Nonlinear Diffusion Equations with Cross-Diffusion: Reaction-Diffusion Equations Appearing in Mathematical Ecology

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Abstract. This is a survey article on reaction-diffusion systems with nonlinear diffusion which model the spatiotemporal behavior of population densities of two biological species. Such a system was introduced by Shigesada, Kawasaki and Teramoto to describe the habitat segregation and it is called SKT model. This model has strongly coupled cross-diffusion effects in addition to the usual linear diffusion. We will focus ourselves on the study of Lotka-Volterra competition model with cross-diffusion by putting Dirichlet and/or Neumann boundary conditions. Our purpose is to give some recent results on the nonstationary and stationary problems for SKT model. In particular, we will give some sufficient conditoins on the existence of global solutions of SKT model for any nonnegative initial functions. As to the stationary problem, we will state some typical results on the structure of positive solutions. The results are depending on the boundary conditions and far from complete. So we will study limiting behavior of positive soolutions when one of cross-diffusion coefficients goes to infinity.

1. Introduction

1.1. Diffusion appearing in mathematical ecology. A lot of phenomena appearing in ecology are accompanied with diffusion and most of such phenomena can be described in the form of partial differential equations with diffusion terms. Among them, reaction-diffusion equations and their related systems have attracted interests of many researchers and have been investigated very extensively. Moreover, new mathematical models with nonlinear diffusion have been proposed in recent years; so that the importance of the study of reaction-diffusion equations are increasing from the ecological view-point as well as the mathematical one.

We now briefly explain how we can formulate diffusion phenomena appearing in the area of mathematical ecology. Consider a certain biological species which lives in a fixed habitat. Let $S(\boldsymbol{x}, t)$ be a population density of the species at point $\boldsymbol{x} = (x_1, x_2, x_3)$ and time t. Here we assume that the spatiotemporal behavior of $S(\boldsymbol{x}, t)$ is governed by a diffusion equation. The total population of the species in any region V is represented by $\iiint_V S(\boldsymbol{x}, t)dx_1dx_2dx_3$. Let vector

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$\boldsymbol{J}(\boldsymbol{x},t) = (J_1(\boldsymbol{x},t), J_2(\boldsymbol{x},t), J_3(\boldsymbol{x},t))$ denote the number of individual species which pass through position $\boldsymbol{x}$ at time t in a unit time. Then the change of the population in V is identical with the number of individual species which pass through ∂V; its formulation is given by

$$\frac{d}{dt}\iiint_V S(\boldsymbol{x},t)\ dx_1dx_2dx_3 = -\iint_{\partial V} \boldsymbol{J}(\boldsymbol{x},t)\cdot \boldsymbol{n}\ d\sigma,$$

where $\boldsymbol{n}$ is the outward normal vector on ∂V and $d\sigma$ is a surface element on ∂V. Vector $\boldsymbol{J}$ is sometimes called a flow or flux. If we use the Gauss divergence theorem in the above identity, then we get

$$\frac{d}{dt}\iiint_V S(\boldsymbol{x},t)\ dx_1dx_2dx_3 = -\iiint_V \mathrm{div}\ \boldsymbol{J}(\boldsymbol{x},t)\ dx_1dx_2dx_3.$$

Since V can be chosen arbitrarily in the above expression, it follows that

$$\frac{\partial}{\partial t}S(\boldsymbol{x},t) = -\mathrm{div}\boldsymbol{J}(\boldsymbol{x},t). \tag{1.1}$$

We assume that, under Fick's law, flow $\boldsymbol{J}$ is proportional to the gradient $\nabla S(\boldsymbol{x},t)$ of population density $S(\boldsymbol{x},t)$ and it moves from a higher density area to a lower density area. In this situation,

$$\boldsymbol{J}(\boldsymbol{x},t) = -C\nabla S(\boldsymbol{x},t), \qquad C: \text{constant};$$

so that equation (1.1) is written as

$$\frac{\partial}{\partial t}S = C\Delta S, \qquad \text{with } \Delta = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}. \tag{1.2}$$

Here C is a diffusion coefficient and (1.2) is a usual linear diffusion equation.

We will explain how one can derive the population flow $\boldsymbol{J}$ of a biological species by following the arguments developed in the famous monograph of Okubo–Levin [**41**]. To simplify the arguments, a one-dimensional population model is discussed here. We denote by $S(1,t)$ (resp. $S(2,t)$) the population density of species at point 1 (resp. point 2) in a one-dimensional interval at time t. Let $k_{1\to 2}$ be the transition probability that an individual species moves from point 1 to point 2 in a short time τ and, on the contrary, let $k_{2\to 1}$ be the transition probability that an individual species moves from point 2 to point 1 in time τ. Then the population of species which move from point 1 to point 2 in time τ is equal to $k_{1\to 2}S(1,t) - k_{2\to 1}S(2,t)$. Therefore, if λ is the distance between point 1 and point 2, then the flow between these two points is given by

$$J = \frac{\lambda}{\tau}\{k_{1\to 2}S(1,t) - k_{2\to 1}S(2,t)\} = -\frac{\lambda^2}{\tau}\cdot\frac{\{k_{2\to 1}S(2,t) - k_{1\to 2}S(1,t)\}}{\lambda}. \tag{1.3}$$

Here we can consider three types of transition by taking account of what transition probabilities $k_{1\to 2}$ and $k_{2\to 1}$ of individual species are depending upon.

(i) $k_{1\to 2} = k_{2\to 1} = k(1;2)$. In this case, the transition probability is independent of staring and arriving points of transition and J in (1.3) is expressed as

$$J = -\frac{\lambda^2 k(1;2)}{\tau}\cdot\frac{S(2,t) - S(1,t)}{\lambda}.$$

Taking $(S(2,t) - S(1,t))/\lambda \to \partial S(x,t)/\partial x$, $\lambda^2 k(1;2)/\tau \to D(x)$ as $\lambda, \tau \to 0$, one can derive

$$J(x) = -D(x)\frac{\partial S}{\partial x}(x,t).$$

A higher-dimensional extension of this relation is given by

$$\boldsymbol{J}(\boldsymbol{x}) = -D(\boldsymbol{x})\nabla S(\boldsymbol{x}). \tag{1.4}$$

This is the case where J obeys Fick's law as is stated before.

(ii) $k_{1\to 2} = k(1), k_{2\to 1} = k(2)$. In this case, the transition probability between two points depends on an environment of a starting point. Then (1.3) is expressed as follows:

$$J = -\frac{\lambda^2}{\tau} \cdot \frac{k(2)S(2,t) - k(1)S(1,t)}{\lambda}.$$

Letting $\lambda, \tau \to 0$ in the above expression leads to

$$J = -\frac{\partial}{\partial x}\{D(x)S(x)\},$$

where $D(x) = \lim_{\lambda,\tau\to 0} \lambda^2 k(1)/\tau$. A higher dimensional version is given by

$$\boldsymbol{J}(\boldsymbol{x}) = -\nabla\{D(\boldsymbol{x})S(\boldsymbol{x},t)\}. \tag{1.5}$$

In this model, the transition probability is depending upon an environment around a starting point; so we can say that diffusion is caused by a "repulsive force" from the starting point. Note that (1.5) is expressed as

$$\boldsymbol{J}(\boldsymbol{x}) = -D(\boldsymbol{x})\nabla S(\boldsymbol{x},t) - (\nabla D(\boldsymbol{x}))S(\boldsymbol{x},t).$$

Then we may think that flow vector $\boldsymbol{J}(\boldsymbol{x})$ contains a drift term which moves to the smaller direction of $D(\boldsymbol{x})$ as well as the standard term obeying Fick's law.

(iii) $k_{1\to 2} = k(2), k_{2\to 1} = k(1)$. In this case, the transition probability between two points is depending upon an environment around an arriving point. In other words, one can say that diffusion is caused by an "attractive force at the arriving point. Then (1.3) is expressed as follow:

$$J = \frac{\lambda^2}{\tau} \cdot \frac{k(2)S(1,t) - k(1)S(2,t)}{\lambda} = -\frac{\lambda^2 k(1)k(2)}{\tau} \cdot \frac{1}{\lambda} \cdot \left\{\frac{S(2,t)}{k(2)} - \frac{S(1,t)}{k(1)}\right\}.$$

Taking $\lambda^2 k(1)/\tau \to D(x)$ as $\lambda, \tau \to 0$ in this expression, we are led to $J = -D(x)^2(\partial/\partial x)(S(x,t)/D(x))$. In a higher dimensional case, we have

$$\boldsymbol{J}(\boldsymbol{x}) = -D(\boldsymbol{x})^2\nabla\left(\frac{S(\boldsymbol{x},t)}{D(\boldsymbol{x})}\right). \tag{1.6}$$

So (1.6) is given by

$$\boldsymbol{J}(\boldsymbol{x}) = -D(\boldsymbol{x})\nabla S(\boldsymbol{x},t) + (\nabla D(\boldsymbol{x}))S(\boldsymbol{x},t).$$

This expression implies that flow vector $\boldsymbol{J}(\boldsymbol{x})$ has a drift term which moves to the larger direction of $D(\boldsymbol{x})$ as well as the standard term obeying Fick's law.

It should be noted here that, in diffusion of types (ii) and (iii), the flow of a biological species is depending on some forces other than the gradient of the population density itself.

1.2. Lotka-Volterra equations with nonlinear diffusion. We consider two competing species which live in the same habitat. Let u and v denote the population densities of two species. When we do not take account of diffusion effects, we assume that the dynamics of u and v is controlled by the following Lotka-Volterra system

$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v), & u(0) = u_0 > 0, \\ v_t = v(a_2 - b_2 u - c_2 v), & v(0) = v_0 > 0, \end{cases} \tag{1.7}$$

where all coefficients a_i, b_i, c_i $(i = 1, 2)$ are positive constants. Then the asymptotic behaviors of solutions of (1.7) are completely determined by conditions satisfied by coefficients ([**12**], [**64**]).
(I) If $a_2/a_1 > \max\{b_2/b_1, c_2/c_1\}$, then every solution (u, v) of (1.7) satisfies

$$\lim_{t\to\infty} (u(t), v(t)) = \left(0, \frac{a_2}{c_2}\right).$$

(II) If $a_2/a_1 < \min\{b_2/b_1, c_2/c_1\}$, then every solution (u, v) of (1.7) satisfies

$$\lim_{t\to\infty} (u(t), v(t)) = \left(\frac{a_1}{b_1}, 0\right).$$

(III) If $b_2/b_1 < a_2/a_1 < c_2/c_1$, then every solution (u, v) of (1.7) satisfies

$$\lim_{t\to\infty} (u(t), v(t)) = (u^*, v^*),$$

where $u^* = (a_1c_2 - a_2c_1)/(b_1c_2 - b_2c_1)$ and $v^* = (a_2b_1 - a_1b_2)/(b_1c_2 - b_2c_1)$.
(IV) If $c_2/c_1 < a_2/a_1 < b_2/b_1$, then two equilibrium points $(a_1/b_1, 0)$ and $(0, a_2/c_2)$ are asymptotically stable and (u^*, v^*) is an unstable equilibrium point.

