; \phi, \ell) < 1 \qquad \text{for all} \ \ t \geq 0.$$

For the sake of simplicity, we write $(u(t,x), s(t))$ in place of $(u(t,x;\phi,\ell), s(t;\phi,\ell))$. We use the following identity:

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\mu_1}{d_1} \int_0^{s(t)} x u dx \right\} &= \frac{\mu_1}{d_1} \int_0^{s(t)} x u_t dx = \mu_1 \int_0^{s(t)} x u_{xx} dx + \frac{\mu_1}{d_1} \int_0^{s(t)} x u f(u) dx \\ &= \mu_1 s(t) u_x(t, s(t) - 0) + \frac{\mu_1}{d_1} \int_0^{s(t)} x u f(u) dx. \end{aligned} \tag{4.30}$$

Similarly,

$$\frac{d}{dt}\left\{\frac{\mu_2}{d_2}\int_{s(t)}^1 xudx\right\} = \mu_2 u_x(t,1) - \mu_2 s(t) u_x(t, s(t)+0) + \frac{\mu_2}{d_2}\int_{s(t)}^1 xug(u)dx. \tag{4.31}$$

Owing to the Stefan condition (1.7), addition of (4.30) and (4.31) leads us to

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\mu_1}{d_1}\int_0^{s(t)} xudx + \frac{\mu_2}{d_2}\int_{s(t)}^1 xudx + \frac{1}{2}s(t)^2\right\} \\ =&\mu_2 u_x(t,1) + \frac{\mu_1}{d_1}\int_0^{s(t)} xuf(u)dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 xug(u)dx. \end{aligned} \tag{4.32}$$

In view of $u_x(t,1) > 0$, integrating (4.32) over $(0,t)$ we get

$$\begin{aligned} \frac{\mu_1}{d_1}\int_0^{s(t)} xudx + \frac{\mu_2}{d_2}\int_{s(t)}^1 xudx + \frac{1}{2}s(t)^2 > \frac{\mu_1}{d_1}\int_0^{\ell} x\phi dx + \frac{\mu_2}{d_2}\int_{\ell}^1 x\phi dx + \frac{\ell^2}{2} \\ + \int_0^t \left\{\frac{\mu_1}{d_1}\int_0^{s(\tau)} xuf(u)dx + \frac{\mu_2}{d_2}\int_{s(\tau)}^1 xug(u)dx\right\} d\tau. \end{aligned} \tag{4.33}$$

Note

$$\left|\int_{s(t)}^1 xug(u)dx\right| \le N \max_{-N\le u\le 0}|g(u)|(1-s(t)) \to 0 \quad \text{as} \quad t\to\infty$$

(by Theorem 2.3) and

$$\int_0^{s(t)} xuf(u)dx \to \int_0^1 xu_1^* f(u_1^*)dx \quad \text{as} \quad t\to\infty.$$

Here u_1^* is a positive solution of (4.17); so that

$$\int_0^1 xu_1^* f(u_1^*)dx = -d_1\int_0^1 x(u_1^*)_{xx}dx = -d_1\left[x(u_1^*)_x\right]_0^1 + d_1\int_0^1 (u_1^*)_x dx = -d_1 u_x^*(1) := c_0 d_1 > 0.$$

Therefore, there exists a large number $T_1 > 0$ such that

$$\frac{\mu_1}{d_1}\int_0^{s(t)} xuf(u)dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 xug(u)dx \ge \frac{\mu_1 c_0}{2} \qquad \text{for all} \quad t \ge T_1.$$

Hence it follows from (4.33) that

$$\frac{\mu_1}{d_1}\int_0^{s(t)} xuf(u)dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 xug(u)dx + \frac{s(t)^2}{2} \ge \frac{\mu_1 c_0}{2}(t-T_1) + C_1 \tag{4.34}$$

with some C_1. Since the first and second terms in the left-hand side of (4.34) are uniformly bounded for all $t \ge 0$, (4.34) implies $s(t)\to\infty$ as $t\to\infty$, which is a contradiction. So $s(t)$ must arrive at $x = 1$ in a finite time. This fact implies (4.21).

□

5 Stationary problem

We will give complete information on the structure of solutions of (SP). For this purpose, we introduce the following auxiliary problem: for any given $\xi \in (0,1)$, consider

$$
\text{(AP)} \qquad \begin{cases} d_1 v_{xx} + v f(v) = 0, & v > 0 \quad \text{in } (0,\xi), \\ d_2 v_{xx} + v g(v) = 0, & v < 0 \quad \text{in } (\xi,1), \\ v(0) = m_1, \quad v(\xi) = 0, \quad v(1) = -m_2. \end{cases}
$$

Let $v(x;\xi)$ be a solution of (AP) (if it exists). Our strategy for solving (SP) is to seek an appropriate number $\xi \in (0,1)$ such that $v(\cdot,\xi)$ satisfies the last equation of (SP); that is,

$$
-\mu_1 v_x(\xi - 0; \xi) + \mu_2 v_x(\xi + 0; \xi) = 0. \tag{5.1}
$$

For the sake of simplicity, we will study (AP) under the following stronger conditions than (A.1) and (A.2):

(A.1)* f is locally Lipschitz continuous in $[0,\infty)$, monotone decreasing in $[0,1]$ and satisfies

$$
f(u) > 0 \quad \text{for} \quad u \in [0,1), \quad f(1) = 0, \quad f(u) < 0 \quad \text{for} \quad u \in (0,\infty).
$$

(A.2)* As a function of $u \in [0,\infty)$, $g(-u)$ has the same properties as f in (A.1)*.

Case $m_1 > 0$ and $m_2 > 0$

As the first step to handle (AP) we study the following boundary value problem

$$
\begin{cases} d v_{xx} + v f(v) = 0, & v > 0 \quad \text{in } (0,\xi), \\ v(0) = m, \quad v(\xi) = 0, \end{cases} \tag{5.2}
$$

where $\xi > 0$ is any given number, d and m are positive numbers and f is a function satisfying (A-1)*. The phase plane analysis is available to solve (5.2).

Consider the following initial value problem for the second-order ordinary differential equation

$$
\begin{cases} d w'' + w f(w) = 0, \quad x > 0, \\ w(0) = m, \quad w'(0) = p \in \mathbb{R}. \end{cases} \tag{5.3}
$$

Let $w(x;p)$ is the solution of (5.3). Then it satisfies

$$
\frac{d w'(x)^2}{2} + F(w(x)) = \frac{dp^2}{2} + F(m), \tag{5.4}
$$

where $F(w) = \displaystyle\int_0^w u f(u) du$. For $p \in \mathbb{R}$, define

$$
X(p) = \inf\{x > 0 : \ w(x;p) = 0\}. \tag{5.5}
$$

If $p \le 0$, then it follows from (5.4) that

$$
\sqrt{\frac{d}{2}} \frac{dw}{dx} = -\left(\frac{dp^2}{2} + F(m) - F(w) \right)^{1/2}.
$$

In view of (5.5), one can get from the above relation

$$X(p) = \sqrt{\frac{d}{2}} \int_0^m \frac{dw}{\sqrt{dp^2/2 + F(m) - F(w)}} \qquad \text{for } \ p \leq 0. \tag{5.6}$$

In order to study (5.3) for $p \geq 0$, define $p_* > 0$ by

$$\frac{dp_*^2}{2} + F(m) = F(1).$$

For $0 \leq p < p_*$, the phase plane analysis of (5.3) gives

$$X(p) = \sqrt{2d} \int_0^{w_p} \frac{dw}{\sqrt{F(w_p) - F(w)}} - \sqrt{\frac{d}{2}} \int_0^m \frac{dw}{\sqrt{dp^2/2 + F(m) - F(w)}}, \tag{5.7}$$

where $w_p > 0$ is a positive number satisfying $F(w_p) = dp^2/2 + F(m)$. Here, by virtue of (A¿1)*, one can prove that the first term in the right-hand side of (5.7) is strictly increasing with respect to p (see the corresponding arguments in case $m_1 = m_2 = 0$). Therefore, it follows from (5.6) and (5.7) that $p \mapsto X(p)$ is continuous and strictly increasing in $(-\infty, p_*)$. Moreover, it is possible to see

$$\lim_{p \to -\infty} X(p) = 0, \quad X(0) = \sqrt{\frac{d}{2}} \int_0^m \frac{dw}{\sqrt{F(m) - F(w)}} := \Lambda, \quad \lim_{p \to p_*} X(p) = +\infty. \tag{5.8}$$

One can solve (5.2) by a shooting method. Indeed, it is sufficient to find p satisfying $X(p) = \xi$ in order to solve (5.2). By virtue of the strictly increasing property of $X(p)$ and (5.8), for every $\xi > 0$ there exists a unique $p(\xi) \in (-\infty, p^*)$ such that

$$X(p(\xi)) = \xi. \tag{5.9}$$

This fact implies the uniqueness of a solution $v(x;\xi)$ of (5.2) and it is represented in terms of $w(x;p)$ as follows:

$$v(x;\xi) = w(x;p(\xi)).$$

Here we should note from (5.9) that $p(\xi)$ is continuous and strictly increasing, $p(\xi) < 0$ for $\xi < \Lambda$, $p(\xi) > 0$ for $\xi > \Lambda$ and possesses the following properties:

$$\lim_{\xi \to 0} p(\xi) = -\infty, \quad p(\Lambda) = 0, \quad \lim_{\xi \to \infty} p(\xi) = p_*.$$

Our next step is to study $\Phi(\xi) := -\mu v_x(\xi - 0; \xi)$ in order to solve (5.1). Since $v(x;\xi) = w(x;p(\xi))$, the phase plane analysis enables us to get

$$\Phi(\xi) = -\mu w_x(\xi; p(\xi)) = \mu \sqrt{p(\xi)^2 + 2F(m)/d}. \tag{5.10}$$

Recalling the properties of $p(\xi)$ we can show the following lemma.

