

WEIGHT BALANCING ON BOUNDARIES

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ABSTRACT. Given a polygonal region containing a target point (which we assume is the origin), it is not hard to see that there are two points on the perimeter that are antipodal, that is, whose midpoint is the origin. We prove three generalizations of this fact. (1) For any polygon (or any compact planar set) containing the origin, it is possible to place a given set of weights on the boundary so that their barycenter (center of mass) coincides with the origin, provided that the largest weight does not exceed the sum of the other weights. (2) On the boundary of any 3-dimensional compact set containing the origin, there exist three points that form an equilateral triangle centered at the origin. (3) For any d -dimensional bounded convex polyhedron containing the origin, there exists a pair of antipodal points consisting of a point on a $\lfloor d/2 \rfloor$ -face and a point on a $\lceil d/2 \rceil$ -face.

1 Introduction

We will discuss three generalizations of the following observation (in this paper, a polygon or a polyhedron is always closed and bounded).

Theorem 0. *On the perimeter of any polygon containing the origin, there are two points that are antipodal, that is, whose midpoint is the origin.*

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In other words, we have

$$2P \subseteq \partial P \oplus \partial P$$

for any polygon P , where ∂P denotes its boundary, $A \oplus B = \{x + y \mid x \in A, y \in B\}$ is the Minkowski sum of regions A and B , and $\alpha A = \{\alpha x \mid x \in A\}$ is the copy of A scaled (about the origin) by a real number α .

Proof of Theorem 0. Consider $-P$, the copy of the given polygon P reflected about the origin. Since P and $-P$ cannot be properly contained in the other (and they both contain the origin), their boundaries intersect at some point $q \in \partial P \cap (-\partial P)$. Then q and $-q$ form the desired pair of points. \square

Distinct weights

An interpretation of Theorem 0 is that we can put two equal weights on the perimeter and balance them about the origin. Generalizing this to different sets of weights, we prove the following in Section 2 (note that this subsumes Theorem 0).

Theorem 1. *Suppose that k weights $w_1 \geq w_2 \geq \dots \geq w_k$ satisfy $w_1 \leq w_2 + \dots + w_k$. Then for any polygon (or any compact set) $P \subseteq \mathbb{R}^2$ containing the origin, the weights can be placed on the boundary ∂P so that their center of mass is the origin.*

In terms of the Minkowski sum, the theorem says that

$$(w_1 + \dots + w_k)P \subseteq w_1\partial P \oplus w_2\partial P \oplus \dots \oplus w_k\partial P$$

if none of the weights is bigger than the sum of the rest.

If P is the unit disk, Theorem 1 is related to a reachability problem of a chain of links (or a robot arm) of lengths w_1, w_2, \dots, w_k where one end is placed at the origin, each link can be rotated around the joints, and the links are allowed to cross each other. In order to reach every point of the disk of radius $w_1 + \dots + w_k$ centered at the origin, it is known that the condition $w_1 \leq w_2 + \dots + w_k$ is sufficient (and necessary) [8]. Theorem 1 generalizes this to arbitrary P .

Our proof is constructive and leads to an efficient algorithm to find such a location of points for a given polygon P . On the other hand, if we drop the condition $w_1 \leq w_2 + \dots + w_k$, then the conclusion does not hold in general (just let P be a disk centered at the origin), and we show that it is NP-hard to decide whether it holds for a given polygon P .

Tripodal points

Our next result concerns the 3-dimensional setting. Generalizing the notion of antipodal points in Theorem 0, we prove the following in Section 3.

Theorem 2. *On the boundary of any 3-dimensional compact set containing the origin, there are tripodal points, that is, three points forming an equilateral triangle centered at the origin.*

After publication of the conference version of the present paper [2], we discovered that Theorem 2 follows from a result by Yamabe and Yujobô [12], with an extension to star-shaped polyhedra in dimensions higher than three studied by Gordon [7]. Since the first of these papers is hard to find and the second is not available in English, we present our alternative proof.