However, competing biological species are not distributed spatially homogeneously in a real world. We can frequently observe habitat segregation of biological species. If we formulate such a spatially inhomogeneous model in the form of differential equations, it becomes important to rewrite and study (1.7) in the form of partial differential equations with diffusion terms. In particular, when we add diffusion terms to ordinary differential equations, it is a very attracting idea to handle ecological models from the view-point of Turing's instability that a constant stationary state becomes unstable under the presence of diffusion. The first approach to discuss (1.7) in this direction was to add linear diffusion terms like $d_1\Delta u$ and $d_2\Delta v$ to (1.7). But it was shown by Kishimoto and Weinberger [**17**] that the Lotka-Volterra competition system with linear diffusion does not have a stable non-constant stationary solution provided that the habitat is a convex set. Therefore, if we consider only linear diffusion, we cannot expect a stable non-constant stationary solution corresponding to habitat segregation of biological species.

It is necessary to take account of nonlinear diffusion in order to realize a stable non-constant stationary solution for Lotka-Volterra competition models. In this situation, a group of mathematical ecologists, Shigesada–Kawasaki–Teramoto, thought that the following three forces have crucial effects on biological movements of individual species which induce biodiffusion:

(i) a dispersive force due to random movements of individual species,
(ii) a population pressure due to interferences between individual species,
(iii) an attractive force which induces directed movements toward favorable environments.

When we discuss biodiffusion for two competing species and, in particular, we consider a population pressure, it is natural to adopt the flow of type (1.5) and formulate diffusion terms. In 1979, Shigesada–Kawasaki–Teramoto [**50**] proposed the following system of equations with nonlinear diffusion in a one-dimensional region:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2}{\partial x^2}\{(d_1 + \gamma_1 u + \alpha v)u\} + u(a_1 - b_1 u - c_1 v), \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial^2}{\partial x^2}\{(d_2 + \beta u + \gamma_2 v)v\} + v(a_2 - b_2 u - c_2 v), \end{cases} \tag{1.8}$$

where a_i, b_i, c_i, d_i $(i = 1, 2)$ are positive constants, α, β, γ_i $(i = 1, 2)$ are nonnegative constants. In this model, we are taking account of diffusion of type (ii) based on population pressures of inter- and intra-species as well as usual diffusion of type (i): α, β are called **cross diffusion coefficients** and γ_1, γ_2 are called **self-diffusion coefficients.** This system (1.8) was considered in $(x, t) \in (0, 1) \times (0, \infty)$ with boundary conditions at $x = 0, 1$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0, \qquad \frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(1, t) = 0 \tag{1.9}$$

and an initial condition

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0. \tag{1.10}$$

Shigesada–Kawasaki–Teramoto carried out numerical simulations for (1.8)–(1.10) and showed that, even if initial functions are close to constant functions, solutions for (1.8)–(1.10) become spatially inhomogeneous for sufficiently large time under some suitable conditions. In this sense, they could observe segregation phenomena for (1.8)–(1.10). These simulations also exhibited the importance of the contribution of cross-diffusion coefficients.

The work of Shigesada–Kawasaki–Teramoto [**50**] suggested by numerical simulations that the cross-diffusion induces the instability of constant stationary solution which is stable as an equilibrium of ordinary differential equations and, at the same time, it also produces a stable non-constant stationary solution. So this work became an important article which has proposed a new ecological model. Hereafter, the analysis of (1.8) and its related systems with nonlinear diffusion in higher space dimensions has attracted interests of lots of mathematicians.

1.3. Destabilization of constant solutions and bifurcation of non-constant solutions. We will call the system proposed by Shigesada–Kawasaki–Teramoto [**50**] **SKT model**. Suppose that a constant state is stable as a stationary solution of a reaction-diffusion system with linear diffusion. Then what will happen if we add an effect of cross-diffusion to this system? Numerical simulations for SKT model suggest the appearance of stable segregated solutions. We will analyze this observation from the view-point of bifurcation theory. As a one-dimensional SKT model in $\Omega = (0, 1)$, consider the following stationary problem:

$$\begin{cases} ((d + \alpha v)u)_{xx} + u(a_1 - u - cv) = 0, & x \in (0, 1), \\ d v_{xx} + v(a_2 - bu - v) = 0, & x \in (0, 1), \\ u_x(0) = u_x(1) = 0, \ v_x(0) = v_x(1) = 0. \end{cases} \tag{1.11}$$

Here $\alpha \geq 0, a_1, a_2, b, c, d > 0$ and assume that these coefficients satisfy

$$b < \frac{a_2}{a_1} < \frac{1}{c}. \tag{1.12}$$

Then it is well known that the following positive stationary solution

$$(u^*, v^*) := \left(\frac{a_1 - a_2 c}{1 - bc}, \frac{a_2 - a_1 b}{1 - bc} \right)$$

is asymptotically stable. We now consider the linearized stability of (u^*, v^*) as a stationary solution of (1.11). The eigenvalue problem for the linearized operator is given by

$$\begin{cases} -(d + \alpha v^*)\hat{u}_{xx} - \alpha u^* \hat{v}_{xx} + u^* \hat{u} + c u^* \hat{v} = \lambda \hat{u}, & x \in (0,1), \\ -d\hat{v}_{xx} + v^* \hat{v} + b v^* \hat{u} = \lambda \hat{v}, & x \in (0,1), \\ \hat{u}_x(0) = \hat{u}_x(1) = 0, \ \hat{v}_x(0) = \hat{v}_x(1) = 0. \end{cases} \tag{1.13}$$

We will use the result for the following eigenvalue problem:

$$-w_{xx} = \mu w, \quad x \in (0,1), \qquad w_x(0) = w_x(1) = 0. \tag{1.14}$$

The eigenvalues for (1.14) are equal to $\mu_n = (n\pi)^2, \ n = 0, 1, 2, \cdots$, and denote the eigenfunctions corresponding to μ_n by $\varphi_0(x) = 1$ and $\varphi_n(x) = \sqrt{2}\cos n\pi x, \ n = 1, 2, 3, \cdots$. Then $\{\varphi_n\}_{n=0}^{\infty}$ becomes a completely orthonormal system in $L^2(0,1)$. Therefore, it is possible to look for solutions of (1.13) in the form of Fourier series expansion with use of $\{\varphi_n\}$. As a result it can be proved that λ is an eigenvalue of (1.13) if and only if it satisfies

$$\det \begin{pmatrix} \lambda - (d + \alpha v^*)\mu_n - u^* & -\alpha u^* \mu_n - c u^* \\ -b v^* & \lambda - d\mu_n - v^* \end{pmatrix} = 0, \quad n = 0, 1, 2, \cdots.$$

It follows from the above expression that all eigenvalues for (1.13) are obtained by solving

$$\begin{aligned} &\lambda^2 - \{(2d + \alpha v^*)\mu_n + u^* + v^*\}\lambda + d(d + \alpha v^*)\mu_n^2 \\ &\quad + \{d(u^* + v^*) - \alpha v^*(b u^* - v^*)\}\mu_n + (1 - bc)u^* v^* = 0, \ n = 0, 1, 2, \cdots. \end{aligned} \tag{1.15}$$

Here we should recall the linearized stability theorem, which asserts that the stability of (u^*, v^*) is determined by the signs of real parts of eigenvalues for (1.13). In particular, (u^*, v^*) is asymptotically stable provided that real parts of all eigenvalues are positive. Therefore, if $\alpha = 0$ or $\alpha > 0$ and $v^* \geq b u^*$, every second order polynomial (1.15) has a negative coefficient of λ and a positive constant term; so that real part of every solution of (1.15) is positive. This fact implies the asymptotic stability of (u^*, v^*).

On the other hand, since

$$b u^* - v^* = \frac{b(a_1 - a_2 c) - (a_2 - a_1 b)}{1 - bc} = \frac{2a_1 b - a_2(1 + bc)}{1 - bc},$$

the following condition

$$2a_1 b > a_2(1 + bc) \tag{1.16}$$

implies $b u^* - v^* > 0$. Then (u^*, v^*) is unstable if there exists at least one $\mu_n, n \geq 1$, satisfying

$$d(d + \alpha v^*)\mu_n^2 + \{d(u^* + v^*) - \alpha v^*(b u^* - v^*)\}\mu_n + (1 - bc)u^* v^* < 0.$$

Note that this condition is equivalent to the following one for α:

$$\alpha v^* \{(b u^* - v^*) - d\mu_n\}\mu_n > d^2 \mu_n^2 + (u^* + v^*)d\mu_n + (1 - bc)u^* v^*.$$

Take a sufficiently small d such that

$$bu^* - v^* > d\mu_1 = d\pi^2 \tag{1.17}$$

and put $N = [\sqrt{(bu^* - v^*)/d\pi^2}]$, where $[p]$ denotes the largest integer n satisfying $n \le p$. If α satisfies

$$\alpha > \min_{1\le n\le N} \left\{ \frac{d^2\mu_n^2 + (u^* + v^*)d\mu_n + (1 - bc)u^*v^*}{v^*(bu^* - v^* - d\mu_n)\mu_n} \right\} =: \alpha^*(d), \tag{1.18}$$

then (u^*, v^*) becomes unstable. In other words, if conditions (1.16), (1.17) are satisfied and α is a sufficiently large number satisfying (1.18), then the constant stationary solution is unstable. Indeed, let d satisfying (1.17) be fixed and regard α as a parameter. Then as α becomes larger and larger, it can be shown that(u^*, v^*) is destabilized at $\alpha = \alpha^*(d)$ and that a new non-constant stationary solution bifurcate there. See Mimura–Kawasaki [**35**] for an application of the bifurcation theory to one-dimensional model (1.11).

As to non-constant stationary solutions in the case that (1.12) does not hold, Mimura–Nishiura–Tesei–Tsujikawa [**36**] proved by singular perturbation method that (1.11) has a stationary solution with sharp transition layers. As is stated later, it is also known that self-diffusion coefficients do not induce the destabilization of constant stationary solutions. Thus cross-diffusion coefficients among nonlinear diffusion coefficients play an important role in the formation of patterns.