Lemma 5.1. *Define $\Phi(\xi)$ by $\Phi(\xi) := -\mu v_x(\xi - 0; \xi)$ or, equivalently, by* (5.10)*. Then $\Phi(\xi)$ is a continuous function in $(0, \infty)$ such that it is strictly decreasing in $(0, \Lambda)$ and strictly increasing in (Λ, ∞). Moreover,*

$$\lim_{\xi \to 0} \Phi(\xi) = +\infty \qquad \textit{and} \quad \lim_{\xi \to \infty} \Phi(\xi) = \mu \sqrt{2F(1)/d}.$$

As the third step, we are will solve (AP). Since f and g satisfy (A.1)* and (A.2)*, the preceding considerations allow us to show that, for each $\xi \in (0,1)$, (AP) has a unique solution $v^*(x;\xi)$. Define

$$\Phi_1(\xi) = -\mu_1 v_x^*(\xi - 0;\xi) \qquad \text{and} \qquad \Phi_2(\xi) = -\mu_2 v_x^*(\xi + 0;\xi) \tag{5.11}$$

Define $\Lambda_i, i = 1,2$, by (5.8) with d and m replaced by d_i and m_i, respectively (for Λ_2 use $G(w) = \int_0^w vg(-v)dv$ un place of $F(w)$). Owing to Lemma 5.1, $\Phi_1(\xi)$ attains its minimum at $\xi = \Lambda_1 > 0$ and is strictly decreasing in $(0,\Lambda_1)$ (resp. increasing in (Λ_1,∞)). Moreover,

$$\lim_{\xi\to 0} \Phi_1(\xi) = +\infty \qquad \text{and} \qquad \lim_{\xi\to\infty} \Phi_1(\xi) = \mu_1\sqrt{2F(1)/d_1}. \tag{5.12}$$

Similarly, one can prove that $\Phi_2(\xi)$ attains its minimum at $\xi = 1 - \Lambda_2 < 1$ and is strictly increasing in $(1-\Lambda_2, 1)$ (resp. decreasing in $(-\infty, 1-\Lambda_2)$). Moreover,

$$\lim_{\xi\to 1} \Phi_2(\xi) = +\infty \qquad \text{and} \qquad \lim_{\xi\to -\infty} \Phi_2(\xi) = \mu_2\sqrt{2G(1)/d_2}. \tag{5.13}$$

Here it should be noted that any solution (u^*, s^*) of (SP) can be obtained by looking for $s^* \in (0,1)$ such that (5.1) holds true for $\xi = s^*$. This is equivalent to find s^* such that $\Phi_1(s^*) = \Phi_2(s^*)$. Once such s^* is found, the solution u^* is given by $u^*(x) = v^*(x;s^*)$. On account of (5.12) and (5.13), it can be seen that (SP) always has at least one solution.

We will summarize results concerning (AP) and (SP) in case $m_1 > 0$ and $m_2 > 0$.

Theorem 5.1. *Assume* (A.1)* *and* (A.2)* *in place of* (A.1) *and* (A.2). *Then it holds that the following properties hold true:*

(i) *For every* $\xi \in (0,1)$, (AP) *possesses a unique solution* $v^*(x;\xi)$.

(ii) *If* $0 < \xi_1 < \xi_2 < 1$, *then* $v^*(x;\xi_1) < v^*(x;\xi_2)$ *in* I.

(iii) $(v^*(x;\xi),\xi)$ *is a solution of* (SP) *if and only if* ξ *is a zero point of* $V(\xi) := \Phi_1(\xi) - \Phi_2(\xi)$, *where* $\Phi_i, i = 1,2$ *are defined by* (5.11).

(iv) (SP) *admits a maximal solution* $(\bar{u},\bar{s}) := (v^*(\cdot;\bar{s}),\bar{s})$ *and a minimal solution* $(\underline{u},\underline{s}) := (v^*(\cdot;\underline{s}),\underline{s})$ *such that any solution* (u^*,s^*) *of* (SP) *satisfies*

$$\underline{s} \le s^* \le \bar{s} \quad \text{and} \quad \underline{u} \le u^* \le \bar{u} \quad \text{in } I.$$

Here $\bar{s}$ *is the largest zero point of* V *and* $\underline{s}$ *is the smallest zero point of* V *in* $(0,1)$.

Remark 5.1. *When (A.1) and (A.2) are imposed on* f *and* g, *the uniqueness of solutions to (AP) does not hold any longer. Therefore, the assertions corresponding to Theorem 5.1 become more complicate.*

Case $m_1 = 0$ and $m_2 = 0$

The analysis in case $m_1 = 0$ (resp. $m_2 = 0$) is slightly different from the case $m_1 > 0$ (resp. $m_2 > 0$). We will give the idea and method for the analysis of the case $m_1 = m_2 = 0$ in detail.

When we discuss (SP-0), $(u^*, s^*) = (0,\xi)$ for any $\xi \in [0,1]$ always satisfies (SP-0). Such a solution is called a *trivial solution.* Consider

$$\begin{cases} dv_{xx} + vf(v) = 0, \qquad v > 0 \qquad \text{in} \quad (0,\xi), \\ v(0) = v(\xi) = 0, \end{cases} \tag{5.14}$$

in place of (5.2). In order to solve (5.14) we also employ the phase plane method and study the following initial value problem

$$\begin{cases} dw'' + wf(w) = 0, & x > 0, \\ w(0) = 0, \quad w'(0) = p > 0. \end{cases} \tag{5.15}$$

When $w(x;p)$ denotes the solution of (5.15), it satisfies

$$\frac{d}{2}\, w'(x;p)^2 + F(w(x;p)) = \frac{dp^2}{2}. \tag{5.16}$$

Set $p^* = \sqrt{2F(1)/d}$ and, for $0 < p < p^*$, defnie $X(p)$ by (5.5). On account of (5.16) the phase plane analysis enables us to derive

$$X(p) = \sqrt{2d} \int_0^{w_p} \frac{dw}{\sqrt{F(w_p) - F(w)}}\ ,$$

where $w_p \in (0,1)$ is defined by $F(w_p) = dp^2/2$. Observe that

$$X(p) = \sqrt{2d} \int_0^1 \frac{dv}{\sqrt{F^*(v;p)}} \qquad \text{with} \qquad F^*(v;p) = \int_v^1 \sigma f(w_p\sigma) d\sigma. \tag{5.17}$$

Since $f(u)$ is deceasing, we see from (5.17) that $p \mapsto X(p)$ is a continuous and strictly increasing function such that

$$\lim_{p\to 0} X(p) = \pi\sqrt{d/f(0)} \quad \text{and} \quad \lim_{p\to p^*} X(p) = +\infty.$$

These properties allow us to show that $X(p) = \xi$ has a (unique) solution $p = p(\xi)$ if and only if $\xi > \pi\sqrt{d/f(0)}$. Therefore, if $\xi \in (0, \pi\sqrt{d/f(0)}]$, then (5.14) has no solution, while, if $\xi \in (\pi\sqrt{d/f(0)}, \infty)$, then (5.14) admits a unique solution $v(x;\xi) = w(x;p(\xi))$. When $v(x;\xi)$ exists, define $\Phi(\xi) = -\mu v_x(\xi - 0; p(\xi))$. Clearly,

$$\Phi(\xi) = \mu v_x(0;\xi) = \mu w_x(0;p(\xi)) = \mu p(\xi) \quad \text{for} \ \ \xi > \pi\sqrt{d/f(0)}.$$

Then we can prove the following result.

Lemma 5.2. *Define $\Phi(\xi) := -\mu v_x(\xi - 0;\xi)$. Then $\Phi(\xi)$ is a continuous and strictly increasing function in $\xi \in (\pi\sqrt{d/f(0)}, \infty)$ such that*

$$\lim_{\xi\to\pi\sqrt{d/f(0)}} \Phi(\xi) = 0 \qquad \text{and} \qquad \lim_{\xi\to\infty} \Phi(\xi) = \mu p^*.$$

Lemma 5.2 is very useful for the study of (SP-0). As a part of (AP) with $m_1 = m_2 = 0$, consider

$$\begin{cases} d_1 v_{xx} + vf(v) = 0, & v > 0 \quad \text{in} \ \ (0,\xi), \\ v(0) = v(\xi) = 0. \end{cases} \tag{5.18}$$

Set $a = \pi\sqrt{d_1(0)/f(0)}$. The preceding arguments show us that (5.18) has no solution in case $0 < \xi \le a$, while it has a unique solution $v^*(x;\xi)$, $0 \le x \le \xi$, in case $\xi > a$. If we define

$$\Phi_1(\xi) = -\mu_1 v_x^*(\xi - 0;\xi) \qquad \text{for} \ \ \xi > a, \tag{5.19}$$

we can apply Lemma 5.2 to Φ_1. Similarly, consider

$$\begin{cases} d_2 v_{xx} + v g(v) = 0, & v < 0 \quad \text{in} \quad (\xi, 1), \\ v(\xi) = v(1) = 0. \end{cases} \tag{5.20}$$

If we set $b = \pi\sqrt{d_2/g(0)}$, then we can prove that (5.20) has no solution in case $0 \le 1 - \xi \le b$, while (5.20) has a unique solution $v^*(x;\xi)$, $\xi \le x \le 1$, in case $1 - \xi > b$. When $\Phi_2(\xi)$ is defined by

$$\Phi_2(\xi) = -\mu_2 v_x^*(\xi + 0; \xi) \qquad \text{for} \quad \xi < 1 - b, \tag{5.21}$$

it can be seen from Lemma 5.2 that $\Phi_2(\xi)$ is continuous and strictly decreasing for $\xi < 1 - b$ and satisfies $\lim_{\xi \to 1-b} \Phi_2(\xi) = 0$.

