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Variable range random walk

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ABSTRACT

Exploiting the coherent medium approximation, I investigate a random walk on objects distributed randomly in a continuous space when the jump rate depends on the distance between two adjacent objects. In one dimension, it is shown that when the jump rate decays exponentially in the long distance limit, a non-diffusive to diffusive transition occurs as the density of sites is increased. In three dimensions, the transition exists when the jump rate has a super Gaussian decay.

1. Introduction

Random walk is a highly robust process which has been applied in almost all areas of science including biology, materials sciences, economics and socio-physics [1]. It has also been exploited in the description of structural evolution by the free energy landscape theory of non-equilibrium systems [2].

Conventionally, a random walk is defined as a process that a walker makes consecutive jumps on a lattice or on a complex network or in a continuous space and the length of a single step depends on traits of the walker. On a lattice, the step length is usually set equal to the bond length between adjacent lattice sites, and on a complex network jumps are allowed along a link between nodes. Theoretically, it is possible to assume a longer step length.

The step length of a random walk in a continuous space can be a constant or can obey a certain distribution. The random walk whose step length obeys the Lévy distribution with a long tail is called Lévy flight [3] and has been applied in various problems including the analysis of flight trajectories of an albatross [4] and the description of slow dynamics of ions in alkali silicate glasses [5].

There has been a different setting for a random walk where a walker moves on objects distributed randomly in a space. The walker can move from one object to other object when the distance between these objects is less than the longest step-length of the walker. This type of random walk has been applied to the hopping conduction [6–8]

In recent years, softwares such as FireChat and Bridgefy have been developed which enable people to send messages to a designated person via network of mobile phones without using the Internet [9]. This network is called a mesh network. Namely, a message moves from one phone to nearby phones successively and will eventually reach the target. In this process, information makes jumps from one phone to nearby phones and the jump distances may be different among jumps. When the transmission process of information is regarded as a random walk of information among phones, a simple model to describe it is a random walk on objects distributed randomly in a space and the step length depends on the

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transmission range of the electro-magnetic wave from a mobile phone and the density of objects. Therefore, this process is essentially identical to the hopping conduction problem described above. Differences between the mesh network and the hopping conduction problem are in the transmission range and in the time dependence of the distribution of objects. It is apparent that this random walk on the mesh network is different from the Lévy flight [3] since the step length does not obey the Lèvy distribution and the walker does not move in a free space but on objects distributed randomly.

In this paper, I focus on the spatial effects of the random walk in a mesh network, leaving the effect of the time dependent structure in the future work. Introducing a simple model relevant to the process described above, I investigate a random walk among randomly distributed objects, where a walker makes a jump from an object to its adjacent neighbor on a limited solid angle along a fixed number of directions. The jump rate is assumed to depend on the distance between two objects. Since objects are distributed randomly, the walker makes jumps with variable ranges. Here, I focus on the diffusion constant, which is the most fundamental property, within the coherent medium approximation [8] and discuss the possibility of a transition from non-diffusive to diffusive states. Within the dynamical definition of percolation process [8], this transition corresponds to percolation transition. In general, the time dependence of a mean squared displacement and the probability distribution contains important information of the process, which will be studied in the future.

In Section 2, I explain the model in some detail and the method of analysis based on the coherent medium approximation. In Section 3, I study the random walk in one dimension for three different types of the jump rate and discuss non-diffusive to diffusive transition for an extended percolation model. The random walk in three dimensions is investigated in Section 4 and results are discussed in Section 5,