2. Nonstationary problem for SKT model

2.1. Local solutions for nonstationary problem. We will extend the initial boundary value problem (1.8)–(1.10) for one-dimensional SKT model to the problem in a domain of general dimension. Let Ω be a bounded domain of $\boldsymbol{R}^n$ with sufficiently smooth boundary $\partial\Omega$ and consider

$$\begin{cases} u_t = \Delta\{(d_1 + \gamma_1 u + \alpha v)u\} + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta\{(d_2 + \gamma_2 v + \beta u)v\} + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0, \ v(\cdot, 0) = v_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here a_i, b_i, c_i, d_i $(i = 1, 2)$ are positive constants $\alpha, \beta, \gamma_1, \gamma_2$ are nonnegative constants and $\partial/\partial\boldsymbol{n}$ denotes the outward normal derivative on the boundary. The boundary condition in the above system is called no-flux condition and it implies that there is no migration of biological species across the boundary. Mathematically it is important to show the existence of solutions to (2.1) for given nonnegative initial functions and investigate some properties of solutions.

The first step in order to analyze initial boundary value problem (2.1) is to show the existence and uniqueness of (local) solutions for any initial data u_0 and v_0. A lot of theories, methods and ideas have been developed to solve such nonlinear partial differential equations. It is an effective idea to choose a suitable function space and deal with (2.1) as an initial value problem in the framework of this function space. We will prepare function spaces which will be used in this article.

In what follows, we write $x \in \boldsymbol{R}^m$ in place of $\boldsymbol{x} \in \boldsymbol{R}^m$. For $1 \le p < \infty$, denote by $L^p(U)$ the space of all p-th integrable functions in domain U of $\boldsymbol{R}^m$ and define its norm by

$$\|u\|_{L^p(U)} = \left(\int_U |u(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, let $L^\infty(U)$ denote the space of all essentially bounded measurable functions in U and define its norm by

$$\|u\|_{L^\infty(U)} = \text{ess sup}_{x \in U} |u(x)|.$$

Then $L^p(U)$ $(1 \le p \le \infty)$ becomes a Banach space for each $1 \le p \le \infty$. For multi-index $\rho = (\rho_1, \rho_2, \cdots, \rho_m)$ with $\rho_i \in \boldsymbol{N}$, we set $|\rho| = \rho_1 + \rho_2 + \cdots + \rho_m$ and define

$$D^\rho u = \left(\frac{\partial}{\partial x_1} \right)^{\rho_1} \left(\frac{\partial}{\partial x_2} \right)^{\rho_2} \cdots \left(\frac{\partial}{\partial x_m} \right)^{\rho_m}.$$

Sobolev space $W_p^k(U)$ is defined by $W_p^k(U) = \{u \in L^p(U);\ D^\rho u \in L^p(U)$ for all ρ satisfying $|\rho| \le k\}$ and its norm is given by

$$\|u\|_{W_p^k(U)} = \sum_{|\rho| \le k} \|D^\rho u\|_{L^p(U)}.$$

Then $W_p^k(U)$ also becomes a Banach space.

Let U be a domain of $\boldsymbol{R}^n$ and let I be an interval of $\boldsymbol{R}$. We will use $C^\lambda(U)$, which is the function space generated by Hölder continuous functions $u(x) : U \to \boldsymbol{R}$ and also use $C^{\lambda,\lambda/2}(U \times I)$, which is the function space generated Hölder continuous functions $v(x,t) : U \times I \to \boldsymbol{R}$. For detailed definitions of these function spaces, see [**23**].

For given interval I and Banach space X, $C(I; X)$ denotes the space of all functions u such that $u(t) \in X$ for every $t \in I$ and that $u(t)$ is norm-continuous in X for $t \in I$, and $C^j(I; X)$ denotes the space of all functions $u \in C(I; X)$ such that derivatives of $u(t)$ with respect to t up to order j are norm-continuous in X for $t \in I$.

Under these preparations of function spaces, we have the following theorem due to Amann [**1**], which assures the existence and uniqueness of local solutions to (2.1).

THEOREM 2.1 (Amann [**1**]). *Assume that nonnegative functions u_0 and v_0 satisfy $u_0, v_0 \in W_p^1(\Omega)$ with $p > n$. Then there exists a unique solution (u, v) of* (2.1) *satisfying*

$$u, v \in C([0,T);\ W_p^1(\Omega)) \cap C((0,T);\ W_p^2(\Omega)) \cap C^1((0,T);\ L^p(\Omega)),$$

where $[0,T)$ is a maximal existence interval of solutions. Furthermore, if

$$\sup_{0 \le t < T} \|u(t)\|_{W_p^1(\Omega)} < \infty, \qquad \sup_{0 \le t < T} \|v(t)\|_{W_p^1(\Omega)} < \infty,$$

then $T = \infty$.

Theorem 2.1 is a useful result when we deal with (2.1) in the framework of $L^p(\Omega)$. When we discuss classical solutions of (2.1), we know the following result (see the monograph of Ladyženskaja–Solonnikov–Ural'ceva [**23**]).

THEOREM 2.2. *Assume that nonnegative initial functions u_0 and v_0 satisfy $u_0, v_0 \in C^{2+\lambda}(\overline{\Omega})$ for some $\lambda > 0$ and $\partial u_0/\partial \boldsymbol{n} = \partial v_0/\partial \boldsymbol{n} = 0$ on $\partial\Omega$. Then there exists a unique solution (u, v) of* (2.1) *satisfying*

$$u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0,T)),$$

where T is a positive number corresponding to a maximal existence interval of solutions.

REMARK 2.3. We may replace the boundary conditions of (2.1) by Dirichlet boundary conditions

$$u = v = 0 \qquad \text{in } \partial\Omega \times (0, \infty).$$

Ecologically, these conditions imply that both of two biological species cannot live on the boundary. Even if boundary conditions in (2.1) are replaced by the above Dirichlet boundary conditions, one can derive similar existence and uniqueness results to Theorems 2.1 and 2.2.

We have given two typical existence theorems of local solutions to nonlinear parabolic equations. In order to discuss the extension of a local solution, we need a priori estimates of the solution in suitable function spaces. But it is difficult to get estimates of solutions of differential equations with nonlinear diffusion like SKT model. We will explain where we meet a difficulty in getting good estimates. The diffusion term of the first equation in (2.1) is expanded as follows:

$$\begin{aligned}\Delta\{(d_1 + \gamma_1 u + \alpha v)u\} &= \operatorname{div}\{(d_1 + 2\gamma_1 u + \alpha v)\nabla u + \alpha u \nabla v\} \\ &= (d_1 + 2\gamma_1 u + \alpha v)\Delta u + 2(\gamma_1 \nabla u + \alpha \nabla v, \nabla u) + \alpha u \Delta v.\end{aligned}$$

Hence the first equation of (2.1) is given by

$$u_t = (d_1 + 2\gamma_1 u + \alpha v)\Delta u + \sum_{i=1}^{n} \tilde{a}_i(x,t) u_{x_i} + \tilde{a}(x,t)u, \tag{2.2}$$

where $\tilde{a}_i$ and $\tilde{a}$ are given by

$$\tilde{a}_i(x,t) = 2\gamma_1 u_{x_i}(x,t) + 2\alpha v_{x_i}(x,t) \text{ and } \tilde{a}(x,t) = \alpha\Delta v(x,t) + a_1 - b_1 u(x,t) - c_1 v(x,t).$$

Therefore, if u_0 is nonnegative, then the solution $u(x,t)$ of remains nonnegative as long as it exists. Indeed, assume that the solution (u, v) of (2.1) exists for $0 \le t \le T$ and it takes a negative value somewhere in $[0, T]$. If we consider $w(x,t) = e^{-kt}u(x,t)$ instead of u, then w also takes a negative value at some point $(x,t) \in \overline{\Omega} \times [0, T]$ and it satisfies

$$w_t = (d_1 + 2\gamma_1 u + \alpha v)\Delta w + \sum_{i=1}^{n} \tilde{a}_i(x,t) w_{x_i} + (\tilde{a}(x,t) - k)w. \tag{2.3}$$

Here $\tilde{a}(x,t) - k < 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$ provided that k is sufficiently large. Since w takes a negative value, we have $\min_{(x,t)\in\overline{\Omega}\times[0,T]} w(x,t) = m_* < 0$. Therefore, if w takes its negative minimum at $(x_*, t_*) \in \Omega \times (0, T]$, then

$$w_t(x_*, t_*) \le 0, \quad \Delta w(x_*, t_*) \ge 0, \quad w_{x_i}(x_*, t_*) = 0, \ i = 1, 2, \cdots, n;$$

so that it follows from (2.3) that

$$0 \ge w_t(x_*, t_*) \ge (\tilde{a}(x_*, t_*) - k)w(x_*, t_*) = (\tilde{a}(x_*, t_*) - k)m_* > 0.$$

This is a contradiction. Even if w takes its negative minimum at a point on the boundary $\partial\Omega$, we can also derive a contradiction. Consequently, we see that the minimum of w is nonnegative. Therefore, $u(x,t) \ge 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$. Similarly, one can prove from the second equation that $v(x,t) \ge 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$.

The difficult thing is to estimate u and v from above. Now assume that u takes its positive maximum M at $(x^*, t^*) \in \Omega \times (0, T]$. Since

$$u_t(x^*, t^*) \geq 0, \quad \Delta u(x^*, t^*) \leq 0, \quad u_{x_i}(x^*, t^*) = 0, \ i = 1, 2, \cdots, n,$$

it follows from (2.2) that

$$0 \leq u_t(x^*, t^*) \leq \tilde{a}(x^*, t^*) u(x^*, t^*) = \tilde{a}(x^*, t^*) M.$$

Hence $\tilde{a}(x^*, t^*) \geq 0$; so that M must satisfy

$$b_1 M = b_1 u(x^*, t^*) \leq b_1 u(x^*, t^*) + c_1 v(x^*, t^*) \leq a_1 + \alpha \Delta v(x^*, t^*).$$

If a cross-diffusion coefficient satisfies $\alpha = 0$, then we can get $M \leq a_1/b_1$, while, if $\alpha > 0$, we cannot directly derive any estimate for M from the above relation. So the difficulty of the problem heavily depends on the existence of cross-diffusion coefficients.

As is stated in the preceding paragraph, when cross-diffusion coefficients α and β in (2.1) are positive, it is a very important problem to derive a priori estimates of solutions and, therefore, to show the existence of global solutions. But it is also a very difficult problem.

2.2. Existence of global solutions. We do not have enough satisfactory results about the existence of global solutions of (2.1) when both of cross-diffusion coefficients α and β are positive. As the first result, Kim [**16**] proved the existence of global solutions in the case of $n = 1, d_1 = d_2$ and $\gamma_1 = \gamma_2 = 0$. Then for $n = 2$, Yagi [**58**], [**59**] proved the existence of global solutions under the assumption $0 < \alpha < 8\gamma_1$ and $0 < \beta < 8\gamma_2$. This sufficient condition was weakened by Ichikawa–Yamada [**14**], who showed the global existence under the conditions $\alpha\beta < 64\gamma_1\gamma_2$ or $0 < \alpha\beta = 64\gamma_1\gamma_2$. In higher dimensional space Deuring [**10**] proved that (2.1) has a global solution provided that α and β are small (depending on initial data). In this manner, some restrictions are imposed on the amplitude of cross-diffusion coefficients α, β in order to derive the existence of global solutions to (2.1) for any nonnegative initial functions in case $n \geq 2$.