We are ready to prove the following result on a set $\mathcal{S}$ of non-trivial solutions of (SP-0):

$$\mathcal{S} := \{(u^*, s^*) : \ u^* \not\equiv 0, (u^*, s^*) \text{ satisfies (SP-0)}\}.$$

Theorem 5.2. *Assume* (A.1)* *and*(A.2)* *in place of* (A.1) *and* (A.2). *Set* $a = \pi\sqrt{d_1/f(0)}$ *and* $b = \pi\sqrt{d_2/g(0)}$. *Then the following properties hold true.*

(i) *If* $a \ge 1$ *and* $b \ge 1$, *then* (SP-0) *admits no non-trivial solution.*

(ii) *If* $a < 1$ *and* $b \ge 1$, *then*

$$\mathcal{S} = \{(\bar{u}, 1)\} \quad \textit{with} \quad \bar{u} > 0 \quad \textit{in} \quad I.$$

(iii) *If* $a \ge 1$ *and* $b < 1$, *then*

$$\mathcal{S} = \{(\underline{u}, 0)\} \quad \textit{with} \quad \underline{u} < 0 \quad \textit{in} \quad I.$$

(iv) *If* $a < 1, b < 1$ *and* $a + b > 1$, *then*

$$\mathcal{S} = \{(\underline{u}, 0), \ (\bar{u}, 1)\} \quad \textit{with} \quad \underline{u} < 0 < \bar{u} \quad \textit{in} \quad I.$$

(v) *If* $a + b < 1$, *then there exists a unique number* $c \in (0, 1)$ *such that*

$$\mathcal{S} = \{(\underline{u}, 0), \ (\bar{u}, 1), \ (u_c, c)\}$$

with $\underline{u} < 0 < \bar{u}$ *and* $\underline{u} < u_c < \bar{u}$ *in* I.

Proof. Consider (5.18) with $\xi = 1$; then one can see that (5.18) with $\xi = 1$ admits a unique positive solution if and only if $a < 1$. This fact implies that (SP-0) admits a semi-trivial solution of the form $(\bar{u}, 1)$ with $\bar{u} > 0$ in I if and only if $a < 1$. Similarly, it follows from (5.20) with$\xi = 0$ that (SP-0) admits a semi-trvial solution $(\underline{u}, 0)$ with $\underline{u} < 0$ in I if and only if $b < 1$.

In order to find another non-trivial solution, we have to find $\xi = s^*$ satisfying $\Phi_1(s^*) = \Phi_2(s^*)$. It is easy to show that such s^* exists (uniquely) if and only if $a < 1 - b$ from basic properties of $\Phi_i, i = 1, 2$ (see Lemma 5.2). □

6 Asymptotic behaviors

We will study stability properties of solutions of (SP) or (P-0) in connection with large-time behaviors of smooth solutions of (P). The analysis can be carried out with use of the comparison principle.

Case $m_1 > 0$ and $m_2 > 0$

The first result is concerned with the stability and instability of stationary solutions given in Theorem 5.1 for the case $m_1 > 0$ and $m_2 > 0$. Hereafter we use the following notation for $(u_i, s_i) \in C(\bar{I}) \times \bar{I}, i = 1, 2$: $(u_1, s_1) \geq (u_2.s_2)$ means that $u_1 \geq u_2$ in $\bar{I}$ and $s_1 \geq s_2$.

Theorem 6.1. *In addition to*(A.1)*, (A.2)*, (A.3) - (A.5), *assume* $m_1 > 0$ *and* $m_2 > 0$. *Then the following properties hold true.*

(i) *The maximal stationary solution* $(\bar{u}, \bar{s})$ *given in Theorem 5.1 is globally and asymptotically stable from above in the sense that, if* $(\phi, \ell) \geq (\bar{u}, \bar{s})$, *then the smooth solution of* (P) *satisfies* $(u(t;\cdot;\phi,\ell), s(t;\phi,\ell)) \geq (\bar{u}, \bar{s})$ *and*

$$\lim_{t\to\infty} (u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (\bar{u}, \bar{s}) \qquad \textit{in the sense of } \Omega\textit{-topology.} \tag{6.1}$$

(ii) *The minimal stationary solution* $(\underline{u}, \underline{s})$ *given in Theorem 5.1 is globally and asymptotically stable from below; that is, the assertion of (i) remains true with "*$(\bar{u}, \bar{s})$*" and "*$\geq$ *" replaced by "*$(\underline{u}, \underline{s})$*" and "*$\leq$ *", respectively.*

(iii) *Let* $(0 <)\ \xi_1 < \xi_2\ (< 1)$ *be two adjacent zero points of* V *and let* $v^*(\cdot;\xi_i)$ *be a solution of* (SP) *with* $i = 1, 2$. *If* (ϕ, ℓ) *satisfies*

$$(v^*(\cdot;\xi_1), \xi_1) \leq (\phi, \ell) \leq (v^*(\cdot;\xi_2), \xi_2), \tag{6.2}$$

then

$$(v^*(\cdot;\xi_1), \xi_1) \leq (u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) \leq (v^*(\cdot;\xi_2), \xi_2) \tag{6.3}$$

for all $t \geq 0$. *Moreover, if* $V(\xi) > 0$ *for* $\xi \in (\xi_1, \xi_2)$ *and* $(\phi, \ell) \neq (v^*(\cdot;\xi_1), \xi_1)$, *then*

$$\lim_{t\to\infty} (u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (v^*(\cdot;\xi_2), \xi_2) \quad \textit{in the sense of } \Omega\textit{-topology,} \tag{6.4}$$

while, if $V(\xi) < 0$*for* $\xi \in (\xi_1, \xi_2)$ *and* $(\phi, \ell) \neq (v^*(\cdot;\xi_2), \xi_2)$, *then* (6.4) *holds true with " $(v^*(\cdot;\xi_2), \xi_2)$ " replaced by " $(v^*(\cdot;\xi_1), \xi_1)$ "*

Proof. (i) Take any (ϕ, ℓ) such that $(\phi, \ell) \geq (\bar{u}, \bar{s})$. Theorem 4.2 assures

$$(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) \geq (\bar{u}, \bar{s}) \quad \text{for all} \quad t \geq 0. \tag{6.5}$$

Using Theorem 2.2 and letting $t \to \infty$ along a suitable subsequence in (6.5) we get

$$(u^*, s^*) \geq (\bar{u}, \bar{s}),$$

where (u^*, s^*) is a stationary solution. Since $(\bar{u}, \bar{s})$ is a maximal stationary solution, (u^*, s^*) must be identical with $(\bar{u}, \bar{s})$. This fact together with Theorem 2.2 implies (6.1).

The poof of (ii) is essentially the same as that of (i).

(iii) Since (ϕ, ℓ) satisfies (6.2), it is easy to see (6.3) by the comparison principle (Theorem 4.2). Assume $V > 0$ in (ξ_1, ξ_2) and take any $\xi \in (\xi_1, \xi_2)$. Since $(u_2(t,\cdot), s_2(t)) \equiv (v^*(\cdot;\xi), \xi)$ is a subsolution, Theorem 4.3 implies the strong increasing property of $t \mapsto (u(t,\cdot;v^*(\xi),\xi), s(t;v^*(\xi),\xi))$ and

$$\lim_{t\to\infty} (u(t,\cdot;v^*(\xi),\xi), s(t;v^*(\xi),\xi)) = (u^*, s^*),$$

where (u^*, s^*) is minimal stationary solution in the class satisfying $(u^*, s^*) \geq (v^*(\xi), \xi)$. Since $(u^*, s^*) \leq (v^*(\xi_2), \xi_2)$, the above convergence assures $(u^*, s^*) = (v^*(\xi_2), \xi_2)$; so that

$$\lim_{t\to\infty} (u(t,\cdot;v^*(\xi),\xi), s(t;v^*(\xi),\xi)) = (v^*(\cdot;\xi_2), \xi_2) \tag{6.6}$$

and the above convergence holds in the sense of Ω-topology. If (ϕ, ℓ) satisfies

$$(v^*(\cdot;\xi), \xi) \le (\phi, \ell) \le (v^*(\cdot;\xi_2), \xi_2),$$

Theorem 4.2 shows

$$(u(t, \cdot; v^*(\xi), \xi), s(t; v^*(\xi), \xi)) \le (u(t, \cdot; \phi, \ell), s(t; \phi, \ell)) \le (v(\cdot;\xi_2), \xi_2) \tag{6.7}$$

for all $t \ge 0$. Hence letting $t \to \infty$ in (6.7) and using (6.6) one can get (6.4).

If $(\phi, \ell) \ne (v^*(\xi_1), \xi_1)$, then the strong maximum principle for parabolic equations implies $s(t; \phi, \ell) > \xi_1$ for every $t > 0$ and, therefore, $u(t, \cdot; \phi, \ell) > v^*(\cdot, \xi_1)$ for every $t > 0$. Moreover, Hopf's boundary lemma yields $u_x(t, 0; \phi, \ell) > v^*_x(0; \xi_1)$ and $u_x((t, 1; \phi, \ell) < v^*_x(1; \xi_1)$ for every $t > 0$. In this situation, we simply write $(u(t, \cdot; \phi, \ell), s(t; \phi, \ell) \gg (v^*(\cdot; \xi_1), \xi_1)$. So for any fixed $t_0 > 0$, it is possible to choose $\xi_o \in (\xi_1, \xi_2)$ such that

$$(u(t_0, \cdot; \phi, \ell), s(t_0; \phi, \ell)) > (v^*(\cdot; \xi_0), \xi_0).$$

Hence Theorem 4.2 implies

$$(u(t + t_0, \cdot; \phi, \ell), s(t + t_0; \phi, \ell)) > (u(t, \cdot; v^*(\xi_0), \xi_0), s(t; v^*(\xi_0), \xi_0) \qquad \text{for all } \ t > 0$$

like (6.7). Then repeating the preceding arguments one can derive (6.4) from (6.6). □

Case $m_1 = m_2 = 0$

Our first result on the asymptotic behavior of smooth solutions of (P) is concerned with the case $a := \pi\sqrt{d_1/f(0)} \ge 1$ and $b := \pi\sqrt{d_2/g(0)} \ge 1$.