2. Model and the method of analysis

I consider a random walk on objects, where the position of object n is denoted by $\mathbf{r}_n$ which is called as site $\mathbf{r}_n$ for simplicity. The transition probability $P(\mathbf{r}_n, t|\mathbf{r}_0, 0)$ of the random walker obeys

$$\frac{\partial P(\mathbf{r}_n, t|\mathbf{r}_0, 0)}{\partial t} = -\sum_m w(|\mathbf{r}_m - \mathbf{r}_n|)P(\mathbf{r}_n, t|\mathbf{r}_0, 0) + \sum_m w(|\mathbf{r}_n - \mathbf{r}_m|)P(\mathbf{r}_m, t|\mathbf{r}_0, 0), \tag{1}$$

where $P(\mathbf{r}_n, t|\mathbf{r}_0, 0)$ is the transition probability that a random walker is at the site $\mathbf{r}_n$ at time t when it started $\mathbf{r}_0$ at time $t = 0$, and $w(|\mathbf{r}_m - \mathbf{r}_n|)$ is the jump rate of a random walker from site $\mathbf{r}_n$ to site $\mathbf{r}_m$. I assume that $w(|\mathbf{r}_m - \mathbf{r}_n|)$ is a function of the distance between $\mathbf{r}_n$ and $\mathbf{r}_m$. Usually, $w(r)$ is assumed to be nonzero within a certain distance. For example, the percolation process on lattices is modeled by jumps of a random walker within nearest neighbor sites. In this paper, I introduce an extended percolation model in which a random walker can make a longer jump with smaller rate beyond the limited distance used for the standard percolation model.

The diffusion constant D is given by

$$D = \lim_{u\to 0} \frac{u^2}{2d} \sum_m \langle (\mathbf{r}_m - \mathbf{r}_0)^2 \tilde{P}(\mathbf{r}_m, u|\mathbf{r}_0)\rangle, \tag{2}$$

where

$$\tilde{P}(\mathbf{r}_m, u|\mathbf{r}_0) = \int_0^\infty P(\mathbf{r}_m, t|\mathbf{r}_0, 0)e^{-ut}dt \tag{3}$$

is the Laplace transform of the transition probability $P(\mathbf{r}_m, t|\mathbf{r}_0, 0)$, d is the dimension of the space and $\langle \cdots \rangle$ denotes an ensemble average over the random distribution of sites. It is known that the Laplace transform $\tilde{P}(\mathbf{r}_n, u|\mathbf{r}_0)$ obeys

$$u\tilde{P}(\mathbf{r}_n, u|\mathbf{r}_0) - P(\mathbf{r}_n, 0|\mathbf{r}_0, 0) = -\sum_m w(|\mathbf{r}_m - \mathbf{r}_n|)\tilde{P}(\mathbf{r}_n, u|\mathbf{r}_0) + \sum_m w(|\mathbf{r}_n - \mathbf{r}_m|)\tilde{P}(\mathbf{r}_m, u|\mathbf{r}_0). \tag{4}$$

In order to obtain the ensemble average of $\tilde{P}(\mathbf{r}_m, u|\mathbf{r}_0)$, I employ the coherent medium approximation [8]. In this approximation, a coherent transition probability $\tilde{P}_c(\mathbf{r}_n, u|\mathbf{r}_0)$ is introduced which obeys the same equation as Eq. (4) with $w(|\mathbf{r}_m - \mathbf{r}_n|)$ replaced by a coherent jump rate $w_c(u)$

$$u\tilde{P}_c(\mathbf{r}_n, u|\mathbf{r}_0) - P(\mathbf{r}_n, 0|\mathbf{r}_0, 0) = -\sum_m w_c(u)\tilde{P}_c(\mathbf{r}_n, u|\mathbf{r}_0) + \sum_m w_c(u)\tilde{P}_c(\mathbf{r}_m, u|\mathbf{r}_0), \tag{5}$$

and the coherent jump rate $w_c(u)$ is self-consistently determined by requiring

$$\tilde{P}_c(\mathbf{r}_n, u|\mathbf{r}_0) = \langle \tilde{P}(\mathbf{r}_m, u|\mathbf{r}_0)\rangle. \tag{6}$$