In this article, we want to discuss the existence of global solutions without any restrictions on the amplitude of initial data and cross-diffusion coefficients. For this purpose, we will consider the system in the following simplified form:

$$\text{(P)} \quad \begin{cases} u_t = \Delta\{(d_1 + \gamma_1 u + \alpha v)u\} + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta\{(d_2 + \gamma_2 v)v\} + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 \geq 0, \ v(\cdot, 0) = v_0 \geq 0 & \text{in } \Omega, \end{cases}$$

where $\alpha > 0$. Lou–Ni–Wu [**32**] published the first paper discussing the existence of global solutions for the above SKT model. In 1998, they showed that (P) has a unique global solution

$$u, v \in C([0, \infty); \ W_p^1(\Omega)) \cap C^\infty((0, \infty); \ C^\infty(\Omega))$$

for any $u_0, v_0 \in W_p^1(\Omega), p > 2$, in case of $n = 2, \gamma_1 \geq 0$ and $\gamma_2 \geq 0$. Hereafter, the global existence result was extended to case $n \geq 3$ by Choi-Lui-Yamada [**5, 6**] and Tuộc [**52, 53**] under the condition that self-diffusion coefficient γ_1 is positive.

In this article, we will state fundamental ideas to derive a priori estimates of solutions by following the arguments used in [**6**]. In what follows, initial data u_0 and v_0 are assumed to satisfy

$$\text{(A)} \qquad u_0 \geq 0,\ v_0 \geq 0, \quad u_0, v_0 \in C^{2+\lambda}(\overline{\Omega}) \quad \text{and} \quad \frac{\partial u_0}{\partial \boldsymbol{n}} = \frac{\partial v_0}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega$$

with a positive constant λ.

We will divide the arguments into two cases; the case that the second equation of (P) has a linear diffusion term, i.e., $\gamma_2 = 0$, and the other case that the equation has a nonlinear diffusion term, i.e., $\gamma_2 > 0$.

(a) Case $\gamma_2 = 0$

Theorem 2.4 (Choi-Lui-Yamada[**6**]). *If initial data u_0 and v_0 satisfy* (A)*, then* (P) *has a unique solution (u, v) in the class $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0, \infty))$.*

It is important to derive a priori estimates of the solution (u, v) in order to prove this theorem. As is sated in subsection **2.1**, it follows from the maximum principle for parabolic equations ([**44**]) that the following result holds true.

Lemma 2.5. *Let (u, v) be any solution of* (P) *in $[0, T]$. Then it holds that*

$$u(x,t) \geq 0 \qquad \text{and} \qquad m \geq v(x,t) \geq 0, \quad (x,t) \in Q_T := \overline{\Omega} \times [0, T],$$

where $m = \max\{\|v_0\|_\infty, a_2/b_2\}$ with $\|w\|_\infty = \max_{x\in\overline{\Omega}} |w(x)|$.

Lemma 2.5 assures the boundedness of v. We also need an estimate of u from above. In order to estimate $\max_{(x,t)\in Q_T} |u(x,t)|$, we will rewrite the first equation as follows:

$$u_t = \sum_{i,j=1}^{n} (a^*_{ij}(x,t) u_{x_j})_{x_i} + \sum_{i=1}^{n} (a^*_i(x,t) u)_{x_i} - a^*(x,t) u, \tag{2.4}$$

where

$$a^*_{ij}(x,t) = (d_1 + 2\gamma_1 u(x,t) + \alpha v(x,t))\delta_{ij}, \quad i, j = 1, 2, \cdots, n,$$

and

$$a^*_i(x,t) = \alpha v_{x_i}(x,t), \quad i = 1, 2, \cdots, n, \quad a^*(x,t) = -(a_1 - b_1 u(x,t) - c_1 v(x,t)).$$

We will apply the maximum principle [**23**, Theorem 7.1, p.181] to parabolic equations of the form (2.4). By this result, if coefficients a^*_{ij}, a^*_i, a satisfy appropriate conditions, then one can get an estimate of $\|u\|_{L^\infty(Q_T)}$ for any weak solution u of (2.4) provided that u satisfies

$$\|u\|_{V_2(Q_T)} := \sup_{0\leq t\leq T} \|u(t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)} < \infty.$$

So making use of this maximum principle we can prove the following result (for details, see [**5**, Lemma 3.1]).

Lemma 2.6. *Let (u, v) be any solution of* (P) *in $[0, T]$. If it satisfies $\|u\|_{L^q(Q_T)} \leq C_{q,T}$ and $\|u\|_{V_2(Q_T)} \leq C_T$ for some $q > (n+2)/2$ with positive constants $C_{q,T}$ and C_T, then there exists a positive constant M_T such that*

$$\|u\|_{L^\infty(Q_T)} \leq M_T.$$

Lemma 2.7. *Let (u, v) be any solution of* (P) *in $[0, T]$. Then it holds that*

$$\sup_{0\leq t\leq T} \|u(t)\|_{L^1(\Omega)} \leq \|u\|_{L^1(\Omega)} e^{a_1 T} \qquad \text{and} \qquad \|u\|^2_{L^2(Q_T)} \leq \|u\|_{L^1(\Omega)} e^{a_1 T}/b_1.$$

PROOF. Integrating the first equation of (P) in Ω we see from the divergence theorem that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(x,t)dx &= \int_\Omega \Delta\{(d_1+\gamma_1 u+\alpha v)u\}dx + \int_\Omega u(a_1-b_1u-c_1v)dx \\ &= \int_{\partial\Omega}\frac{\partial}{\partial \boldsymbol{n}}\{(d_1+\gamma_1 u+\alpha v)u\}d\sigma + \int_\Omega u(a_1-b_1u-c_1v)dx \\ &\le a_1\int_\Omega u\ dx - b_1\int_\Omega u^2\ dx.\end{aligned}$$

By integrating the above inequality and using Gronwall's inequality it follows that

$$\begin{aligned}\|u(t)\|_{L^1(\Omega)} + b_1\int_0^t \|u(s)\|^2_{L^2(\Omega)}ds &\le \|u_0\|_{L^1(\Omega)} + a_1\int_0^t \|u(s)\|_{L^1(\Omega)}ds \\ &\le \|u_0\|_{L^1(\Omega)}e^{a_1T}\end{aligned}$$

for $0\le t\le T$. Thus one can get desired estimates of this lemma. $\square$

Lemma 2.7 is a starting point to derive estimates for $\|u\|_{L^q(Q_T)}$ and $\|u\|_{V_2(Q_T)}$. Multiplying the first equation of (P) by u^{q-1} with $q>1$ and integrating the resulting expression we obtain

$$\begin{aligned}&\frac{1}{q}\frac{d}{dt}\int_\Omega u^q\ dx = \int_\Omega u^{q-1}u_t\ dx \\ &= \int_\Omega u^{q-1}\nabla\{(d_1+2\gamma_1u+\alpha v)\nabla u+\alpha u\nabla v\}dx + \int_\Omega u^q(a_1-b_1u-c_1v)dx \\ &= -(q-1)\int_\Omega (d_1+2\gamma_1 u+\alpha v)u^{q-2}|\nabla u|^2\ dx \\ &\quad -(q-1)\alpha\int_\Omega u^{q-1}\nabla u\cdot\nabla v\ dx + \int_\Omega u^q(a_1-b_1u-c_1v)dx.\end{aligned} \tag{2.5}$$

By virtue of $\gamma_1>0$ in the right-hand side of (2.5) one can see

$$\begin{aligned}\int_\Omega (d_1+2\gamma_1u+\alpha v)u^{q-2}|\nabla u|^2\ dx &\ge 2\gamma_1\int_\Omega u^{q-1}|\nabla u|^2\ dx \\ &= \frac{8\gamma_1}{(q+1)^2}\int_\Omega |\nabla(u^{(q+1)/2})|^2\ dx.\end{aligned}$$

It follows from $u,v\ge 0$ that $\max_{u,v\ge 0} u^q(a_1-b_1u-c_1v)\le \max_{u\ge0}u^q(a_1-b_1u) = m_q^*<\infty$. hence

$$\int_\Omega u^q(a_1-b_1u-c_1v)dx \le m_q^*|\Omega|,$$

where $|\Omega|$ denotes the volume of Ω. Furthermore, we note

$$-\int_\Omega u^{q-1}\nabla u\cdot\nabla v\ dx = -\frac{1}{q}\int_\Omega \nabla(u^q)\cdot\nabla v\ dx = \frac{1}{q}\int_\Omega u^q\Delta v\ dx.$$

Therefore, Integration of (2.5) with respect to t leads us to

$$\begin{aligned}&\|u(t)\|^q_{L^q(\Omega)} + c_0\|\nabla(u^{(q+1)/2})\|^2_{L^2(Q_t)} \\ &\le \|u_0\|^q_{L^q(\Omega)} + qm_q^*|\Omega|t + (q-1)\alpha\int_{Q_t}u^q\Delta v\ dxds\end{aligned} \tag{2.6}$$

with a positive constant c_0. We will make use of the maximal regularity (see, e.g., [**2**]) to estimate integral $\int_{Q_t}u^q\Delta v\ dxds$ appearing in the right-hand side of (2.6). If we combine this estimate with Lemmas 2.5, 2.7 and Sobolev's embedding theorem, we can show the following result. For the detailed process of estimation, see [**6**].

PROPOSITION 2.8. *Let* (u, v) *be any solution of* (P) *in* $[0, T]$. *Then for every* $q \geq 1$ *there exists positive constant* $C_{q,T}$ *and* C_T *such that*

$$\|u\|_{L^q(Q_T)} \leq C_{q,T}, \quad \|u\|_{V_2(Q_T)} \leq C_T.$$

Owing to the estimates in Proposition 2.8, Lemma 2.6 is applicable to get

$$\max_{(x,t)\in Q_T} |u(x,t)| \leq M_T^*.$$

This estimate together with Lemma 2.5 enables us to show the boundedness of u and v. We can also derive estimates of derivatives of u and v successively and prove Theorem 2.4 (for details, see [**6**]).

REMARK 2.9. Wang [**54**] has shown that an analogous result as Theorem 2.4 holds true for the system with more general reaction terms.