A classical problem reminiscent of Theorem 2 is the square peg problem of Toeplitz. Given a closed curve in a plane, the problem asks for four points on the curve forming the vertices of a square. It was conjectured by Otto Toeplitz in 1911 that every Jordan curve contains four such points. Although the problem is still open for general Jordan curves, it has been affirmatively solved for curves with some smoothness conditions. As a variant of this problem, Meyerson [11] and Kronheimer and Kronheimer [9] proved that for any triangle T and any Jordan curve C , we can find three points on C forming the vertices of a triangle similar to T (note the contrast to our Theorem 2 where we need the triangle to be equilateral). See a recent survey of Matschke [10] on these problems.

Antipodal points on convex polyhedra

By viewing Theorem 0 again as the balancing of two equal weights, we can consider another generalization to convex polyhedra in dimension d , asking whether there are two antipodal points on the surface of the polyhedron. This is not very interesting if we are allowed to put them anywhere on the surface of the polyhedron: we can then cut the polyhedron by any plane through the origin and apply Theorem 0.

The question becomes interesting if we restrict the points to lie on *lower-dimensional faces* of the polyhedron. In Section 4 we prove the following.

Theorem 3. *For any convex polyhedron $P \subset \mathbb{R}^d$ containing the origin, there is an antipodal pair consisting of a point on a $\lfloor d/2 \rfloor$ -face and a point on a $\lceil d/2 \rceil$ -face.*

In other words,

$$2P \subseteq S_{\lfloor d/2 \rfloor}(P) \oplus S_{\lceil d/2 \rceil}(P).$$

where $S_k(P)$ denotes the k -skeleton of a convex polyhedron P .

We also show that it is not possible to replace the pair of dimensions by $(k, d - k)$ for any $k < \lfloor d/2 \rfloor$.

By repeated application of Theorem 3, it follows that when the dimension d is a power of two, then there are d points on the *edges* (the one-skeleton) of P whose barycenter is the origin. Dobbins has shown using equivariant topology that this statement does indeed hold in any dimension, and in a more general form: When $d = nk$, then there are n points on the k -skeleton whose barycenter is the origin [4]. Blagojević et al. [3] gave an alternative proof of the same result. Finally, Dobbins and Frick [5] generalize Theorem 3 by setting $d = nk + r$, for $0 \leq r < n$, and prove the existence of n points in the k -faces and $(k + 1)$ -faces whose barycenter is the origin. They also consider non-equal weights.

Related work

Bringing the center of mass to a desired point by putting counterweights is a common technique for reduction of vibrations in mechanical engineering [1]. There have been studies on Minkowski operations considering the boundary of objects (see, for instance, Ghosh and Haralick [6]), but our paper seems to be the first to deal with the general question of covering the body with convex linear combinations of the boundary.

2 Distinct weights

We prove Theorem 1. Let us first assume that P is a simple polygon containing the origin in its interior, and let p be a point on ∂P closest to the origin. We first put the biggest weight w_1 at p , and the remaining $k - 1$ weights at $p' = -p \cdot w_1 / (w_2 + \dots + w_k)$. By the assumption $w_1 \leq w_2 + \dots + w_k$ we have $p' \in P$. The barycenter of the chosen k weighted points is the origin. One of the weights, namely w_1 , lies on ∂P , while the remaining weights lie in P .

We will now repeatedly move two of the weights while maintaining the barycenter at the origin, in each step moving one more weight to ∂P . Let q_1, q_2, \dots, q_k be the current position of the k weights, with $q_1, q_2, \dots, q_i \in \partial P$, while $q_{i+1} = \dots = q_k = p'$.

We will now move q_i and q_{i+1} such that both lie on ∂P . We set $r = \sum_{j=1}^{i-1} w_j q_j + \sum_{j=i+2}^k w_j q_j$. By assumption, we have $w_i q_i + w_{i+1} q_{i+1} + r = 0$, and thus $q_{i+1} = -(r + w_i q_i) / w_{i+1}$. If we move the weight w_i to a new position $q'_i \in \partial P$, we can move w_{i+1} to $q'_{i+1} = -(r + w_i q'_i) / w_{i+1}$ to maintain the barycenter at the origin. As q'_i moves along ∂P , the point q'_{i+1} moves along $\partial P'$, where $P' = -(w_i / w_{i+1})P - r / w_{i+1}$, that is, a translated, reflected, and scaled copy of P . Since $q_{i+1} = p' \in \partial P' \cap P$, the boundary $\partial P'$ contains a point in P . Since $w_i / w_{i+1} \geq 1$, P' cannot lie entirely inside P , and so the boundaries ∂P and $\partial P'$ must intersect in a point q'_{i+1} . We move w_{i+1} to $q'_{i+1} \in \partial P$, and move w_i to the corresponding point $q'_i \in \partial P$, see Figure 1.