In order to apply the coherent medium approximation to the present problem, I exploit a technique for positionally disordered systems in which the original problem is mapped onto a lattice problem with random jump rates [8,10]. I first divide the space around a site into z equivalent cones and assume that a random walker makes a jump to the adjacent site in one of the z cones. Within this approximation, the normalized diffusion constant is given by

$$\frac{D}{D_0} = \frac{w_c(0)}{w_0}, \tag{7}$$

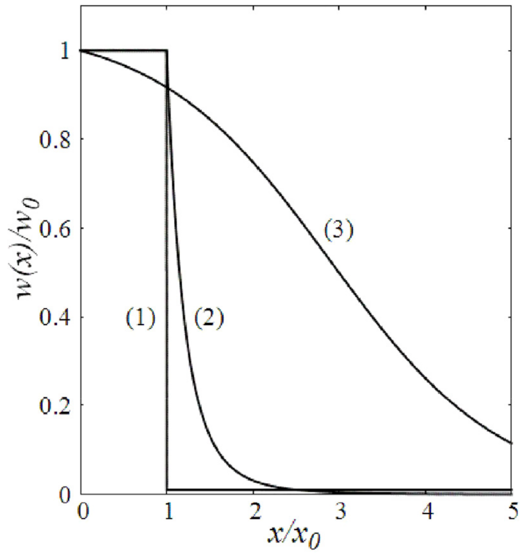

Fig. 1. The dependence of jump rate $w(x)$ on distance x for different models. (1) Eq. (12) with $\epsilon = 0.01$, (2) Eq. (14) with $\alpha = 5$ and (3) Eq. (16) with $x_r/x_0 = 3$.

where D_0 and w_0 are the diffusion constant and jump rate of a reference regular system and Eq. (6) for $w_c(0)$ reduces to [8]

$$\frac{2}{zw_c(0)} = \int_0^\infty \frac{N(r)dr}{(\frac{z}{2}-1)w_c(0)+w(r)}. \tag{8}$$

Here $N(r)$ represents the distribution function of the distance between adjacent neighbors in a cone. In three dimensions,

$$N(r) = \frac{4\pi r^2 n}{z}\exp\left(-\frac{4\pi r^3 n}{3z}\right) \tag{9}$$

when sites are distributed randomly with density n. In Eq. (7), the scale of the length of the system under consideration is assumed to be the same as that of the reference system since it does not play any significant role here.

3. Extended percolation in one dimension

In one dimension, z is set to $z = 2$ in Eq. (8) and the self-consistency equation for the coherent jump rate w_c reads as

$$\frac{1}{w_c(0)} = \int_0^\infty \frac{N(x)dx}{w(x)}, \tag{10}$$

and the distribution function $N(x)$ becomes

$$N(x) = ne^{-nx}. \tag{11}$$

In this section, I investigate several different forms of $w(x)$ which is considered as a percolation model with long range connection, and discuss possibility of a diffusive to non-diffusive transition.

3.1. Simple percolation model

As the simplest model, I first consider a transition probability

$$w(x) = \begin{cases} w_0 & (\text{when } 0 \le x \le x_0) \\ \epsilon w_0 & (\text{when } x > x_0), \end{cases} \tag{12}$$

where the percolation process corresponds to the $\epsilon \to 0$ limit. Curve (1) in Fig. 1 shows x dependence of $w(x)$ given by Eq. (12).

It is straightforward to obtain the diffusion constant from Eqs. (7) and (10)∼(12). I find