(b) Case $\gamma_2 > 0$

We will consider the case when the diffusion term of the second equation of (P) has a self-diffusion coefficient ($\gamma_2 > 0$). In this case, Lemmas 2.5, 2.6 and 2.7 hold true. Moreover, (2.6) is also valid, but it is difficult to estimate Δv this time. So we use

$$\begin{aligned} &\|u(t)\|^q_{L^q(\Omega)} + c_0\|\nabla(u^{(q+1)/2})\|^2_{L^2(Q_t)} \\ &\leq \|u_0\|^q_{L^q(\Omega)} + qm^*_q|\Omega|t - (q-1)\alpha \int_{Q_t} \nabla(u^q)\cdot\nabla v \, dxds. \end{aligned} \tag{2.7}$$

Choi–Lui–Yamada [**6**] estimated $\int_{Q_t} \nabla(u^q)\cdot\nabla v \, dxds$ by using Sobolev's embedding theorem and, consequently, succeeded in proving that Proposition 2.8 is still valid for every q satisfying $1 \leq q \leq 2(n+1)/(n-2)^+$. If we apply Lemma 2.6 under this restriction, then q for $\|u\|_{L^q(Q_T)}$-estimate must satisfy

$$\frac{n+2}{2} < q \leq \frac{2(n+1)}{(n-2)^+}.$$

This inequality brings about the restriction $1 \leq n \leq 5$. Therefore, it was required to put restriction $1 \leq n \leq 5$ in [**6**] in order to show the existence of global solutions for (P).

REMARK 2.10. Using the theory of semi-groups, D. Le–L. Nguyen–T. Nguyen [**26**] discussed (P) in the framework of $W^1_p(\Omega)\times W^1_p(\Omega)$ with $p > n$ and showed, independently of the work of Choi–Lui–Yamada [**6**], that (P) has a global solution in case $1 \leq n \leq 5$.

In 2007 Tuộc presented the PhD thesis and succeeded in weakening the spatial restrictions for the existence of global solutions to $1 \leq n \leq 9$ when γ_2 is positive. His idea is to derive $L^r(Q_t)$-estimate of ∇v in $\int_{Q_t} \nabla(u^q)\cdot\nabla v \, dxds$ directly from the second equation of (P) ([**53**]). As a result, he showed that Proposition 2.8 is still valid for every q satisfying $1 \leq q \leq 4(n+1)/(n-2)^+$. Combination of this condition with the assumption of Lemma 2.6 allows us to put condition $1 \leq n \leq 9$. Summarizing these considerations we have the following result.

THEOREM 2.11 (Tuộc[**53**]). *Assume* $\gamma_2 > 0$ *and* $1 \leq n \leq 9$. *If initial data* u_0 *and* v_0 *satisfy* (A), *then* (P) *admits a unique solution* (u,v) *satisfying* $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0,\infty))$.

There is also a result of Tuộc which asserts the existence of global solutions of (P) without any restrictions on space dimension, but with some restrictions on the amplitude of cross-diffusion coefficients.

THEOREM 2.12 (Tuộc[**52**]). *Assume* $n \geq 1$ *and that initial data* u_0 *and* v_0 *satisfy* (A). *If* $\alpha < 2\gamma_2$ *or* $\alpha = 2\gamma_2, d_1 \leq d_2$, *then* (P) *has a unique solution* (u,v) *satisfying* $u, v \in C^{2+\lambda,(2+\lambda)/2}(\overline{\Omega} \times [0,\infty))$.

In the proof of Theorem 2.12, L^p-estimate is not used. Introducing a new function w of the form $w = G(u,v)$, Tuộc applied the maximum principle ([**44**]) to a parabolic equation satisfied by w. There is also another research of D. Le–T. Nguyen [**27**] for general $n \geq 1$. Their approach is completely different, but Theorem 2.12 improves the result of [**27**]. See also Li–Zhao[**28**] who proved the existence of global solutions for general space dimension with some strong restrictions on cross-diffusion coefficients.

(c) Future issues

In case $\gamma_2 > 0$, we have required some restrictions on the space dimension (Theorem 2.11) or some restrictions on cross-diffusion coefficient (Theorem 2.12) in order to construct global solutions of (P). It is our future issue to show the global existence without these restrictions.

2.3. Related topics. In this subsection we will give some topics which we have not discussed previously.

(a) Bounded global solutions and asymptotic behaviors

Although we have stated the existence of global solutions, we do not have enough information about the uniform boundedness of solutions and their asymptotic behaviors as $t \to \infty$. In case $n = 1$, Shim [**51**] showed the uniform boundedness of solutions of (P). He also proved that, if $b_2/b_1 < a_2/a_1 < c_2/c_1$ and d_1, d_2 are sufficiently large, then the solution (u,v) of (P) satisfies $\lim_{t\to\infty}(u(t),v(t)) = (u^*, v^*)$. In case $n = 2$, D. Le [**24**] and Yagi [**58, 59**] generalized the result of Lou-Ni-Wu [**32**] and proved the existence of global attractors. Moreover, D. Le–L. Nguyen–T. Nguyen [**26**] also discussed the existence of global attractors in case $1 \leq n \leq 5$ (see also [**18**], [**25**]). Among them, it should be noted that Yagi [**60**] has recently succeeded in constructing an exponential attractor (see also [**61**]).

We will be able to expect the development of studies of global attractors and exponential attractors which are related with the dynamical theory for solutions of SKT model (P).

(b) Case $\gamma_1 = 0$

In the proof the existence of global solutions of (P), the positivity of a cross-diffusion coefficient γ_1 has played an important role in case $n \geq 3$. However, in case of $\alpha > 0$ and $\gamma_1 = 0$ in (P), it is difficult to show the existence of global solutions even if the second equation has a linear diffusion term ($\gamma_2 = 0$). Consider the following modified problem:

$$
(2.8)\qquad \begin{cases} u_t = \Delta\{(d_1+\alpha v)u\} + u(a_1 - b_1 u^{\rho_1} - c_1 v) & \text{in } \Omega \times (0,\infty), \\ v_t = d_2 \Delta v + v(a_2 - b_2 u^{\rho_2} - c_2 v) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \geq 0,\ v(\cdot,0) = v_0 \geq 0 & \text{in } \Omega, \end{cases}
$$

where $a_i, b_i, c_i, d_i, \rho_i (i = 1,2), \alpha$ are positive constants. This system was investigated by Pozio–Tesei [**43**], Redlinger [**45**] and Yamada [**62**]. Among them, it was shown in [**45**] and [**62**] that (2.8) admits a global solution under the condition $\rho_1 > \rho_2$. But SKT model (P) is not included in this case. So it is an interesting open problem to show the existence of global solutions of (P) in case of $\alpha > 0$ and $\gamma_1 = 0$. Here we should note the work of Chen–Jüngel [**3, 4**], which asserts that, as far as weak solutions of (P) are concerned, global (weak) solutions can be constructed even if $\gamma_1 = \gamma_2 = 0$.

(c) Prey-predator model

Let u be the population density of a prey species and let v be the population density of a predator species. Taking account of nonlinear diffusion depending on population densities we consider the following model:

$$
(2.9)\qquad \begin{cases} u_t = \Delta\{(d_1 + \gamma_1 u + \alpha v)u\} + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta\{(d_2 + \gamma_2 v)v\} + v(a_2 + b_2 u - c_2 v) & \text{in } \Omega \times (0,\infty). \end{cases}
$$

The existence of global solutions for the initial boundary value problem related to (2.9) is an important open problem.

3. Stationary problem for SKT model

3.1. Stationary problem – Neumann conditions–. We will consider the following stationary problem associated with (2.1) :

$$
\text{(SPN)}\quad \begin{cases} \Delta\{(d_1 + \gamma_1 u + \alpha v)u\} + u(a_1 - b_1 u - c_1 v) = 0 & \text{in } \Omega, \\ \Delta\{(d_2 + \gamma_2 v + \beta u)v\} + v(a_2 - b_2 u - c_2 v) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega, \end{cases}
$$

As nonnegative solutions, (SPN) has $(0,0), (a_1/b_1, 0), (0, a_2/b_2)$ and, moreover, a positive solution (u^*, v^*) provided that coefficients a_i, b_i, c_i satisfy $\min\{b_2/b_1, c_2/c_1\} < a_2/a_1 < \max\{b_2/b_1, c_2/c_1\}$. Our purpose is to answer the following questions:

- Under what conditions does (SPN) admit a non-constant positive solution?
- What is the profile of a non-constant positive solution?
- Is a non-constant positive solution stable?

Consider the case of space dimension $n = 1$. We have shown that non-constant positive solutions bifurcate as α becomes large when coefficients a_i, b_i, c_i satisfy (III) in subsection **1.2** and some additional conditions ([**35**]). Furthermore, Mimura [**34**] also constructed large amplitude solutions near constant solutions. When (III) is not satisfied, Mimura–Nishiura–Tesei–Tsujikawa [**36**] used singular perturbation technique to construct non-constant solutions with transition layers in the interior of Ω or near the boundary when $\alpha > 0, \gamma_1 = \gamma_2 = \beta = 0$ and $a_1/d_1, d_2/d_1$ are sufficiently small. Moreover, the stability of such non-constant solutions was proved by Kan-on [**15**]. Taking account of these results and the fact that (u^*, v^*) is unstable, we can infer that positive solutions with transition layers are obtained not as the first bifurcating solutions from (u^*, v^*), but as the secondary (or more than secondary) bifurcating solutions. As for the existence and stability properties of positive solutions with transition layers and spikes, see the results of Wu [**55, 56**]. Consequently, the structure of non-constant positive solutions of (SPN) is very complicated even in one-dimensional case. We also refer to the paper of Iida–Mimura–Ninomiya [**13**], which exhibits the detailed structure of solutions by numerical simulations in case $n = 1$.

We next study when non-constant positive solutions appear in general space dimension. Our discussions are restricted in the case that coefficients a_i, b_i, c_i satisfy

$$\text{(III)} \qquad \frac{b_2}{b_1} < \frac{a_2}{a_1} < \frac{c_2}{c_1}$$

as in subsection **1.2**. We are interested with this case because it is well known that (u^*, v^*) is a globally asymptotically stable solution under condition (III) if $\alpha = \beta = \gamma_1 = \gamma_2 = 0$. So it is a very interesting problem to investigate what kind of effects are induced by cross-diffusion upon the structure of positive solutions. Lou and Ni proved the following result for this problem.

THEOREM 3.1 (Lou-Ni [**30**]). *Under condition* (III) *the following results hold true.*

(i) *There exists a positive constant* $C_1 = C_1(a_i, b_i, c_i, d_i, \alpha, \beta)$ *such that* (SPN) *has no non-constant positive solution if* $\max\{\gamma_1, \gamma_2\} \geq C_1$.

(ii) *Assume* γ_1 *and* $\gamma_2 > 0$. *Then there exists a positive constant* $C_2 = C_2(a_i, b_i, c_i, \alpha, \beta, \gamma_i)$ *such that* (SPN) *has no non-constant positive solution if* $\max\{d_1, d_2\} \geq C_2$.

The important point in Theorem 3.1 is that there exists no non-constant solution provided that at least one of diffusion coefficients d_1, d_2 or self-diffusion coefficients γ_1, γ_2 is sufficiently large. On the other hand, if we focus on cross-diffusion coefficients α, β, then we have the following result.