Proposition 6.1. *In addition to* (A.1)* *and* (A.2)**, assume* $a \ge 1$ *and* $b \ge 1$*. Then for every* (ϕ, ℓ)*, the smooth solution* $(u(t, \cdot; \phi, \ell), s(t; \phi, \ell))$ *satisfies*

$$\lim_{t\to\infty} u(t, \cdot; \phi, \ell) = 0 \qquad \textit{uniformly in } \ \bar{I}. \tag{6.8}$$

Moreover, if $a > 1$ *and* $b > 1$*, then it holds that*

$$\sup_{x\in\bar{I}} |u(t, x : \phi, \ell)| = O(e^{-\gamma t}) \quad \textit{as} \ \ t \to \infty \tag{6.9}$$

with some $\gamma > 0$ *and*

$$\lim s(t; \phi, \ell) = s^* \tag{6.10}$$

with some $s^* \in [0, 1]$.

Proof. Theorem 2.5 shows that (SP-0) has no non-trivial solution; that is, $\omega(\phi, \ell) \subset \{(0, \xi) : \ \xi \in [0, 1]\}$ for every (ϕ, ℓ). This fact implies (6.8).

We will derive (6.9) in case $a > 1$ and $b > 1$. Set $v(t, x) = \alpha(t) \sin \pi x$ with $\alpha(t) > 0$.; then

$$\begin{aligned} v_t - d_1 v_{xx} - v f(v) &= \sin \pi x\{\dot\alpha(t) + d_1\pi^2\alpha(t) - \alpha(t)f(\alpha(t) \sin \pi x)\} \\ &\ge \sin \pi x\{\dot\alpha(t) + (d_1\pi^2 - f(0))\alpha(t)\}. \end{aligned}$$

If we take $\alpha(t) = \alpha_0 e^{-\alpha^* t}$ with $\alpha^* = (a^2 - 1)f(0) > 0$, then

$$v_t - d_1 v_{xx} - v f(v) \ge 0 \qquad \text{for } \ (t, x) \in [0, \infty) \times I.$$

Therefore, if α_0 is sufficiently large such that $\alpha_0 \sin \pi x \geq \phi$ in I, the comparison principle for parabolic equations gives

$$u(t, x; \phi, \ell) \leq v(t, x) = \alpha_0 e^{-\alpha^* t} \sin \pi x, \qquad (t, x) \in [0, \infty) \times I. \tag{6.11}$$

Similarly, it is also possible to prove

$$u(t, x; \phi, \ell) \geq -\beta_0 e^{-\beta^* t} \sin \pi x, \qquad (t, x) \in [0, \infty) \times I \tag{6.12}$$

with some $\beta_0 > 0$ and $\beta^* = (b^2 - 1)g(0) > 0$. Then (6.9) comes from (6.11) and (6.12).

In order to show (6.10) we assume

$$0 < s(t; \phi, \ell) < 1 \qquad \text{for all } \ t \geq 0$$

because there is nothing left to prove if $s(t; \phi, \ell)$ reaches $x = 0$ or $x = 1$ in a finite time. We will employ (4.32). Since $u(t, 1; \phi, \ell) = 0$ for $t \geq 0$, it follows from (6.12) that

$$0 < u_x(t, 1; \phi, \ell) \leq \beta_0 \pi e^{-\beta^* t} \quad \text{for } \ t \geq 0. \tag{6.13}$$

Owing to (6.11), (6.12) and (6.13), all terms in the right-hand side of (4.32) are integrable with respect to t over $(0, \infty)$. Therefore, (4.32) allows us to conclude that $s(t)$ is convergent as $t \to \infty$. Thus we have shown (6.10). □

Our next result is concerned with the case $a < 1$, for which Proposition 4.1 asserts the stability of $(\bar{u}, 1)$ in the following manner.

Proposition 6.2. *Assume $a < 1$ and, for $\ell^* \in (0, 1)$ satisfying $\ell^* > \max\{a, 1 - b\}$, define ϕ^* by*

$$\phi^*(x) = \begin{cases} \varepsilon_1 \sin \dfrac{\pi x}{\ell^*} & \text{for } \ 0 \leq x \leq \ell^*, \\ -\varepsilon_2 \sin \dfrac{\pi(1 - x)}{1 - \ell^*} & \text{for } \ \ell^* \leq x \leq 1, \end{cases} \tag{6.14}$$

where ε_1 and ε_2 are small numbers satisfying $f(\varepsilon_1) \geq d_1(\pi/\ell^*)^2$ and $\varepsilon_1\mu_1/\ell^* \geq \varepsilon_2\mu_2/(1 - \ell^*)$. Suppose that $(\phi, \ell) \geq (\phi^*, \ell^*)$. Then the smooth solution $(u(t, \cdot; \phi, \ell), s(t; \phi, \ell))$ satisfies

$$\lim_{t \to \infty} (u(t, \cdot; \phi, \ell), s(t; \phi, \ell)) = (\bar{u}, 1) \qquad \text{in the sense of } \Omega\text{-topology}, \tag{6.15}$$

where $(\bar{u}, 1)$ is a (positive) semi-trivial stationary solution given in Theorem 5.2. Moreover, there exists a positive number T^* such that $s(t; \phi, \ell)$ reaches the fixed boundary $x = 1$ at $t = T^*$.

Remark 6.1. *Assume $b < 1$. One can derive analogous stability properties of $(\underline{u}, 0)$ in the same way as p Proposition 6.2. For $\ell_* \in (0, 1)$ satisfying $\ell_* < \min\{a, 1 - b\}$ define ϕ_* by (6.14) with " ϕ^* " and " ℓ^* " replaced by " ϕ_* " and " ℓ_* ", respectively. Here ε_1 and ε_2 are small numbers such that $g(-\varepsilon_2) \geq d_2(\pi/(1 - \ell_*))^2$ and $\varepsilon_2\mu_2/(1 - \ell_*) \geq \varepsilon_1\mu_1/\ell_*$. If (ϕ, ℓ) satisfies $(\phi, \ell) \leq (\phi^*, \ell^*)$, then the smooth solution of* (P) *satisfies*

$$\lim_{t \to \infty} (u(t, \cdot; \phi, \ell), s(t; \phi, \ell)) = (\underline{u}, 0) \qquad \text{in the sense of } \Omega\text{-topology} \tag{6.16}$$

and

$$0 < s(t; \phi, \ell) \leq \ell_* \quad \text{for } \ t \in [0, T_*) \quad \text{and } \ s(t) \equiv 0 \quad \text{for } \ t \geq T_*$$

with some $T_* < \infty$.

Proposition 6.2 and Remark 6.1 also provide us interesting information on the instability (as a set) of trivial solutions $\{(0,\xi);\ a<\xi<1\}$ in case $a<1$ and $\{(0,\xi);\ 0<\xi<1-b\}$ in case $b<1$. We can also discuss the asymptotic stability (as a set) of trivial solutions.

Proposition 6.3. (i) *Assume $a<1$. For any $s_0\in(0,a)$, define*

$$\phi_\eta(x)=\begin{cases}\eta\sin\dfrac{\pi x}{s_0} & \text{for } 0\le x\le s_0,\\ 0 & \text{for } s_0\le x\le 1.\end{cases}\tag{6.17}$$

Then there exists $\eta_1>0$ such that, for every $\eta\in(0,\eta_1]$, the smooth solution of (P) *with initial data (ϕ_η,s_0) satisfies*

$$\lim_{t\to\infty}(u(t,\cdot;\phi_\eta,s_0),s(t;\phi_\eta,s_0))=(0,s^*)\qquad\text{in the sense of }\Omega\text{-topology}$$

with some $s^*\in[0,a)$.

(ii) *Assume $b<1$. For any $s_0\in(1-b,1)$ define*

$$\psi_\eta(x)=\begin{cases}0 & \text{for } 0\le x\le s_0,\\ -\eta\sin\dfrac{\pi(1-x)}{1-s_0} & \text{for } s_0\le x\le 1.\end{cases}\tag{6.18}$$

Then there exists $\eta_2>0$ such that, for every $\eta\in(0,\eta_2]$, the smooth solution of (P) *with initial data (ψ_η,s_0) satisfies*

$$\lim_{t\to\infty}(u(t,\cdot;\psi_\eta,s_0),s(t;\psi_\eta,s_0))=(0,s_*)\qquad\text{in the sense of }\Omega\text{-topology}$$

with some $s_*\in(1-b,1]$.

Proof. We will prove only (i) because the proof of (ii) is essentially the same. We will employ the method in the work of [6].