$$\frac{D}{D_0} = \frac{1}{1+\frac{1-\epsilon}{\epsilon}e^{-nx_0}}, \tag{13}$$

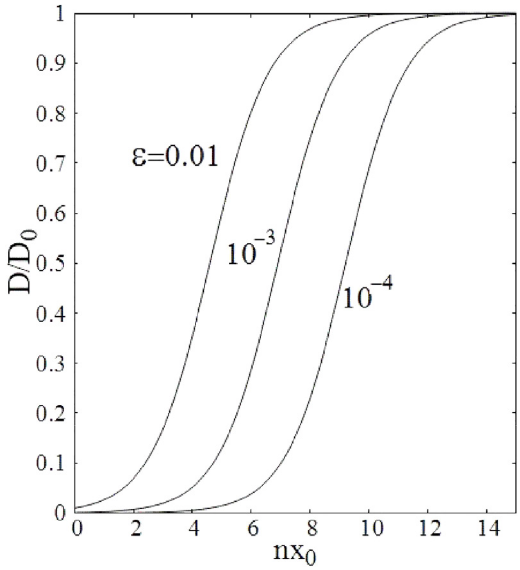

Fig. 2. The scaled diffusion constants of a simple percolation model Eq. (12) are shown as functions of the scaled density for $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$. Note $D = 0$ for the percolation limit $\epsilon = 0$.

and Fig. 2 shows the dependence of D/D_0 on the scaled density nx_0 for $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$. As expected, the diffusion constant is identically zero in the percolation limit $\epsilon = 0$. When ϵ is finite, the diffusion constant is given by a sigmoid function whose inflection point is at $nx_0 = -\ln[\epsilon/(1-\epsilon)]$.

3.2. Extended percolation model

I define an extended percolation model by a jump rate

$$w(x) = \begin{cases} w_0 & (\text{when } 0 \le x \le x_0) \\ w_0 \left(\frac{x_0}{x}\right)^{\alpha} & (\text{when } x > x_0\ ), \end{cases} \tag{14}$$

with $\alpha > 0$, whose x dependence is shown by curve (2) in Fig. 1 for $\alpha = 5$. Namely, in this model, the range of jump beyond x_0 decays as a power-law function with exponent $-\alpha$. From Eqs. (7), (10), (11) and (14), I find

$$\frac{D}{D_0} = \frac{1}{1 - e^{-nx_0} + (nx_0)^{-\alpha}\Gamma(\alpha+1, nx_0)}, \tag{15}$$

where $\Gamma(s,x) \equiv \int_x^{\infty} e^{-t} t^{s-1} dt$ is the upper incomplete Gamma function. Fig. 3 shows the nx_0 dependence of the diffusion constant for $\alpha = 1, 5, 10, 20$.

The diffusion constant becomes identically zero at $\alpha = \infty$, since $\alpha = \infty$ corresponds to the percolation limit.

3.3. Logistic-type model

I consider a smooth function for the jump rate represented by a kind of the logistic curve

$$w(x) = w_0 \frac{e^{x_r/x_0} - 1}{e^{x_r/x_0} + e^{x/x_0} - 2}, \tag{16}$$

which satisfies $w(0) = w_0$, $w(x_r) = w_0/2$ and $w(\infty) = 0$. Curve (3) in Fig. 1 shows x dependence of $w(x)$ given by Eq. (16) for $x_r/x_0 = 3$. The diffusion constant is given by

$$\frac{D}{D_0} = 1 - \frac{1}{1 + (e^{-x_r/x_0} - 1)(nx_0 - 1)}. \tag{17}$$

Fig. 4 shows the diffusion constant as functions of nx_0 for $x_r/x_0 = 1.1, 2, 3$. It is apparent that there is a non-diffusive to diffusive transition at $nx_0 = 1$. Since $D/D_0 \simeq (e^{-x_r/x_0} - 1)(nx_0 - 1)$ when $nx_0 \sim 1$, the critical exponent of the diffusion constant is unity.