THEOREM 3.2 (Lou-Ni [**30**]). *Assume condition* (III). *Then there exists a positive constant* $C_3 = C_3(a_i, b_i, c_i)$ *such that, if one of conditions* (i)–(iii) *is satisfied, then* (u^*, v^*) *is a unique positive solution of* (SPN).

$$\text{(i)} \qquad \max\left\{\frac{\alpha}{d_i}, \frac{\beta}{d_i}, \frac{\gamma_j}{d_i};\ i, j = 1, 2\right\} \leq C_3,$$

$$\text{(ii)} \qquad \max\left\{\frac{\beta}{d_1}\left(1 + \frac{\alpha}{d_1}\right), \frac{\alpha}{d_2}\left(1 + \frac{\beta}{d_2}\right)\right\} \leq C_3,$$

$$\text{(iii)} \qquad \max\left\{\frac{\beta}{\sqrt{d_1 d_2}}\left(1 + \frac{\alpha}{d_1}\right), \frac{\alpha}{\sqrt{d_1 d_2}}\left(1 + \frac{\beta}{d_2}\right)\right\} \leq C_3.$$

Theorem 3.2 implies that, under condition (III), (SPN) has no non-constant positive solution if cross-diffusion coefficients α, β are small compared with d_1, d_2. So we will study what will happen for large α or β. Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of $-\Delta$ with homogeneous Neumann boundary condition and let m_k be the multiplicity of μ_k. Then the following theorem was shown by Lou and Ni.

THEOREM 3.3 (Lou–Ni [**30**]). *Assume that there exists an integer* $k \in \boldsymbol{N}$ *such that* m_k *is an odd number. If* a_i, b_i, c_i *satisfy* $a_1/a_2 > (b_1/b_2 + c_1/c_2)/2$, *then* *there exist positive constants* $C_4 = C_4(a_i, b_i, c_i) < C_5 = C_5(a_i, b_i, c_i)$ *and* $\Lambda_1 = \Lambda_1(a_i, b_i, c_i, d_i, \beta, \gamma_i)$ *such that* (SPN) *has a non-constant positive solution for* α, d_2 *and* γ_2 *satisfying* $\alpha \geq \Lambda_1$ *and* $d_2 + 2v^*\gamma_2 \in (C_4, C_5)$.

Let β be fixed. Theorem 3.3 asserts that, as α becomes large, non-constant positive solutions appear under suitable conditions. Here it should be noted that an analogous result as Theorem 3.3 still holds true even if condition $a_1/a_2 > (b_1/b_2 + c_1/c_2)/2$ is replaced by another condition $a_2/a_1 > (b_2/b_1 + c_2/c_1)/2$. Moreover, even if condition (IV) is assumed in place of (III), one can prove the existence of non-constant positive solutions (see [**30**]).

We have shown that (SPN) has at least one non-constant positive solution if one of cross-diffusion coefficients is sufficiently large. However, it is difficult to get precise information on the profiles or the number of positive solutions. In order to get further information on non-constant positive solutions, there is a useful idea to let one of cross-diffusion coefficients tend to infinity and derive a limit system which is satisfied by limit functions of positive solutions. By studying this limit system, it will be possible to get satisfactory information on positive solutions of the original system (when one of cross-diffusion coefficients is sufficiently large). In what follows, we will fix $\beta \geq 0$ and make α sufficiently large by setting $\gamma_1 = \gamma_2 = 0$ for the sake of simplicity. Then our system is written as follows:

$$(3.1)\qquad \begin{cases} \Delta\{(d_1 + \alpha v)u\} + u(a_1 - b_1 u - c_1 v) = 0 & \text{in } \Omega, \quad \partial u/\partial \boldsymbol{n} = 0 \text{ on } \partial\Omega, \\ \Delta\{(d_2 + \beta u)v\} + v(a_2 - b_2 u - c_2 v) = 0 & \text{in } \Omega, \quad \partial v/\partial \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{cases}$$

Theorem 3.3 assures the existence of non-constant positive solutions of (3.1) for sufficiently large α. Denote such a positive solution by (u_α, v_α) and consider the asymptotic behavior of (u_α, v_α) as $\alpha \to \infty$. The first step of analysis is to obtain estimates of (u_α, v_α), which are uniform with respect to α. Lou and Ni showed the following theorem.

Theorem 3.4 (Lou–Ni [**31**]). *Assume* $1 \leq n \leq 3$ *and let* $\eta > 0$ *be an arbitrary number. Then for any* $d_2 \geq \eta$ *and* $0 < b_2 \leq 1/\eta$ *there exists a positive number* $\delta_0 = \delta_0(\eta, a_i, b_1, c_2)$ *such that, if* $\beta/d_2 \leq \delta_0$, *then every solution* (u, v) *of* (3.1) *satisfies* $\|u\|_\infty \leq 1/\delta_0$ *and* $\|v\|_\infty \leq 1/\delta_0$.

In this theorem it should be noted that estimates of positive solutions are independent of α. Hence combining these estimates with regularity estimates for elliptic equations and Sobolev's embedding theorem ([**11**]) one can discuss the asymptotic behavior of positive solutions as $\alpha \to \infty$.

Let $\beta \geq 0$ be fixed and let $\{\alpha_n\}$ be any sequence satisfying $\alpha_n \to \infty$. By (u_n, v_n) we denote any non-constant positive solution of (3.1) with $\alpha = \alpha_n$.

Theorem 3.5 (Lou–Ni [**31**]). *Assume* $1 \leq n \leq 3$, $a_2/a_1 \neq b_2/b_1$, $a_2/a_1 \neq c_2/c_1$ *and* $a_2/d_2 \neq \mu_k$ *for* $k = 1, 2, \cdots$. *Then there exists a positive number* $\delta_1 = \delta_1(a_i, b_i, c_i, d_i) > 0$ *such that, if* $0 \leq \beta \leq \delta_1$, *then* $\{(u_n, v_n)\}$, *by choosing a suitable subsequence, satisfies either* (i) *or* (ii).

(i) $\lim_{n\to\infty}(u_n, \alpha_n v_n) = (u, w)$ *uniformly in* Ω, *where* (u, w) *is a pair of positive functions satisfying*

$$(3.2)\qquad \begin{cases} \Delta\{(d_1 + w)u\} + u(a_1 - b_1 u) = 0 & \text{in } \Omega, \quad \partial u/\partial \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \\ \Delta\{(d_2 + \beta u)w\} + w(a_2 - b_2 u) = 0 & \text{in } \Omega, \quad \partial w/\partial \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

(ii) $\lim_{n\to\infty}(u_n, v_n) = (v/\tau, v)$ *uniformly in* Ω, *where* τ *is a positive constant,* v *is a positive function and they satisfy*

$$(3.3)\qquad \begin{cases} d_2\Delta v + v(a_2 - c_2 v) - b_2\tau = 0 & \text{in } \Omega, \quad \partial v/\partial \boldsymbol{n} = 0 \text{ on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{v}\left(a_1 - \frac{b_1\tau}{v} - c_1 v\right) dx = 0. \end{cases}$$

According to Theorem3.5, there are two types of asymptotic behaviors of positive solutions (u_α, v_α) as $\alpha \to \infty$. In case of (i), αv_α converges to a positive function, while v_α converges to 0 as $\alpha \to \infty$. In case of (ii), product $u_\alpha v_\alpha$ converges to a positive constant τ. Therefore, this fact implies the possibility that both u_α and v_α coexist and that they become spatially inhomogeneous for large α. Moreover, it was shown by Lou–Ni [**31**] that both properties (i) and (ii) can be realized under appropriate conditions. See also the works of Ni [**39, 40**] which discuss non-constant positive solutions of (SPN) and their profiles.

Remark 3.6. It is possible to remove the spatial restrictions $1 \le n \le 3$ in Theorem 3.5 if we get uniform a priori estimates of positive solutions independently of n in Theorem 3.4. The restriction comes from the assumption of Harnack's inequality for nonnegative solutions to elliptic equation $\Delta w + c(x)w = 0$. Its proof requires the condition $c \in L^p(\Omega)$ with $p > \max\{n/2, 1\}$(see [**29**]).

In the study of limit systems, it is important to investigate whether the limit system (3.3) in Theorem 3.5 has non-constant positive solutions or not. In case of $n = 1$, we have better understanding on the structure of solutions of (3.3) (see the works of Lou–Ni–Yotsutani [**33**] and Wu–Xu [**57**]). In particular, Lou–Ni–Yotsutani [**33**] have obtained precise results with use of the theory of elliptic functions. Furthermore, Yotsutani [**65**] has recently got complete information on the structure of positive solutions of (3.3). On the other hand, when d_1 and d_2 are very small, Wu–Xu [**57**] have discussed the existence of positive solutions with spikes and their instability properties. For detailed information, refer to these articles.

3.2. Stationary problem – Dirichlet boundary conditions–. When we study the nonstationary problem for SKT model, ideas and methods of analysis do not depend on boundary conditions: they are applicable to both Neumann boundary conditions and Dirichlet boundary conditions. However, the situation is totally different in the analysis of stationary problems. We will replace Neumann boundary conditions in (SPN) by Dirichlet boundary conditions:

$$(\text{SPD})\quad \begin{cases} \Delta\{(d_1 + \gamma_1 u + \alpha v)u\} + u(a_1 - b_1 u - c_1 v) = 0 & \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \\ \Delta\{(d_2 + \gamma_2 v + \beta u)v\} + v(a_2 - b_2 u - c_2 v) = 0 & \text{in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \end{cases}$$

Then (SPD) has no trivial solutions other than $(0,0)$. Moreover, (SPD) also has no trivial positive solution. So it is an important issue to study how to look for a positive solution. We now set

$$\tilde{u} = \frac{b_1}{d_1}u,\ \tilde{v} = \frac{c_2}{d_2}v,\ a = \frac{a_1}{d_1},\ b = \frac{a_2}{d_2},\ c = \frac{c_1 d_2}{c_2 d_1},\ d = \frac{b_2 d_1}{b_1 d_2},$$
$$\tilde{\alpha} = \frac{d_2\alpha}{c_2 d_1},\ \tilde{\beta} = \frac{d_1\beta}{b_1 d_2},\ \gamma_1 = \gamma_2 = 0$$

and rewrite (SPD) as a system of equations for $\tilde{u}$ and $\tilde{v}$. For the sake of simplicity of notation, we use (u, v, α, β) instead of $(\tilde{u}, \tilde{v}, \tilde{\alpha}, \tilde{\beta})$. Then (SPD) is expressed in the following form

$$\begin{cases} \Delta\{(1+\alpha v)u\} + u(a-u-cv) = 0 & \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \\ \Delta\{(1+\beta u)v\} + v(b-du-v) = 0 & \text{in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

When we consider semitrivial solutions of (3.4) in the form $(u^*, 0)$ with $u^* > 0$ and $(0, v^*)$ with $v^* > 0$, the we get a logistic equation with linear diffusion as follows:

$$\Delta w + w(a-w) = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega, \tag{3.5}$$

Let λ_1 be the least eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition. Then it is well known that $a > \lambda_1$ is a necessary and sufficient condition for the existence of a positive solution of (3.5) and that such a positive solution is uniquely determined ([**9**]). If we denote this positive solution by θ_a, then we see that (3.4) has, as semitrivial solutions, $(\theta_a, 0)$ if $a > \lambda_1$ and $(0, \theta_b)$ if $b > \lambda_1$. Moreover, it is also known that (3.4) has a positive solution only if

$$a > \lambda_1 \quad \text{and} \quad b > \lambda_1. \tag{3.6}$$

We will prepare some notation which will be used in this subsection. For continuous function $q \in C(\overline{\Omega})$, consider the eigenvalue problem

$$-\Delta u + q(x)u = \lambda u \qquad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{3.7}$$

and denote the least eigenvalue by $\lambda_1(q)$. Then it is known that $\lambda_1(q)$ is characterized as follows:

$$\lambda_1(q) = \inf_{u \in H_0^1(\Omega), \|u\|=1} \left\{ \|\nabla u\|^2 + \int_\Omega q(x)u(x)^2 \, dx \right\},$$

where $H_0^1(\Omega) = \{u \in W_2^1(\Omega);\ u|_{\partial\Omega} = 0\}$ and $\|\cdot\| = L^2(\Omega)$-norm. In particular, for $q \equiv 0$ we simply write $\lambda_1(0) = \lambda_1$.