In case $a<1$, let $s_0\in(0,a)$ be fixed and choose $\delta>0$ satisfying $s_0(1+\delta)<a$. Set

$$s(t)=s_0\{1+\delta(1-e^{-\alpha t})\},$$

where α is a positive number to be determined later. Define $\bar u(x)$ by

$$\bar u(t,x)=\varepsilon e^{-\beta t}\sin\left(\frac{(\pi-\gamma)x}{s(t)}+\gamma\right)\qquad\text{for } t\ge 0 \text{ and } 0\le x\le s(t),$$

where ε,β and $\gamma\in(0,\pi)$ are also positive numbers to be determined later. If we put

$$X(t,x;\gamma)=\frac{(\pi-\gamma)x}{s(t)}+\gamma,$$

then

$$\frac{\partial X}{\partial x}=\frac{\pi-\gamma}{s(t)}>0,\qquad \frac{\partial X}{\partial t}=-\frac{(\pi-\gamma)x\dot s(t)}{s(t)^2}=-\frac{(\pi-\gamma)s_0\alpha\delta e^{-\alpha t}x}{s(t)^2}<0$$

and

$$\gamma=X(t,0;\gamma)\le X(t,x;\gamma)\le X(t,s(t);\gamma)=\pi$$

for $t \geq 0$ and $0 \leq x \leq s(t)$. Then $\bar{u}$ satisfies the following relations:

$$\bar{u}_t = -\beta \bar{u} - \frac{(\pi-\gamma)s_0 \alpha\delta e^{-\alpha t} x}{s(t)^2} \cdot \varepsilon e^{-\beta t} \cos X(t,x;\gamma),$$
$$\bar{u}_{xx} = -\left(\frac{\pi-\gamma}{s(t)}\right)^2 \bar{u}.$$

Therefore,

$$\bar{u}_t - d_1 \bar{u}_{xx} - \bar{u} f(\bar{u}) = \left\{ d_1 \left(\frac{\pi-\gamma}{s(t)^2}\right)^2 - \beta - f(\bar{u}) \right\} \bar{u} - \frac{(\pi-\gamma)s_0\alpha\delta e^{-\alpha t}x}{s(t)^2} \cdot \varepsilon e^{-\beta t} \cos X(t,x;\gamma). \tag{6.19}$$

Here

$$d_1 \left(\frac{\pi-\gamma}{s(t)^2}\right)^2 - \beta - f(\bar{u}) \geq d_1 \left(\frac{\pi-\gamma}{s_0(1+\delta)}\right)^2 - \beta - f(0). \tag{6.20}$$

Siince $s_0(1+\delta) < a = \pi\sqrt{d_1/f(0)}$, it is possible to choose sufficiently small $\beta > 0$ and $\gamma > 0$ satisfying

$$d_1 \left(\frac{\pi-\gamma}{s_0(1+\delta)}\right)^2 > \beta + f(0). \tag{6.21}$$

If $\pi/2 \leq X(x,t;\gamma) \leq \pi$, it is easily seen fron (6.19)-(6.21) that

$$\bar{u}_t - d_1 \bar{u}_{xx} - \bar{u} f(\bar{u}) \geq 0. \tag{6.22}$$

If $\gamma \leq X(x,t;\gamma) \leq \pi/2$, then

$$\sin X(x,t;\gamma) \geq \sin\gamma \quad \text{and} \quad \cos X(x,t;\gamma) \leq \cos\gamma.$$

So it follows from (6.19)-(6.21) that

$$\bar{u}_t - d_1 \bar{u}_{xx} - \bar{u} f(\bar{u}) \geq \varepsilon e^{-\alpha t} \left[\left\{ d_1 \left(\frac{\pi-\gamma}{s_0(1+\delta)}\right)^2 - \beta - f(0) \right\} \sin\gamma - \pi\alpha\delta\cos\gamma \right].$$

When we choose sufficiently small α satisfying

$$\left\{ d_1 \left(\frac{\pi-\gamma}{s_0(1+\delta)}\right)^2 - \beta - f(0) \right\} \sin\gamma - \pi\alpha\delta\cos\gamma > 0, \tag{6.23}$$

we have

$$\bar{u}_t - d_1 \bar{u}_{xx} - \bar{u} f(\bar{u}) \geq 0 \tag{6.24}$$

if $\gamma \leq X(t,x;\gamma) \leq \pi/2$.

We now define

$$u_1(t,x) = \begin{cases} \bar{u}(t,x) & \text{for } \ 0 \leq x \leq s(t), \\ 0 & \text{for } \ s(t) \leq x \leq 1. \end{cases}$$

Note

$$-\mu_1 u_{1,x}(t, s(t)-0) + \mu_2 u_{1,x}(t, s(t)+0) = \mu_1 \varepsilon e^{-\beta t} \cdot \frac{\pi-\gamma}{s(t)} \leq \frac{\pi\mu_1\varepsilon}{s_0} \cdot e^{-\beta t}$$

and $\dot{s}(t) = s_0 \alpha \delta e^{-\alpha t}$. We choose sufficiently small α and ε such that

$$\beta \geq \alpha \quad \text{and} \quad \pi \mu_1 \varepsilon \leq s_0^2 \alpha \delta.$$

Then it is easy to see that

$$\dot{s}(t) \geq -\mu_1 u_{1,x}(t, s(t)-0) + \mu_2 u_{1,x}(t, s(t)+0) \qquad \text{for } \ t \geq 0. \tag{6.25}$$

Owing to (6.22), (6.24) and (6.25) we have shown that (u_1, s) is a supersolution.

We are ready to apply Theorem 4.2. Define

$$\eta_1 = \sup\{\eta > 0;\ \phi_\eta(x) \leq u_1(0,x) = \varepsilon \sin\left(\frac{(\pi-\gamma)x}{s_0} + \gamma\right) \quad \text{for } \ 0 \leq x \leq s_0\}.$$

Since $(\phi_\eta, s_0) \leq (u_1(0,\cdot), s_0)$ for every $\eta \leq \eta_1$, Theroem 4.2 yields

$$(0, s_0) \leq (u(t,\cdot;\phi_\eta, s_0), s(t;\phi_\eta, s_0)) \leq (u_1(t,\cdot), s(t)) \qquad \text{for all } \ t \geq 0.$$

Letting $t \to \infty$ in the above relation leads us to

$$\lim_{t\to\infty} u(t,\cdot;\phi_\eta, s_0) = 0 \qquad \text{uniformly in} \quad I.$$

The convergence of $s(t;\phi_\eta, s_0)$ as $t \to \infty$ can be proved in the same way as Proposition 6.1. □

Finally we will give important and useful results on the stability or instability of a non-trivial stationary solution (u^*, s^*) with $0 < s^* < 1$. By Theorem 5.2 such a solution exists if and only if $a + b < 1$. For each $\xi \in (a, 1-b)$ let $v^*(x;\xi)$ be a (unique) solution of (AP) whose existence is assured by the results given in §5. We define

$$V(\xi) = \Phi_1(\xi) - \Phi_2(\xi),$$

where $\Phi_1(\xi) = -\mu_1 v_x^*(\xi - 0;\xi)$ and $\Phi_2(\xi) = -\mu_2 v_x^*(\xi + 0;\xi)$ (see also (5.19) and (5.21)). Recalling the properties of Φ_1 and Φ_2 we know that $V(\xi)$ has a unique zero point $\xi = c$.

Proposition 6.4. *Assume $a+b<1$ and $V(\xi_0) > 0$ (resp. $V(\xi_0) < 0$) for $\xi_0 \in (a, 1-b)$. Then the smooth solution of* (P) *with initial data $(v^*(\cdot;\xi_0), \xi_0)$ satisfies*

$$\lim_{t\to\infty} (u(t,\cdot;v^*(\xi_0),\xi_0), s(t;v^*(\xi_0),\xi_0)) = (\bar{u}, 1) \qquad (\textit{resp.} \ \ (\underline{u}, 0)) \tag{6.26}$$

in the sense of Ω-topology. Moreover, there exists a positive constant T^* such that $s(T^*;v^*(\xi_0),\xi_0) = 1$ (resp. $s(T^*;v^*(\xi_0),\xi_0) = 0$) and $0 < s(t;v^*(\xi_0),\xi_0) < 1$ for $0 \leq t < T^*$.

Proof. We will prove this proposition in case $V(\xi_0) > 0$. Since $V(\xi)$ is strictly increasing, note that $(v(\xi_0), \xi_0) > (v_c, c)$. It is easy to see that $(v^*(\cdot;\xi_0), \xi_0)$ satisfies the assumptions of Theorem 4.3. So this theorem implies that $t \mapsto (u(t,\cdot;v^*(\xi_0),\xi_0), s(t;v^*(\xi_0),\xi_0))$ is increasing and that

$$\lim_{t\to\infty} (u(t,\cdot;v^*(\xi_0),\xi_0), s(t;v^*(\xi_0),\xi_0)) = (u^*, s^*) \quad \text{in the sense of } \Omega\text{-topology},$$

where (u^*, s^*) is a minimal of (SP) in the class of $(u^*, s^*) \geq (v^*(\xi_0), \xi_0)(> (v_c, c))$. Then it follows from Theorem 5.2 taht $(u^* s^*)$ must be identical with $(\bar{u}, 1)$. In order to show the last assertion, it is sufficient to repeat the argument of Step 4 in the proof of Proposition 4.1. □

Theorem 6.1 and Propositions 6.1-6.4 allow us to obtain the following theorem.

Theorem 6.2. *Assume (A.1*), (A.2*) and $m_1 = m_2 = 0$.*

(i) *Every semi-trivial stationary solution of the form $(\bar{u}, 1)$ or $(\underline{u}, 0)$ is asymptotically stable whenever it exists. Moreover, if the smooth solution (u, s) of* (P) *satisfies*

$$\lim_{t\to\infty}(u(t,\cdot), s(t)) = (\bar{u}, 1) \quad (\textit{resp. } (\underline{u}, 0)) \quad \textit{in the sense of } \Omega\textit{-topology},$$

then there exists $T^* > 0$ such that $s(T^*) = 1$ (resp. $s(T^*) = 0$) and $s(t) \in (0, 1)$ for $0 \le t < T^*$.