Repeating this step $k - 1$ times, we bring all weights to ∂P , proving Theorem 1 for the case where P is a simple polygon.

We now consider the case that P is an arbitrary compact set containing the origin in \mathbb{R}^2 . For $m = 1, 2, \dots$ we partition the plane with an axis-aligned grid whose cells have side length $1/m$, and let $A_m \supseteq P$ be the union of all grid cells intersecting P (where a “grid cell” is to be understood as including its boundary). Note that each point in ∂A_m is within distance $\sqrt{2}/m$ from ∂P . Let $B_m \subseteq A_m$ be the union of all grid cells reachable from the origin by a path in the interior of A_m (this step is necessary because we do not require P to be connected). Let X_m be the unique unbounded connected component of $\mathbb{R}^2 \setminus B_m$, and set $C_m = \mathbb{R}^2 \setminus X_m$ (in other words, C_m is B_m with all “holes filled in”).

Since $\partial C_m \subseteq \partial B_m \subseteq \partial A_m$, each point in ∂C_m lies at distance at most $\sqrt{2}/m$ from ∂P . We observe that C_m is a simple polygon containing the origin, and so we can apply the above special case and obtain a k -tuple of locations $\mathbf{q}^{(m)} = (q_1^{(m)}, \dots, q_k^{(m)}) \in (\partial C_m)^k$ such that putting the weight w_i at $q_i^{(m)}$ (for $i = 1, \dots, k$) brings the barycenter to the origin.

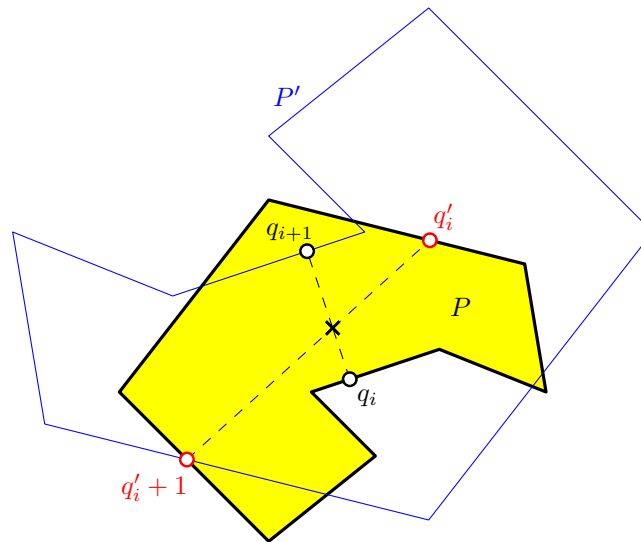


Figure 1: Proof of Theorem 1. The weights w_i and w_{i+1} are initially at $q_i \in \partial P$ and $q_{i+1} = p' \in P$. As w_i moves along ∂P , w_{i+1} moves along a magnified (and reflected) copy $\partial P'$ of ∂P , which intersects ∂P at some point q'_{i+1} .

Since the sequence $\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \dots$ is in the compact space U^k , where U is a sufficiently large compact set containing P , it has a subsequence that converges to some $\mathbf{q} = (q_1, \dots, q_k)$. Since each $q_i^{(m)}$ is within distance $\sqrt{2}/m$ from ∂P , each q_i is in ∂P . Furthermore, since the barycenter is a continuous function of the location of the weights, we conclude that putting the weight w_i at q_i for each i brings the barycenter to the origin, proving the general form of Theorem 1.

Algorithmic aspects

We consider the computational problem that corresponds to Theorem 1: Given a region P and a set of k weights, we want to determine whether we can balance the weights by putting them on ∂P , and if so, to find such a location. We restrict ourselves to the case where P is a simple polygon with n vertices, and design algorithms in terms of n and k .