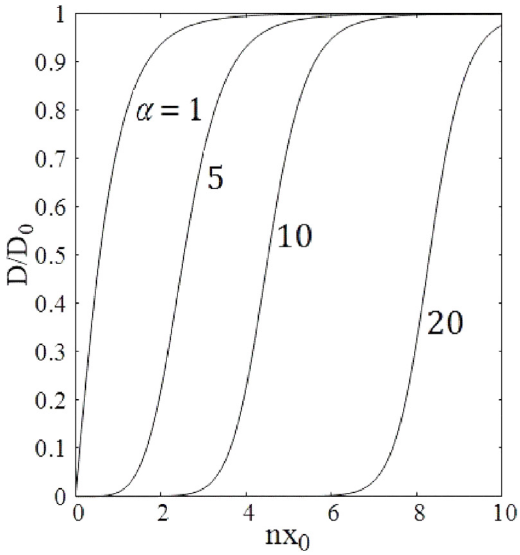

Fig. 3. Scaled diffusion constants for an extended percolation model Eq. (14) are shown as functions of the scaled density for $\alpha = 1, 5, 10, 20$. Note $D = 0$ for the percolation limit $\alpha = \infty$.

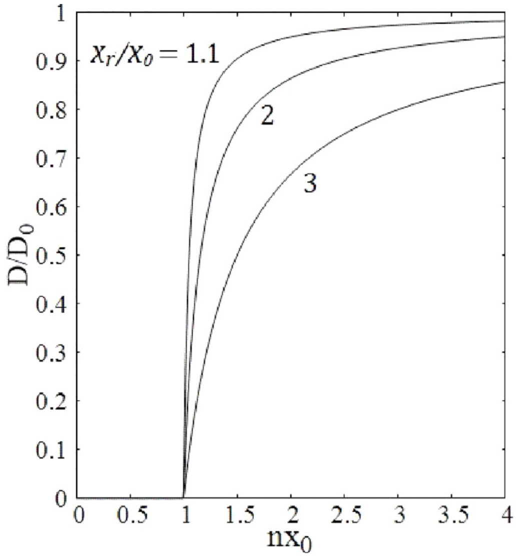

Fig. 4. Scaled diffusion constants for a logistic-type model Eq. (16) are shown as functions of the scaled density for $x_r/x_0 = 1.1, 2, 3$.

4. Extended percolation in three dimensions

4.1. Extended percolation model

I consider an extended percolation model in three dimensions where sites are distributed randomly with density n in a three dimensional space and the jump rate $w(r)$ is given by

$$w(r) = \begin{cases} w_0 & \text{(when } 0 \le r \le r_0) \\ w_0 \left(\frac{r_0}{r}\right)^\alpha & \text{(when } r > r_0 \text{).} \end{cases} \tag{18}$$

Self-consistency Eqs. (8) and (9) for w_c with Eq. (18) are solved numerically. Fig. 5 shows the dependence of the scaled diffusion constant D/D_0 on the scaled density $(4\pi r_0^3/3)n$ for $\alpha = 10$ and ∞, where $z = 6$ is used as an example. There are no percolation transition for $\alpha < \infty$. The case $\alpha = \infty$ is the simple percolation model, the diffusion constant of which is

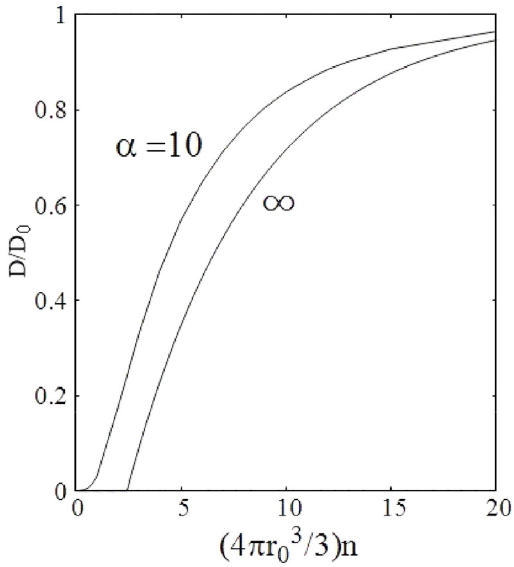

Fig. 5. The dependence of the scaled diffusion constant on the scaled density for the extended percolation model for $\alpha = 10, \infty$. Note $\alpha = \infty$ corresponds to the standard percolation model.

given by

$$\frac{D}{D_0} = 1 - \frac{z}{z-2} \exp\left(-\frac{4\pi r_0^3}{3z} n\right) \quad \text{(when } n > n_c\text{)}, \tag{19}$$

where the critical percolation density n_c is given by

$$\frac{4\pi r_0^3}{3} n_c = z \ln \frac{z}{z-2} \tag{20}$$

When $z = 6$, $\frac{4\pi r_0^3}{3} n_c = 2.43$.