(ii) *Assume $a \ge 1$ and $b \ge 1$. Then the set of trivial solutions $\{(0, \xi) :\ 0 \le \xi \le 1\}$ is globally asymptotically sable in the sense of Proposition 6.1.*

(iii) *Assume $a < 1$ and $b \ge 1$. Then the set $\{(0, \xi) :\ a < \xi < 1\}$ is unstable in the sense that, if $(\phi, \ell) \ge (\phi^*, \ell^*)$, where (ϕ^*, ℓ^*) is defined by* (6.14) *with $\ell^* \in (a, 1)$ as in Proposition 6.2, then the smooth solution of* (P) *with initial data (ϕ, ℓ) satisfies*

$$\lim_{t\to\infty}(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (\bar{u}, 1) \quad \textit{in the sense of } \Omega\textit{-topology}. \tag{6.27}$$

On the other hand, the set $\{(0, \xi);: 0 < \xi < a\}$ is asymptotically stable (as a set) in the sense that, if $(\phi, \ell) \le (\phi_\eta, s_0)$, where ϕ_η is defined by (6.17) *with $s_0 \in (0, a)$ as in (i) of Proposition 6.3, then*

$$\lim_{t\to\infty}(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (0, s^*) \quad \textit{in the sense of } \Omega\textit{-topology} \tag{6.28}$$

with some $s^* \in [0, a)$.

(iv) *Assume $a \ge 1$ and $b < 1$. Then the set $\{(0, \xi) :\ 0 < \xi < 1 - b\}$ is unstable in the sense that , if $(\phi, \ell) \le (\phi_*, \ell_*)$, where (ϕ_*, ℓ_*) is defined as in Remark 6.1 with $\ell_* \in (0, 1 - b)$, then the smooth solution of* (P) *with initial data (ϕ, ℓ) satisfies*

$$\lim_{t\to\infty}(u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (\underline{u}, 0) \quad \textit{in the sense of } \Omega\textit{-topology}. \tag{6.29}$$

On the other hand, the set $\{(0, \xi) :\ 1 - b < \xi < 1\}$ is asymptotically stable (as a set) in the sense that, if $(\phi, \ell) \ge (-\psi_\eta, s_0)$, where ψ_η is defined by (6.18) *with $s_0 \in (1 - b, 1)$ as in (ii) of Proposition 6.3, then the smooth solution $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ satisfies* (6.28) *with "s^*" replaced by "$s_* \in (1 - b, 1]$".*

(v) *Assume $a < 1, b < 1$ and $a + b \ge 1$. Then the set $\{(0, \xi) :\ 0 < \xi < 1 - b$ or $a < \xi < 1\}$ is unstable in the sense as stated in (iii) and (iv). On the other hand, the set $\{(0, \xi) :\ 1 - b < \xi < a\}$ is asymptotically stable in the sense that, if $(-\psi_\eta, s_1) \le (\phi, \ell) \le (\psi_\eta, s_2)$ with $s_1 < s_2$, where ψ_η is defined by* (6.18) *with s_0 replaced by $s_1 \in (1 - b, a)$ and ϕ_η is defined by* (6.17) *with s_0 replaced by $s_2 \in (1 - b, a)$, then* (6.28) *holds true with some $s^* \in (1 - b, a)$.*

(vi) *Assume $a + b < 1$. Then the set $\{(, \xi) : 0 < \xi < 1 - b$ or $a < \xi < 1\}$ is unstable in the sense as stated in (iii) and (iv). Moreover, (v_c, c) is unstable in the sense that, if $(\phi, \ell) \gg (v_c, c)$ (resp. $(\phi, \ell) \ll (v_c, c)$), the smooth solution $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ of* (P) *satisfies* (6.27)*(resp.* (6.29)*).*

7 Some remarks

We will consider a similar free boundary problem for (1.1) and (1.3) with the Dirichlet conditions on the fixed boundaries in (1.2) replaced by the homogeneous Neumann conditions

$$u_{1,x}(t, 0) = u_{2,x}(t, L) = 0 \qquad \text{for} \quad t > 0.$$

Then it is possible to rewrite the corresponding free boundary problem in the following form:

$$(\mathrm{P}-\mathrm{N})\quad \begin{cases} u_t = d_1 u_{xx} + uf(u) & \text{for } (t,x) \in S^-, \\ u_t = d_2 u_{xx} + ug(u) & \text{for } (t,x) \in S^+, \\ u_x(t,0) = u_x(t,1) = 0 & \text{for } t \in \{\tau > 0:\ 0 < s(\tau) < 1\}, \\ u(t,s(t)) = 0 & \text{for } t > 0, \\ \dot{s}(t) = -\mu_1 u_x(t,s(t)-0) + \mu_2 u_x(t,s(t)+0) & \text{for } t \in \{\tau > 0:\ 0 < s(\tau) < 1\}, \\ u(0,x) = \phi(x) & \text{for } 0 \le x \le 1, \\ s(0) = \ell. \end{cases}$$

Here we assume (A.1)-(A.4) and

(A.5)* $\phi \in C^1(I)$ satisfies $\phi(\ell) = 0, \phi_x(0) = \phi_x(1) = 0$ and $(\ell - x)\phi(x) \ge 0$ for $x \in I$.

On the existence of a smooth solution (u, s) for (P-N), we can prove the same result as Theorem 2.3. So the free boundary $x = s(t)$ may reach the fixed boundary $x = 0$ or $x = 1$ at a finite time $T^* > 0$. As in the case $m_1 = m_2 = 0$ for (P), we will continue to solve (P-N) after $t = T^*$. Set $s(t) \equiv 1$ (resp. $s(t) \equiv 0$) if $s(T^*) = 1$ (resp. $s(T^*) = 0$) and solve the standard boundary value problem for $u_t = d_1 u_{xx} + uf(u)$ (resp. $u_t = d_2 u_{xx} + ug(u)$) with boundary conditions

$$u_x(t,0) = u(t,1) = 0 \qquad (\text{resp. } u(t,0) = u_x(t,1) = 0).$$

Therefore, the global existence of a unique smooth solution to (P-N) can be stated in the following form:

Theorem 7.1. *Under assumptions* (A.1)-(A.4) *and* (A.5)*, *there exists a unique solution* $(u, s) \in C(\bar{Q}) \times C[0, \infty)$ *with the following properties:*

(i) (u, s) *satisfies initial conditions in* (P-N) *and*

$$u_x(t,0) = 0 \qquad \text{for } t \in \{\tau > 0:\ 0 < s(\tau) \le 1\},$$
$$u_x(t,1) = 0 \qquad \text{for } t \in \{\tau > 0:\ 0 \le s(\tau) < 1\}.$$

(ii) $\dot{s} \in L^3(0, \infty)$ *and* s *satisfies one of the following conditions;* $0 < s(t) < 1$ *for all* $t > 0$, $s(t) = 0$ *for all* $t \ge T^*$ *or* $s(t) = 1$ *for all* $t \ge T^*$ *with some* $T^* \in (0, \infty)$.

(iii) (u, s) *satisfies*

$$0 \le u \le M := \max\{1, \sup_{0 \le x \le \ell} \phi(x)\} \quad \text{in } \overline{S^-},$$
$$0 \ge u \ge -N := \min\{-1, \inf_{\ell \le x \le 1} \phi(x)\} \quad \text{in } \overline{S^+}.$$

(iv) $u^\pm \in C([0, \infty);\ H^1(I))$.

(v) $(u^+)_x \in L^\infty(S^-_{\delta,\infty})$, $(u^-)_x \in L^\infty(S^+_{\delta,\infty})$ *and* $\dot{s} \in L^\infty(\delta, \infty)$ *for any* $\delta > 0$.

(vi) $u_t \in L^2(S^-) \cap L^2(S^+)$.

(vii) $u_t, u_{xx} \in C(S^-) \cap C(S^+)$ *and* (u, s) *satisfies the first and second equations of* (P-N).

(viii) *For any* $\delta > 0$, u_x *is Hölder continuous with respect to* (t, x) *in* $\{(y, \tau) \in \overline{S^-_{\delta,\infty}} :\ s(\tau) \ge \delta\}$ *and* $\{(y, \tau) \in \overline{S^+_{\delta,\infty}} :\ s(\tau) \le 1 - \delta\}$ *and* $\dot{s}$ *is Hölder continuous in* $t \in [\delta, \infty)$.

(ix) (u, s) *satisfies the fourth and fifth equations of* (P-N).

Theorem 7.1 assures the existence of a global smooth solution $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ of (P-N) for any initial data (ϕ,ℓ). We define the ω-limit set $\omega(\phi,\ell)$ by (2.1). Then we can also obtain the following result.

Theorem 7.2. *Let $\omega(\phi,\ell)$ be the ω-limit set associated with the smooth solution $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$ of* (P-N). *Then the following properties hold true.*

(i) *$\omega(\phi,\ell)$ is a non-empty, connected and compact set in $H^1(I)\times\bar{I}$.*

(ii) *$\omega(\phi,\ell)$ is positively invariant: if $(u^*,s^*)\in\omega(\phi\,\ell)$, then $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))\in\omega(\phi,\ell)$ for every $t\geq 0$.*

(iii) *If $(u^*,s^*)\in\omega(\phi,\ell)$, then it satisfies*

$$(\mathrm{SP-N})\qquad \begin{cases} d_1u^*_{xx}+u^*f(u^*)=0, \quad u^*\geq 0 & \textit{in} \quad (0,s^*),\\ d_2u^*_{xx}+u^*g(u^*)=0, \quad u^*\leq 0 & \textit{in} \quad (s^*,1),\\ u^*_x(0)=u^*(s^*)=u^*_x(1)=0, & \textit{if} \ \ 0<s^*<1,\\ -\mu_1u^*_x(s^*-0)+\mu_2u^*_x(s^*+0)=0 & \textit{if} \ \ 0<s^*<1,\\ u^*(0)=u^*_x(1)=0 & \textit{if} \ \ s^*=0,\\ u^*_x(0)=u^*(1)=0 & \textit{if} \ \ s^*=1. \end{cases}$$

Here it should be noted that Propositions 3.1 and 3.2 are still valid for (P-N); so that they are applicable to derive some estimates in Theorem 7.1. Moreover, the comparison principle is also available for (P-N) with slight modifications of definitions of a supersolution and a subsolution as follows:

Definition 7.1. *Let $(u,s)\in C(\bar{Q})\times C(\bar{I})$ possess the property* (R) *given in §4. Then (u,s) is called a* **supersolution** *of* (P-N) *with initial data (ϕ,ℓ) if it satisfies the following:*

$$\begin{cases} u_t\geq d_1u_{xx}+uf(u) & \textit{for} \ \ (t,x)\in S^-,\\ u_t\geq d_2u_{xx}+ug(u), \quad u\leq 0 & \textit{for} \ \ (t,x)\in S^+,\\ u(t,s(t))=0 & \textit{for} \ \ t>0,\\ u_x(t,0)\leq 0 & \textit{for} \ \ t\in\{\tau>0:\ 0<s(\tau)\leq 1\},\\ u_x(t,1)\geq 0 & \textit{for} \ \ t\in\{\tau>0:\ 0\leq s(\tau)<1\},\\ \dot{s}(t)\geq -\mu_1u_x(t,s(t)-0)+\mu_2u_x(t,s(t)+0) & \textit{for} \ \ t\in\{\tau>0:\ 0<s(\tau)<1\},\\ u(0,x)=\phi(x) & \textit{for} \ \ 0\leq x\leq 1,\\ s(0)=\ell. \end{cases}$$

On the other hand, if (u,s) satisfies the above relations by reversing the inequality signs except for " $u\leq 0$ in S^+" and replacing " $u\leq 0$ in S^+" by " $u\geq 0$ in S^-", then it is called a **subsolution** *of* (P) *with initial data (ϕ,ℓ).*

The we can prove the same comparison results as Theorems 4.1 and 4.2. Correspondingly to Theorem 7.3, it is possible to show the following result:

Theorem 7.3. *Let $(\phi,\ell)\in C([0,1])\times(0,1)$ possess the following properties: $\phi\in(C^2((0,\ell))\cap C^2((\ell,1)))\cap(C^1([0,\ell])\cap C^1([\ell,1]))$ satisfies*

$$\begin{cases} d_1\phi_{xx}+\phi f(\phi)\geq 0, \qquad \phi\geq 0 & \textit{in} \ \ (0,\ell),\\ d_2\phi_{xx}+\phi g(\phi)\geq 0, \qquad \phi\leq 0 & \textit{in} \ \ (\ell,1),\\ \phi_x(0)\geq 0, \quad \phi(\ell)=0, \quad \phi_x(1)\leq 0,\\ 0\leq -\mu_1\lim\limits_{x\to\ell-0}\phi_x(x)+\mu_2\lim\limits_{x\to\ell+0}\phi_x(x). \end{cases}$$

Then the smooth solution $(u(t;\phi,\ell), s(t;\phi,\ell))$ *of* (P-N) *with initial data* (ϕ,ℓ) *fulfills the following properties:*

(i) $u_t(t;\phi,\ell) \geq 0$ *in* Q *and* $\dot{s}(t) \geq 0$ *for* $t \geq 0$. *In particular, if* (ϕ,ℓ) *is not a solution of* (SP-N)*, then* $t \mapsto u(t;\phi,\ell)$ *is strictly increasing in* $(0,\infty)$ *and* $t \mapsto s(t;\phi,\ell)$ *is strictly increasing as long as* $\ell \leq s(t;\phi,\ell) < 1$.

(ii) $\lim_{t\to\infty} u^{\pm}(t;\phi,\ell) = (u^*)^{\pm}$ *in* $H^1(I)$ *and* $\lim_{t\to\infty} s(t;\phi,\ell) = s^*$, *where* (u^*, s^*) *is a minimal solution of* (SP-N) *in the class satisfying* $u^* \geq \phi$ *in* I *and* $s^* \geq \ell$.

Making use of Theorem 7.3 we will prove the following result which is similar to Proposition 4.1.

Proposition 7.1. *In addition to* (A.1) *and* (A.2)*, assume that* f *and* g *satisfy* $f(0) > \pi^2 d_1/4, g(0) > 0$ *and that* g *is increasing near* $u = 0$. *Let* $\ell^* \in (0,1)$ *satisfy*

$$\ell^* > \max\left\{\frac{\pi}{2}\sqrt{\frac{d_1}{f(0)}}, 1 - \frac{\pi}{2}\sqrt{\frac{d_2}{g(0)}}\right\}$$

and define ϕ^* *by*

$$\phi^*(x) = \begin{cases} \varepsilon_1 \cos\dfrac{\pi x}{2\ell^*} & \text{for } 0 \leq x \leq \ell^*, \\ -\varepsilon_2 \cos\dfrac{\pi(1-x)}{2(1-\ell^*)} & \text{for } \ell^* \leq x \leq 1, \end{cases}$$

where $\varepsilon_i > 0, i = 1,2$ *are sufficiently small numbers such that* $\varepsilon_1\mu_1/\ell^* \geq \varepsilon_2\mu_2/(1-\ell^*)$. *Suppose that* $(\phi,\ell) \geq (\phi^*,\ell^*)$. *Then the smooth solution of* (P-N) *with initial data* (ϕ,ℓ) *satisfies*

$$\lim_{t\to\infty} (u(t,\cdot;\phi,\ell), s(t;\phi,\ell)) = (u_1^*, 1) \qquad \text{in the sense of } \Omega\text{-topology}, \tag{7.1}$$

where $(u_1^*, 1)$ *is a minimal positive solution of* (SP-N)*. Moreover, there exists* $T^* > 0$ *such that*

$$s(t;\phi,\ell) = 1 \quad \text{for } t \geq T^* \quad \text{and} \quad s(t;\phi,\ell) < 1 \quad \text{for } 0 \leq t < T^*. \tag{7.2}$$

Proof. The proof can be carried out essentially in the same manner as Proposition 4.1. One can show that $(u_1(t,x), s_1(t,x)) := (\phi^*(x), \ell^*)$ is a subsolution for (P-N) provided that ε_1 and ε_2 satisfy

$$\min_{0\leq u\leq\varepsilon_1} f(u) \geq \frac{d_1\pi^2}{4(\ell^*)^2} \quad \text{and} \quad \frac{\varepsilon_1\mu_1}{\ell^*} \geq \frac{\varepsilon_2\mu_2}{1-\ell^*}.$$

So the discussions in Steps 2 and 3 in the proof of Proposition 4.1are still valid with obvious modifications. Therefore, we can prove (7.1).

Finally, in order to show (7.2) we use the following identity in place of (4.33):

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\mu_1}{d_1}\int_0^{s(t)} u(t,x)dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 u(t,x)dx + s(t)\right\} \\ =&\frac{\mu_1}{d_1}\int_0^{s(t)} u(t,x)f(u(t,x))dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 u(t,x)g(u(t,x))dx \end{aligned} \tag{7.3}$$

as long as $0 < s(t) < 1$. Here we have simply written $(u(t,\cdot), s(t))$ in place of $(u(t,\cdot;\phi,\ell), s(t;\phi,\ell))$. Assume $0 < s(t) < 1$ for all $t \geq 0$. Observe that $\lim_{t\to\infty} s(t) = 1$ and that $\lim_{t\to\infty} u(t,\cdot) = u_1^*$ uniformly in $\bar{I}$, where u_1^* is a positive solution of

$$d_1 u_{1,xx}^* + u_1^* f(u_1^*) = 0 \quad \text{with} \quad u_{1,x}^*(0) = u_1^*(1) = 0.$$

Therefore, $\lim_{t\to\infty}\int_{s(t)}^1 u(t,x)g(u(t,x))dx = 0$ and

$$\lim_{t\to\infty}\int_0^{s(t)} u(t,x)f(u(t,x))dx = \int_0^1 u_1^* f(u_1^*)dx = -d_1\int_0^1 u_{1,xx}^* dx = -d_1 u_{1,x}^*(1) > 0.$$

Hence there exist positive constants c_1 and T_1 such that

$$\frac{\mu_1}{d_1}\int_0^{s(t)} u(t,x)f(u(t,x))dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 u(t,x)g(u(t,x))dx \geq c_1 \quad \text{for all} \quad t \geq T_1.$$

This fact together with (7.3) implies

$$s(t) \to \infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction. Thus we have shown (7.2). □

We will study the structure of the set of solutions of (SP-N). Our first task is to study the following auxiliary problem: for any $\xi \in (0,1)$ condider

$$(\mathrm{AP-N}) \qquad \begin{cases} d_1 v_{xx} + vf(v) = 0, \quad v > 0 & \text{in} \ (0,\xi), \\ d_2 v_{xx} + vg(v) = 0, \quad v < 0 & \text{in} \ (\xi,1), \\ v_x(0) = v(\xi) = v_x(1) = 0. \end{cases}$$

As in §5, we assume that f and g satisfy (A¿1)* and (A.2)*. We will employ the phase plane method for the study of (AP-N). Note that (AP-N) consists of two types of boundary value problems:

$$\begin{cases} d_1 v_{xx} + vf(v) = 0, \quad v > 0 & \text{in} \ (0,\xi), \\ v_x(0) = v(\xi) = 0 \end{cases} \tag{7.4}$$

and

$$\begin{cases} d_2 v_{xx} + vg(v) = 0, \quad v < 0 & \text{in} \ (\xi,1), \\ v(\xi) = v_x(1) = 0. \end{cases} \tag{7.5}$$

In order to solve (7.4) (and (7.5)), consider the following initial value problem

$$\begin{cases} d_1 w_{xx} + wf(w) = 0, \qquad x > 0, \\ w(0) = q > 0, \quad w_x(0) = 0. \end{cases} \tag{7.6}$$