If none of the weights exceeds the sum of the others, the proof of Theorem 1 implies a polynomial-time algorithm. In order to replace q_i and q_{i+1} with a pair of boundary points q'_i and q'_{i+1} (Figure 1), we need to find an intersection point of ∂P and $\partial P'$. This can be done in $O(n \log n)$ time. The initial location of the largest weight can be found in $O(n)$ time. Thus, we have an $O(kn \log n)$ time algorithm.

We can design a faster algorithm as follows. We greedily divide the weights into three groups so that no group weighs more than the sum of the rest. This is always possible in $O(k)$ time (as long as no single weight exceeds the sum of the others). Thus, we have an instance for $k = 3$, which we solve in $O(n \log n)$ time. This gives an $O(k + n \log n)$ -time algorithm, although the output may look a little artificial since all weights will be located

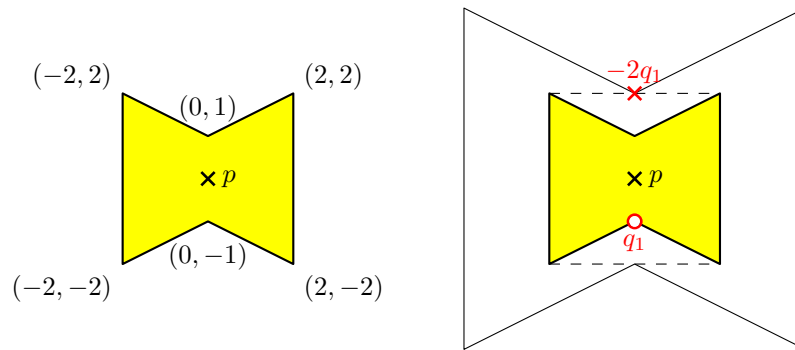


Figure 2: A hard instance of the balancing location problem.

at (at most) three points.

On the other hand, if we are given a set of an unknown number of weights that may contain a weight exceeding the sum of the rest, then the problem is NP-hard.

Proposition 1. *There exists a polygon $P \subseteq \mathbb{R}^2$ containing the origin such that it is NP-hard to determine if a given set of weights can be placed on the boundary ∂P so that their barycenter is at the origin.*

Proof. We prove NP-hardness by reducing the PARTITION problem to this problem. The input to PARTITION is a set of N nonnegative integers a_1, a_2, \dots, a_N , and the problem asks whether there is a subset $X \subset \{1, \dots, N\}$ such that

$$\sum_{i \in X} a_i = \sum_{i \in \{1, \dots, N\} \setminus X} a_i.$$

We transform the problem into a weight balancing problem as follows: we set $k = N + 1$, $w_i = a_{i-1}$ for $i = 2, 3, \dots, k$, and $w_1 = 2 \sum_{i=1}^N a_i$.

Let P be the non-convex polygon with vertices $(0, 1)$, $(2, 2)$, $(2, -2)$, $(0, -1)$, $(-2, -2)$, and $(-2, 2)$, see Figure 2(left). Note that $P = -P$ and $-2P$ contains the convex hull $\text{conv}(P)$ of P . Moreover, the two reflex vertices of $-2P$ are the only points of $\partial 2P \cap \text{conv}(P)$, and each of the points is the midpoint of an edge of $\text{conv}(P)$ as shown in Figure 2(right).

Observe that the only possible location q_1 of weight w_1 is one of the two reflex vertices since its reflection $-2q_1$ can be written as a convex combination of other points on ∂P , and is hence contained in $\text{conv}(P)$. That is, $-2q_1$ lies on the midpoint of an edge of $\text{conv}(P)$ (without loss of generality, we may assume it is the edge e from $(-2, 2)$ to $(2, 2)$). In particular, there is a solution if and only if we can place the remaining points in a way that their barycenter lies on the midpoint of e .

Since the new target point lies on the edge e of $\text{conv}(P)$, the only possible location for the remaining points is $(-2, 2)$ or $(2, 2)$. Moreover, the barycenter becomes the midpoint if and only if the weights are equally divided. Thus, the PARTITION problem is reduced to the balancing location problem. Since PARTITION is NP-complete, detecting the existence of a balancing location is NP-hard. \square

3 Tripodal points

In this section, we consider a 3-dimensional compact set P and prove Theorem 2, which states that there are tripodal points on the boundary ∂P . Note that tripodal points are a natural analogue of antipodal points: saying that three points are tripodal is equivalent to requiring that they are at the same distance from the origin and their barycenter is the origin.