Under homogeneous Dirichlet boundary conditions, the study of positive solutions of (3.4) has been done by lots of researchers ([**21**], [**42**], [**46**], [**47**], [**48**], [**49**], [**63**]). In order to prove the existence of positive solutions for (3.4), there are two typical methods; one is based on the degree theory (see, e.g., the articles of Dancer [**8**], [**9**]) and the other is based on the bifurcation theory ([**7**]). However, it is far from complete for the understanding of the structure of positive solutions.

We will follow the idea of the analysis of [**63**] and explain under what conditions positive solutions of (3.4) exist. Independently of the methods of proof (the degree theory or the bifurcation theory), the existence of positive solutions of (3.4) is closely related with the stability of semitrivial solutions $(\theta_a, 0)$ and $(0, \theta_b)$. Regarding a and b as parameters we will consider the stability of semitrivial solutions in ab plane. The stability of $(\theta_a, 0)$ changes at a curve defined by

$$S_1(\beta) = \left\{ (a, b) \in \boldsymbol{R}^2;\ \lambda_1\left(\frac{d\theta_a - b}{1 + \beta\theta_a}\right) = 0,\ a \geq \lambda_1 \right\}$$

and the stability of $(0, \theta_b)$ changes at a curve defined by

$$S_2(\alpha) = \left\{ (a, b) \in \boldsymbol{R}^2;\ \lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) = 0,\ b \geq \lambda_1 \right\}.$$

Furthermore, it is known that the first curve is expressed by

$$S_1(\beta) = \{(a, b) \in \boldsymbol{R}^2;\ b = f(a; \beta) \quad \text{for } a \geq \lambda_1\},$$

where $f(a;\beta)$ is a strictly monotone increasing C^1-function with respect to a such that $f(\lambda_1;\beta) = \lambda_1$ and $\lim_{a\to\infty} f(a;\beta) = \infty$ for each $\beta \geq 0$. Similarly, $S_2(\alpha)$ can be expressed as

$$S_2(\alpha) = \{(a,b) \in \boldsymbol{R}^2;\ b = g(a;\alpha) \quad \text{for } a \geq \lambda_1\},$$

where $g(a;\alpha)$ has analogous properties as $f(a;\beta)$. We now prepare the following sets

$$\Sigma^+(\alpha,\beta) = \{(a,b) \in \boldsymbol{R}^2;\ f(a;\beta) < b < g(a;\alpha) \quad \text{for } a > \lambda_1\},$$
$$\Sigma^-(\alpha,\beta) = \{(a,b) \in \boldsymbol{R}^2;\ g(a;\alpha) < b < f(a;\beta) \quad \text{for } a > \lambda_1\}$$

to state the existence theorem. Here $\Sigma^+(\alpha,\beta)$ is a region which is surrounded by S_1 and S_2 curves and S_2 is located above S_1. For each $(a,b) \in \Sigma^+(\alpha,\beta)$, both of two semitrivial solutions $(\theta_a, 0)$ and $(0, \theta_b)$ are unstable. On the other hand, for $(a,b) \in \Sigma^-(\alpha,\beta)$, both of two semitrivial solutions $(\theta_a, 0)$ and $(0,\theta_b)$ are asymptotically stable. Under these preparations, one can show the following results.

Theorem 3.7 (Yamada[**63**]). *Set* $\Sigma(\alpha,\beta) := \Sigma^-(\alpha,\beta) \cup \Sigma^+(\alpha,\beta)$. *Then* (3.4) *has a positive solution if* $(a,b) \in \Sigma(\alpha,\beta)$.

In Theorem 3.7, $\Sigma(\alpha,\beta)$ is a region surrounded by S_1 and S_2 curves. When (a,b) lies in this region, there exists a positive solution of (3.4). However, this theorem gives no information whether (3.4) has a positive solution or not when (a,b) lies outside of $\Sigma(\alpha,\beta)$. Regard a (or b) as a bifurcation parameter. The existence region of positive solutions depends upon not only the direction of bifurcation of positive solutions from S_1 or S_2, but also the location of turning points of bifurcation on a branch of positive solutions, and the existence or nonexistence of the secondary or tertiary bifurcation. In this manner, there are still many open problems for the understanding of the structure of positive solutions.

It is also not easy to investigate what kind of effects are given by cross-diffusion coefficients α,β on the structure of positive solutions. Letting a cross-diffusion coefficient tend to infty as in subsection **3.1**, we will study a limit problem which is satisfied by limit functions of positive solutions. We first study the dependence of $S_1(\beta)$ on β. Let $a = f^{-1}(b;\beta)$ be the inverse function of $b = f(a;\beta)$. Then one can see

$$\lim_{\beta\to\infty} f^{-1}(b;\beta) = \lambda_1 \quad \text{for each } b > \lambda_1$$

(see [**21**]). Therefore, $\Sigma^{-1}(\alpha,\beta)$ expands to $\Sigma_\infty =: \{(a,b) \in \boldsymbol{R}^2;\ b > g(a;\alpha),\ a > \lambda_1\}$ as $\beta \to \infty$. So it follows that $\Sigma(\alpha,\beta)$ also expands to $\Sigma_\infty =: \{(a,b) \in \boldsymbol{R}^2;\ b > g(a;\alpha),\ a > \lambda_1\}$ as $\beta \to \infty$. This fact implies that, if $(a,b) \in \Sigma_\infty$, then (3.4) has a positive solution for sufficiently large β.

As a next step we will study asymptotic behavior of positive solutions as $\beta \to \infty$. Correspondingly to Theorem 3.4 in case of Neumann boundary conditions, we can obtain the following estimates of positive solutions independently of cross-diffusion coefficients.

Theorem 3.8. *Let* (u,v) *be any positive solution of* (3.4). *Then there exist positive constants* C_1 *and* C_2*, independent of* α *and* β*, such that*

$$0 \leq u(x) \leq C_1, \qquad 0 \leq v(x) \leq C_2, \quad \textit{for } x \in \Omega.$$

PROOF. If we set $U = (1 + \alpha v)u$, then U satisfies

$$-\Delta U = u(a - u - cv) \quad \text{in } \Omega, \quad U = 0 \text{ on } \partial\Omega. \tag{3.8}$$

Since the right-hand side of the above equation is bounded by $u(a-u-cv) \le m_1 := a^2/4$, application of the comparison theorem to (3.8) gives the following estimate:

$$0 \le U(x) \le U^*(x), \quad x \in \Omega, \tag{3.9}$$

where U^* is a solution of $-\Delta U^* = m_1$ in Ω, $U^*|_{\partial\Omega} = 0$. Hence setting $C_1^* = \max_{x\in\overline{\Omega}} U^*(x)$ one can derive from (3.9)

$$0 \le u(x) \le U(x) \le C_1^*, \quad x \in \Omega,$$

with a positive constant C_1^* independent of α. We can derive similar estimate for v. □

Theorem 3.8 holds independently of the space dimension; so that it is quite different from the estimates in case of Neumann boundary conditions (Theorem 3.4). Hereafter, we set $\alpha = 0$ and study asymptotic behavior of solutions as $\beta \to \infty$. The following results hold true.

THEOREM 3.9 (Kuto–Yamada[**21, 22**]). *Assume $(a, b) \in \Sigma_\infty$. Let $\{\beta_n\}$ be any sequence satisfying $\beta_n \to \infty$ and let (u_n, v_n) be any positive solution of* (3.4) *with $(\alpha, \beta) = (0, \beta_n)$. Then $\{\|\beta_n u_n\|_\infty\}$ is uniformly bounded and $\{(u_n, v_n)\}$ satisfies, by choosing a subsequence if necessary,*

$$\lim_{n\to\infty} (\beta_n u_n, v_n) = (w^*, v^*) \quad \textit{uniformly in } \Omega,$$

where w^* and v^* are positive functions satisfying the following system :

$$\begin{cases} \Delta w + w(a - cv) = 0, & \textit{in } \Omega, \quad w = 0 \textit{ on } \partial\Omega, \\ \Delta\{(1 + w)v\} + v(b - v) = 0, & \textit{in } \Omega, \quad v = 0 \textit{ on } \partial\Omega. \end{cases} \tag{3.10}$$

It follows from Theorem 3.9 that $u \to 0$ as $\beta \to \infty$; so that, for any positive solution, u is very small when β is sufficiently large. This solution is not corresponding to "segregation" phenomenon. Therefore, we cannot observe segregation when one of cross-diffusion coefficients is large. But it is still an open problem to derive a limit system and study it when both of cross-diffusion coefficients are large.

See Kuto–Yamada [**21**] as to the study of the structure of positive solutions to a limit system (3.10).

4. Concluding remarks

In this article we have focused on the competitive two-species systems as ecological models with cross-diffusion. However, there are still many important topics which we have not discussed here. We will briefly introduce some typical models.

In the study of reaction-diffusion systems with cross-diffusion, lots of researches are concerned with not only competitive models but also prey-predator models and cooperative models. In particular, as to the stationary problem for prey-predator systems with cross-diffusion, we should refer to interesting works of Kuto [**19, 20**] who has discussed the structure of positive solutions, profiles of solutions and their stability properties. See also [**63**] and other related works in this direction.