Let $w(x;q)$ be the solution of (7.6). Define

$$Y(q) = \inf\{x > 0 : \ w(x;q) = 0\}.$$

By the phase plane analysis one can derive

$$Y(q) = \sqrt{\frac{d_1}{2}}\int_0^q \frac{dw}{\sqrt{F(q)-F(w)}} = \sqrt{\frac{d_1}{2}}\int_0^1 \frac{d\sigma}{\sqrt{H(\sigma,q)}}$$

where $H(\sigma,q) = \displaystyle\int_\sigma^1 \tau f(q\tau)d\tau$ (for details, see §5). By (A.1)*, $Y(q)$ is a continuous and increasing function in $q \in (0,1)$ such that

$$\lim_{q\to 0} Y(q) = \frac{\pi}{2}\sqrt{\frac{d_1}{f(0)}} \quad \text{and} \quad \lim_{q\to 1} Y(q) = +\infty.$$

When we study (7.4), the above considerations allow us to show that, if $\xi > a := (\pi/2)\sqrt{d_1/f(0)}$, then (7.4) has a unique solution $v_1(x;\xi)$, while, if $\xi \le a$, then (7.4) admits no solution. Here it should be noted that $v_1(x;\xi)$ is given by

$$v_1(x;\xi) = w(x;q(\xi)),$$

where $q(\xi)$ is determined from $Y(q(\xi)) = \xi$ in case $\xi > a$. Moreover, if we set $\Phi_1(\xi) := -\mu_1 v_{1,x}(\xi - 0;\xi)$, then we see

$$\Phi_1(\xi) = \mu_1\{2F(q(\xi))/d_1\}^{1/2};$$

so that Φ_1 is a continuously increasing function of $\xi \in (a,\infty)$ such that

$$\lim_{\xi\to a}\Phi_1(\xi) = 0 \qquad \text{and} \qquad \lim_{\xi\to\infty}\Phi_1(\xi) = \mu_1\sqrt{2F(1)/d_1}.$$

One can obtain analogous results for (7.5). Set $b := (\pi/2)\sqrt{g(0)/d_2}$. If $\xi < 1-b$, then (7.5) has a unique solution $v_2(x;\xi)$, while, if $\xi \ge 1-b$, then (7.5) admits no solution. Moreover, when we define

$$\Phi_2(\xi) = -\mu_2 v_{2,x}(\xi + 0;\xi)$$

in case $v_2(x;\xi)$ exists, we can also show that Φ_2 is a continuously decreasing function of $\xi \in (-\infty, 1-b)$ such that

$$\lim_{\xi\to 1-b}\Phi_2(\xi) = 0 \qquad \text{and} \qquad \lim_{\xi\to-\infty}\Phi_2(\xi) = \mu_2\sqrt{2G(1)/d_2}.$$

We are ready to study (SP-N). Observe that (SP-N) has trivial solutions $(0,s^*)$ with $0 \le s^* \le 1$. Additionally, (SP-N) has two types of semi-trivial solutions $(u^*,1)$ and $(u_*,0)$, where u^* and u_*, respectively, satisfy

$$\begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, \quad u^* > 0 \quad \text{in} \ \ (0,1), \\ u^*_x(0) = u^*(1) = 0 \end{cases} \tag{7.7}$$

and

$$\begin{cases} d_2 u_{*,xx} + u_* g(u_*) = 0, \quad u_* < 0 \quad \text{in} \ \ (0,1), \\ u_*(0) = u_{*,x}(1) = 0. \end{cases} \tag{7.8}$$

When we intend to solve (7.7), it is sufficient to look for a solution $w(x;q)$of (7.6) such that $Y(q) = 1$. Therefore, it is easily seen that (7.7) has a unique solution u^* if and only if $a < 1$. Similarly, (7.8) has a unique solution u_* if and only if $b < 1$.

In order to find a non-trivial solution of (SP-N), we have only to find $\xi \in (0,1)$ satisfying $\Phi_1(\xi) = \Phi_2(\xi)$. Clearly, such ξ exists (uniquely) if and only if $a + b < 1$.

The above considerations enable us to prove the following theorem.

Theorem 7.4. *Assume* (A.1)*, (A.2)* *in place of* (A.1), (A.2) *and set* $a = (\pi/2)\sqrt{d_1/f(0)}$, $b = (\pi/2)\sqrt{d_2/g(0)}$. *Let* $\mathcal{S}$ *denote the set of non-trivial solutions of* (SP-N). *Then the following properties hold true.*

(i) *If* $a \ge 1$ *and* $b \ge 1$, *then* (SP-N) *admits no non-trivial solution.*

(ii) *If* $a < 1$ *and* $b \ge 1$, *then*

$$\mathcal{S} := \{(u^*,1)\},$$

where u^* *is a solution of* (7.7).

(iii) *If* $a \ge 1$ *and* $b < 1$, *then*

$$\mathcal{S} := \{(u_*,0)\},$$

where u_* is a solution of (7.8).

(iv) *If $a < 1, b < 1$ and $a + b > 1$, then*

$$\mathcal{S} := \{(u^*, 0),\ (u_*, 1)\}.$$

(v) *If $a + b < 1$, then there exists a unique number $c \in (0, 1)$ such that*

$$\mathcal{S} := \{(u_*, 0),\ (u^*, 1),\ (u_c, c)\} \quad \textit{with} \ \ u_* < u_c < u^* \quad \textit{in} \ \ I.$$

When we study the asymptotic behavior of the smooth solution of (P-N), the comparison principle is a very important tool for the analysis. Repeating the essentially same arguments as in §6 we can sow similar results to Theorem 6.1 and Propositions 6.1-6.4 with "$a = \pi\sqrt{d_1/f(0)}$ and "$b = \pi\sqrt{d_2/g(0)}$" replaced by "$a = (\pi/2)\sqrt{d_1/f(0)}$ and "$b = (\pi/2)\sqrt{d_2/g(0)}$", respectively.

As to the global attractivity of the trivial solutions, Proposition 6.1 is still valid. It is sufficient to repeat the proof of Proposition 6.1 with use of $v(t) = \alpha(t)\cos(\pi x/2)$ in place of $v(t) = \alpha(t)\sin \pi x$.

Owing to Proposition 7.1, Proposition 6.2 holds true with use of

$$\phi^*(x) = \begin{cases} \varepsilon_1 \cos \dfrac{\pi x}{2\ell^*} & \text{for} \ \ 0 \le x \le \ell^*, \\ -\varepsilon_2 \cos \dfrac{\pi(1-x)}{2(1-\ell^*)} & \text{for} \ \ \ell^* \le x \le 1, \end{cases}$$

instead of (6.14). This proposition implies not only the global stability of $(u^*, 1)$ but also the instability of the set $\{(0, s^*) :\ 1 \ge s^* > \max\{a, 1-b\}\}$.

We will discuss the stability of the set $\{(0, s^*);\ s^* \text{ satisfies a certain condition}\}$ when $a < 1$. We take $s_0 \in (0, a)$ and $\delta > 0$ such that $s_0(1 + \delta) < a$. Define

$$s_1(t) = s_0\{1 + \delta(1 - e^{-\alpha t})\},$$

where $\alpha > 0$ is to be determined later. We also define

$$u_(t, x) = \begin{cases} \varepsilon e^{-\beta t} \cos \frac{\pi x}{2s_1(t)} & \text{for} \ \ 0 \le x \le s_1(t), \\ 0 & \text{for} \ \ s_1(t) \le x \le 1. \end{cases}$$

We will show that(u_1, s_1) is a supersolution for (P-N). For $0 \le x \le s_1(t)$,

$$\begin{aligned} u_{1,t} - d_1 u_{1,xx} - u_1 f(u_1) &\ge \left\{ d_1 \left(\frac{\pi}{2s_1(t)} \right)^2 - \beta - f(u_1) \right\} u_1 \\ &\ge \left\{ d_1 \left(\frac{\pi}{2s_0(1+\delta)} \right)^2 - \beta - f(0) \right\} u_1. \end{aligned}$$

Therefore, if we choose a sufficiently smallβ satisfying $d_1(\pi/2s_0(1+\delta))^2 \ge f(0) + \beta$, then we see $u_{1,t} - d_1 u_{1,xx} - u_1 f(u_1) \ge 0$.

Since $u_1(t, s_1(t)) = 0$ and $u_{1,x}(t, 0) = 0$, it remains to show

$$\dot{s}_1(t) = s_0 \alpha \delta e^{-\alpha t} \ge -\mu_1 u_{1,x}(t, s_1(t) - 0) + \mu_2 u_{1,x}(t, s_1(t) + 0) = \mu_1 \varepsilon e^{-\beta t} \cdot \frac{\pi}{2s_1(t)}.$$

This inequality can be shown provided that

$$\beta \ge \alpha > 0 \quad \text{and} \quad 2s_0^2 \alpha \delta \ge \varepsilon \mu_1 \pi.$$

Then it is possible to obtain the following proposition.

Proposition 7.2. *Assume $a < 1$. For any fixed $s_0 \in (0, a)$, define*

$$\phi_\varepsilon(x) = \begin{cases} \varepsilon \cos \dfrac{\pi x}{2s_0} & \text{for } \ 0 \le x \le s_0, \\ 0 & \text{for } \ s_0 \le x \le 1. \end{cases}$$

Then there exists a sufficiently small $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1)$, the smooth solution of (P-N) *with initial data (ϕ_ε, s_0) satisfies*

$$\lim_{t\to\infty} (u(t, \cdot; \phi_\varepsilon, s_0), s(t; \phi_\varepsilon, s_0)) = (0, s^*) \qquad \text{in the sense of } \Omega\text{-topology}$$

with some $s^* \in [0, a)$.

Remark 7.1. *It should be noted that a similar result holds true in case $b < 1$ as is stated in (ii) of Proposition 6.3.*

Finally it is easy to see that Proposition 6.4 holds true; so that (u_c, c) in Theorem 7.4 is unstable.

Taking account of the above discussions we can complete information on the asymptotic behavior of the smooth solution of (P-N). Correspondingly to Theorem 6.2, we can get analogous precise results on the stability and instability of all solutions of (SP-N).

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