We will first assume that P is the union of a finite number of convex polyhedra such that the boundary ∂P is connected. Let p_0 and p_1 be a nearest and a farthest point on ∂P , respectively, from the origin o . They exist because ∂P is compact. By our assumption on P , there is a simple piecewise-linear path L from p_0 to p_1 on ∂P , parametrized by a one-to-one continuous function $\gamma: [0, 1] \rightarrow L$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.

We claim that there exist three points $a \in L$, $b \in \partial P$, and $c \in \partial P$ that are tripodal. For each $q \in L$, let $H(q)$ be the set of vectors perpendicular to the line through o to q . We use the following fact.

Lemma 1. *There exists a continuous piecewise algebraic function $\mathbf{v}: L \rightarrow \mathbb{S}^2$ such that $\mathbf{v}(q) \in H(q)$ for all $q \in L$.*

Proof. Consider the points of L as vectors in \mathbb{R}^3 , and let $\mathcal{F} \subset \mathbb{S}^2$ be the set obtained by normalizing all those vectors. Since L consists of a finite number of linear segments, \mathcal{F} has measure zero. Thus, there must exist some vector $\mathbf{w}_0 \in \mathbb{S}^2 \setminus (\mathcal{F} \cup -\mathcal{F})$.

We define the function \mathbf{v} as follows: for any point $q \in L$, let $\mathbf{v}(q)$ be the normalized projection of \mathbf{w}_0 into $H(q)$. By construction, \mathbf{w}_0 is not parallel to any vector of L , hence its projection is nonzero and its normalization is properly defined. Since $H(q)$ is a continuous function of q , the projection of \mathbf{w}_0 on $H(q)$ is continuous. Furthermore, on a single linear segment of L this projection is an algebraic function of q . Since normalization is continuous and algebraic, the function \mathbf{v} is continuous and piecewise algebraic. \square

We fix such a function $\mathbf{v}: L \rightarrow \mathbb{S}^2$. For each $t \in [0, 1]$ and each angle $\theta \in [0, \pi]$, let $b(t, \theta)$ and $c(t, \theta)$ be the unique pair of points such that $\gamma(t)$, $b(t, \theta)$, $c(t, \theta)$ are tripodal points and the vector $b(t, \theta) - c(t, \theta) \in H(\gamma(t))$ makes an angle of $+\theta$ with $\mathbf{v}(\gamma(t))$ (using the vector $\gamma(t)$ to determine the sign of the rotation). Define $f_1(t, \theta) \in \{+, -, 0\}$ by whether the point $b(t, \theta)$ lies inside (the interior of) P , outside P , or on ∂P . Define $f_2(t, \theta)$ analogously using the point $c(t, \theta)$.

If there is (t, θ) such that $f_1(t, \theta) = f_2(t, \theta) = 0$, then $\gamma(t)$, $b(t, \theta)$, $c(t, \theta)$ are tripodal points and we are done. Suppose otherwise. Define the *signature* of (t, θ) , denoted $F(t, \theta)$, as

$$F(t, \theta) = \begin{cases} ++ & \text{if } (f_1(t, \theta), f_2(t, \theta)) \in \{(+, +), (+, 0), (0, +)\}, \\ -- & \text{if } (f_1(t, \theta), f_2(t, \theta)) \in \{(-, -), (-, 0), (0, -)\}, \\ +- & \text{if } (f_1(t, \theta), f_2(t, \theta)) = (+, -), \\ -+ & \text{if } (f_1(t, \theta), f_2(t, \theta)) = (-, +) \end{cases}$$

(Figure 3). Since p_0 and p_1 are the nearest and the farthest points, it holds that $F(0, \theta) =$

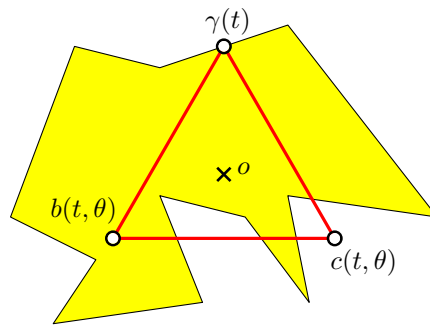


Figure 3: The signature is $+-$ for this (t, θ) .