In order to investigate properties of nonlinear diffusion terms, it will be a valuable and interesting idea to use a lot of knowledge, information and methods which have been developed for the study of linear diffusion. Along this line, Iida–Ninomiya–Mimura [**13**] have succeeded in approximating to SKT model (P) with $\gamma_1 = \gamma_2 = 0$ with use of suitable reaction-diffusion systems with only linear diffusion terms. This idea is also available in the area of numerical computations. Murakawa [**37**] has carried out numerical analysis of ecological models with cross-diffusion and got interesting results (see also [**38**]).

References

[1] H. Amann, *Dynamic theory of quasilinear parabolic equations II. Reaction-diffusion systems*, Differential Integral Equations, **3** (1990), 13–75.

[2] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. 1 Abstract Linear Theory*, Birkhäuser Verlag, Basel-Boston-Berlin, 1995.

[3] L. Chen and A. Jüngel, *Analysis of multi-dimensional parabolic population model with strong cross-diffusion*, SIAM J. Math. Anal., **36**(2004), 301–322.

[4] L. Chen and A. Jüngel, *Analysis of a parabolic cross-diffusion population model without self-diffusion*, J. Differential Equations, **224**(2006), 39-59.

[5] Y. S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with weak cross-diffusion*, Discrete Contin. Dyn. Syst., **9**(2003), 1193–1200.

[6] Y. S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, Discrete Contin. Dyn. Syst., **10**(2004), 719–730.

[7] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal., **8** (1971), 321–340.

[8] E. N. Dancer, *On the indices of fixed points of mappings on cones and applications*, J. Math. Anal. Appl., **91**(1983), 131–151.

[9] E. N. Dancer, *On positive solutions of some pairs of differential equations*, Trans. Amer. Math. Soc., **284** (1984), 729–743.

[10] P. Deuring, *An initial-boundary-value problem for a certain density-dependent diffusion system*, Math. Z., **194**(1987), 375–396.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, Berlin-Heidelberg, 1983.

[12] R. Haberman, *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow*, Classics in Applied Mathematics, **21**, SIAM, Philadelphia, 1998.

[13] M. Iida, M. Mimura and H. Ninomiya, *Diffusion, cross-diffusion and competitive interaction*, J. Math. Biology, **53** (2006), 617–641.

[14] T. Ichikawa and Y. Yamada, *Some remarks on global solutions to quasilinear parabolic system with cross-diffusion*, Funkcial. Ekvac., **43**(2000), 285–301.

[15] Y. Kan-on, *Stability of singularly perturbed solutions to nonlinear diffusion systems arising in population dynamics*, Hiroshima Math. J., **23** (1993), 509–536.

[16] J. U. Kim, *Smooth solutions to a quasilinear system of diffusion equations for a certain population model*, Nonlinear Anal. **8** (1984), 1121–1144.

[17] K. Kishimoto and H. F. Weinberger, *The spatial homogeneity of stable equilibria of some reaction diffusion systems on convex domains*, J. Differential Equations, **58**(1985), 15–21.

[18] H. Kuiper and Le Dung, *Global attractors for cross-diffusion systems on domains of arbitrary dimension*, Rocky Mountain J. Math., **37** (2007), 1645–1668.

[19] K. Kuto, *Bifurcation branch of stationary solutions for a Lotka-volterra cross-diffusion system in a spatially heterogeneous environment*, Nonlinear Anal. RWA, **10**(2009), 943–965.

[20] K. Kuto, *Stability and Hopf bifurcation of coexistence steady-states to an SKT model in spatially heterogeneous environment*, Discrete Contin. Dyn. Syst., **24**(2009), 489–509.

[21] K. Kuto and Y. Yamada, *Positive solutions for Lotka-Volterra competition systems with cross-diffusion*, Applicable Anal. **89**(2010), 1037–1066.

[22] K. Kuto and Y. Yamada, *On limit systems for some population models with cross-diffusion*, Discrete Contin. Dyn. Syst. Ser. B **17** (2012), 2745–2769.

[23] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, Rhode Island, 1988.

[24] D. Le, *Cross-diffusion systems on n spatial dimensional domains*, Indiana Univ. Math. J, **51** (2002), 625–643.

[25] D. Le, *Global existence for a class of strongly coupled parabolic systems*, Ann. Mat. Pura Appl., **185** (2006), 133–154.

[26] D. Le, L. V. Nguyen, T. T. Nguyen, *Shigesada-Kawasaki-Teramoto model on higher dimensional domains*, Electron J. Differential Equations **2003**, No. 72, 12pp.

[27] D. Le and T. T. Nguyen, *Global existence for a class of triangular parabolic systems on domains of arbitrary dimension*, Proc. Amer. Math. Soc., **133** (2005), 1985–1992.

[28] Y. Li and C. Zhao, *Global existence of solutions to a cross-diffusion system in higher dimensional domains*, Discrete Contin. Dyn. Syst., **12** (2005), 185-192.

[29] C. S. Lin, W. -M. Ni and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations, **72**(1988), 1–27.

[30] Y. Lou and W. -M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, **131**(1996), 79–131.

[31] Y. Lou and W. -M. Ni, *Diffusion vs cross-diffusion: an elliptic approach*, J. Differential Equations, **154**(1999), 157–190.

[32] Y. Lou, W. -M. Ni and Y. Wu, *On the global existence of a cross-diffusion system*, Discrete Contin. Dynam. Systems, **4** (1998), 193–203.

[33] Y. Lou, W. -M. Ni and S. Yotsutani, *On a limiting system in the Lotka -Volterra competition with cross-diffusion*, Discrete Contin. Dyn. Syst., **10** (2004), 435-458.

[34] M. Mimura, *Stationary pattern of some density-dependent diffusion system with competitive dynamics*, Hiroshima Math. J., **11** (1981), 621–635.

[35] M. Mimura and K. Kawasaki, *Spatial segregation in competitive interaction-diffusion equations*, J. Math. Biology, **9** (1980), 49–64.

[36] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, *Coexistence problem for two competing species models with density-dependent diffusion*, Hiroshima Math. J. **14** (1984), 425–449.

[37] H. Murakawa, *A linear scheme to approximate nonlinear cross-diffusion system*, ESAIM Math. Model Numer. Anal. **45** (2011), 1141-1161.

[38] H. Murakawa, *A relation between cross-diffusion and reaction-diffusion*, Discrete Contin. Dyn. Syst. Ser. S, **5**(2012), 147–158.

[39] W. M. Ni, *Qualitative properties of solutions to elliptic problems*, Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. **1**, Edited by M. Chipot and P. Quittner, Elsevier, Amsterdam (2004), 157-233.

[40] W. M. Ni,*Cross-diffusion in pattern formation*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, Mat. Appl., **15**(2004), 197–214.

[41] A. Okubo, L. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, Second edition, Interdisciplinary Applied Mathematics, **14**, Springer-Verlag, New York, 2001.

[42] C. V. Pao, *Strongly coupled elliptic systems and applications to Lotka-Volterra models with cross-diffusion*, Nonlinear Anal. **60** (2005), 1197-1217.

[43] M. A. Pozio and A. Tesei, *Global existence of solutions for a strongly coupled quasilinear parabolic system*, Nonlinear Anal., **14**(1990), 657–689.

[44] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.

[45] R. Redlinger, *Existence of the global attractor for a strongly coupled parabolic system arising in population dynamics*, J. Differential Equations, **118** (1995), 219–252.

[46] W. H. Ruan, *Positive steady-state solutions of a competing reaction-diffusion system with large cross-diffusion coefficients*, J. Math. Anal. Appl., **197** (1996), 558-578.

[47] K. Ryu and I. Ahn, *Positive steady-states for two interacting species models with linear self-cross diffusions*, Discrete Contin. Dyn. Syst., **9** (2003), 1049–1061.

[48] K. Ryu and I. Ahn, *Coexistence theorem of steady states for nonlinear self-cross diffusion systems with competitive dynamics*, J. Math. Anal. Appl., **283** (2003), 46-65.

[49] K. Ryu and I. Ahn, *Positive solutions of certain nonlinear elliptic systems with diffusion: nondegenerate vs degenerate diffusion*, Nonlinear Anal. **63**(2005), 247-259.

[50] N. Shigesada, K. Kawasaki, E. Teramoto, *Spatial segregation of interacting species*, J. Theor. Biol., **79** (1979), 83–99.

[51] S. -A. Shim, *Uniform boundedness and convergence of solutions to the systems with a single nonzero cross-diffusion*, J. Math, Anal. Appl., **279** (2003), 1–21.

[52] P. V. Tuộc, *Global existence of solutions to Shigesada-Kawasaki-Teramoto cross-diffusion systems on domains of arbitrary dimensions*, Proc. Amer. Math. Soc., **135** (2007), 3933–3941.

[53] P. V. Tuộc, *On global existence of solutions to a cross-diffusion system*, J. Math. Anal. Appl., **343** (2008), 826–834.

[54] Y. Wang, *The global existence of solutions for a cross-diffusion system*, Acta Math. Appl.Sin. Engl. Ser., **21** (2005), 519–528.

[55] Y. Wu, *Existence of stationary solutions with transition layers for a class of cross-diffusion systems*, Proc. Roy. Soc. Edinburgh, Sect A, **132** (2002), 1493–1511.

[56] Y. Wu, *The instability of spiky steady states for a competing species model with cross-diffusion*, J. Differential Equations, **213**(2005), 289–340.

[57] Y. Wu and Q. Xu, *The existence and structure of large spiky steady states for S-K-T competition systems with cross-diffusion*, Discrete Contin. Dyn. Syst., **29**(2011), 367–385.

[58] A. Yagi, *Global solution to some quasilinear parabolic system in population dynamics*, Nonlinear Anal., **21**(1993), 603–630.

[59] A. Yagi, *A priori estimates for some quasilinear parabolic system in population dynamics*, Kobe J. Math., **14**(1997), 91–108.

[60] A. Yagi, *Exponential attractors for competing species model with cross-diffusion*, Discrete Contin. Dyn. Syst. **22** (2008), 1091–1120.

[61] A. Yagi, *Exponential attractors*, Sugaku Expositions **24** (2011), 183–204; Translations of *Exponetial attractors* (Japanese), Sūgaku **61** (2009), 187–208.

[62] Y. Yamada, *Global solutions for quasilinear parabolic systems with cross-diffusion effects*, Nonlinear Anal., **24** (1995), 1395–1412.

[63] Y. Yamada, *Positive solutions for Lotka-Volterra systems with cross-diffusion*, Handbook of Differential Equations, Stationary Partial Differential Equations, Vol. **6** (ed. M. Chipot), Elsevier, Amsterdam (2008), 411–501.

[64] M. Yamaguchi, *Mathematics for Nonlinear Phenomena* (Japanese), Asakura-shoten, Tokyo, 2004.

[65] S. Yotsutani, *Multiplicity of solutions to a limiting system in the Lotka-Volterra competition with cross-diffusion*, RIMS workshop on "Structure of Attractors in Dissipative Systems," 2010.

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