4.2. Super exponential decay model

I consider the jump rate $w(r)$ given by

$$w(r) = \begin{cases} w_0 & \text{(when } 0 \le r \le r_0\text{)} \\ w_0 \exp\{-[k(r - r_0)]^\beta\} & \text{(when } r > r_0 \text{).} \end{cases} \tag{21}$$

Fig. 6 represents the dependence of the scaled diffusion constant D/D_0 on the scaled density $(4\pi r_0^3/3)n$ for $\beta = 1, 4$, where $kr_0 = 3$ and $z = 6$ are used. Apparently, there is a non-diffusive to diffusive transition for $\beta = 4$. In fact, a non-diffusive to diffusive transition exists when $\beta \ge 3$.

5. Discussion

I have studied random walks on objects distributed randomly in a space where a random walker can make long range jumps and obtained characteristic behavior of the diffusion constant within the coherent medium approximation. As for the distance dependence of the jump rate, I investigated different types of extended percolation models. It is shown that a non-diffusive to diffusive transition exists in certain types of the jump rate function in one and three dimensions.

The self-consistency equation, Eq. (8), supports a solution $w_c(0) = 0$ only when

$$\int_0^\infty \frac{N(r)dr}{w(r)} = \infty. \tag{22}$$

Therefore, in one dimension, the non-diffusive to diffusive transition exists when the jump rate function exhibits exponential or faster decay in the long distance limit [10]. In d-dimensional systems, there are no non-diffusive states unless the jump rate function decays faster than e^{-r^d}.

In this paper, I focused on the diffusion constant representing long time behavior in order to introduce a new random walk model. It is an interesting problem to observe the time dependence of the mean squared displacement and the

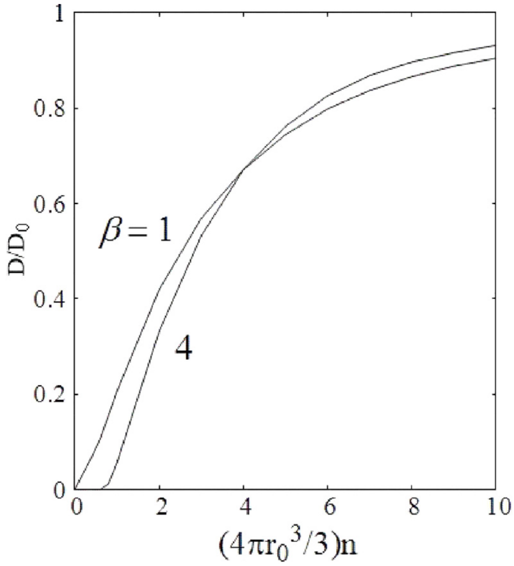

Fig. 6. The dependence of the scaled diffusion constant on the scaled density for the super Gaussian decay model for $\beta = 1, 4$.

probability distribution of a walker, which will be studied theoretically and numerically in the future. It should also be mentioned to that while a walker makes jumps among fixed sites in the present analysis, nodes of the mesh network formed by phones moves in the space. Therefore, in order to analyze the nature of the mesh network, one needs to study a random walk in which a walker makes jumps among moving sites. The situation is similar to the problem of diffusion on diffusing particles [11,12]. Generalization of the present work in this direction will also be pursued in the future.

It is interesting to note that the continuous limit of the present random walk will reduce to a diffusion processes in which the diffusivity depends on position and time [13]. Investigation of the mesh network in this direction will also be pursued in the future. It is also expected that information about the mesh network will give some insights in random walks in a highly complex structure like the free energy landscape [2].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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