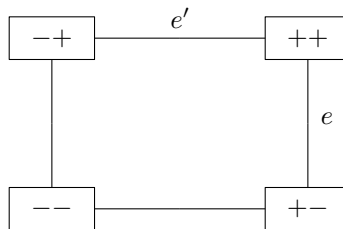


Figure 4: The cycle C .

$++$ and $F(1, \theta) = --$.

Consider the domain $[0, 1] \times [0, \pi]$ of $F(t, \theta)$. It is partitioned into regions corresponding to the four different values of F . Boundaries of the regions consist of points where either $b(t, \theta)$ or $c(t, \theta)$ lies on ∂P . By our assumption, regions for $++$ and $--$ cannot have a common boundary point, and neither can regions for $+-$ and $-+$. Since \mathbf{v} is piecewise algebraic, we can choose a parameterization where the region boundaries are piecewise algebraic, so any line through the domain intersects a finite number of points, and regions have a finite number of points with a tangent parallel to the t -axis.

For each θ , consider the transition of $F(t, \theta)$ as t changes from 0 to 1. This corresponds to the finite sequence of regions intersected by the segment $\{(t, \theta) \mid t \in [0, 1]\}$, and we obtain a finite walk $\mathcal{W}(\theta)$ from $++$ to $--$ in the graph C shown in Figure 4.

Consider the edge e between $++$ and $+-$. A walk from $++$ to $--$ is called *even* if it uses e an even number (possibly zero) of times, and it is called *odd* otherwise. For example, the path $++, +-, --$ is odd and $++, --, --$ is even.

As we increase θ continuously from 0 to π , the walk $\mathcal{W}(\theta)$ changes when the sequence of regions intersected by the segment $\{(t, \theta) \mid t \in [0, 1]\}$ changes, that is, for a finite number of values of θ where the segment is tangent to one of the regions. The walk can change in two possible ways:

- when the segment starts to intersect a new region, an entry a in $\mathcal{W}(\theta)$ is replaced by a sequence a, b, a , where b is a neighbor of a in C ;
- when the segment stops intersecting a region, a sequence a, b, a is replaced by a .

Neither of these events change the parity of the walk.

On the other hand, the walks $\mathcal{W}(0)$ and $\mathcal{W}(\pi)$ have different parities. To see this, let e' be the edge between $++$ and $-+$. Since e and e' form a cut separating $++$ and $--$ in C , each walk must use them an odd number of times in total. Since $b(t, 0) = c(t, \pi)$ and $c(t, 0) = b(t, \pi)$, the walk $\mathcal{W}(\pi)$ is obtained from $\mathcal{W}(0)$ by exchanging $-+$ and $+-$, and hence has opposite parity.

This is a contradiction, so Theorem 2 follows for the special case where P is the union of a finite number of convex polyhedra with connected boundary.

Consider now the general case, where P is an arbitrary compact set containing the origin in \mathbb{R}^3 . We proceed as in the two-dimensional result: for $m = 1, 2, \dots$ we partition \mathbb{R}^3 with an axis-aligned grid of width $1/m$, let $A_m \supseteq P$ be the union of all grid cells intersecting P , let $B_m \subseteq A_m$ be the union of all grid cells reachable from the origin by a path in the interior of A_m , let X_m be the unbounded connected component of $\mathbb{R}^3 \setminus B_m$, and set $C_m = \mathbb{R}^3 \setminus X_m$.

Unlike in the two-dimensional case, the set C_m is not necessarily a topological ball. However, we observe that C_m is the union of a finite number of convex polyhedra (the grid cells), and since $\partial C_m = \partial X_m$, its boundary ∂C_m is connected. This implies that we can apply the above special case and obtain tripodal points $(a_m, b_m, c_m) \in (\partial C_m)^3$. By compactness, the sequence (a_m, b_m, c_m) has a subsequence converging to a triple of points (a, b, c) . By continuity, a, b, c are tripodal points. Since $\partial C_m \subseteq \partial B_m \subseteq \partial A_m$, each point in ∂C_m lies at distance at most $\sqrt{3}/m$ from ∂P , so compactness of ∂P implies that $a, b, c \in \partial P$. This completes the proof of the general form of Theorem 2.

We may ask similar questions for three points forming other shapes, or for higher dimensions.

Conjecture 1. *For a d -dimensional polyhedron P containing the origin, there exist d points on ∂P forming a regular $(d - 1)$ -dimensional simplex centered at the origin.*

Algorithmic aspects need further investigation. It is easy to devise an $O(n^3)$ -time algorithm to find a tripodal location guaranteed by Theorem 2 for a polyhedron with n vertices, just by going through all the triples of faces. It is not clear if this can be improved.

4 Antipodal points on convex polyhedra

A d -dimensional (closed bounded) convex polyhedron P decomposes into faces of dimensions $i = 0, 1, \dots, d$. Let F_i be the set of i -dimensional faces. The union $S_k(P) = \bigcup_{i=0}^k \bigcup_{f \in F_i} f$ of faces of at most k dimensions is called the k -skeleton of P . In particular, the 1-skeleton $S_1(P)$ is the union of edges (including vertices), and the $(d - 1)$ -skeleton is ∂P .

We will now prove Theorem 3 by proving

$$2P \subseteq S_{\lfloor d/2 \rfloor}(P) \oplus S_{\lceil d/2 \rceil}(P).$$

Choose any point of the left-hand side, $2P$. We will show that this point is in the right-hand side. We may assume that this point is in the interior of $2P$, since the right-hand

side is a closed set. Also, without loss of generality, we may assume that this point is the origin. Thus, assuming that P contains the origin in its interior, we need to show that the origin belongs to the right-hand side, or equivalently, that $S_{\lfloor d/2 \rfloor}(P) \cap S_{\lfloor d/2 \rfloor}(-P)$ is nonempty.

For simplicity of notation, we assume that d is even. The odd case is shown identically by replacing $d/2$ by $\lfloor d/2 \rfloor$ and $\lceil d/2 \rceil$ accordingly.

Since P contains the origin in its interior, the intersection $P \cap (-P)$ is a d -dimensional convex polyhedron. Moreover, its boundary C is centrally symmetric (i.e., $C = -C$). It suffices to show that C has a vertex in $S_{d/2}(P) \cap S_{d/2}(-P)$.

A facet ($(d-1)$ -dimensional face) of C is a subset of a facet of either P or $-P$. We start with the special case in which C is simple. That is, every vertex of C is contained in exactly d facets of P or $-P$. A vertex of C is of *type* $(j, d-j)$ if it is contained in j facets of P and $d-j$ facets of $-P$. Let v be any vertex of C , and let $(k, d-k)$ be the type of v . If $k = d/2$, we are done. Thus, we assume without loss of generality that $k < d/2$. Since C is centrally symmetric, $-v \in C$, and $-v$ is of type $(d-k, k)$. Since the 1-skeleton of C is connected, there exists a path P in the skeleton from v to $-v$. Let (x, y) be an edge of P with x and y of type $(i, d-i)$ and $(j, d-j)$, respectively. Then $j \in \{i-1, i, i+1\}$. Thus, there exists a vertex w on P of type $(d/2, d/2)$.

Now, we consider the general case where C might have a vertex that is an intersection of more than d facets. We consider an infinitesimal perturbation of hyperplanes defining facets of P to make C simple. Then, the perturbed version \tilde{C} of C has a vertex \tilde{v} of type $(d/2, d/2)$, which corresponds to a vertex v of C . Thus, v must lie at an intersection of $S_{d/2}(P)$ and $S_{d/2}(-P)$, completing the proof of Theorem 3.

Thus we can always find an antipodal pair of points from $\lfloor d/2 \rfloor$ - and $\lceil d/2 \rceil$ -dimensional faces. However, this does not extend to other pairs of dimensions k and $d-k$.

Proposition 2. *There exists a convex polyhedron $P \subseteq \mathbb{R}^d$ containing the origin such that for any $k < \lfloor d/2 \rfloor$, it holds that $S_k(P) \cap S_{d-k}(-P) = \emptyset$.*

Proof. First, we consider the case where $d = 2m$ is even (thus, $k < m$). Consider an equilateral triangle T centered at the origin. Then, we observe that all three vertices of T lie outside $-T$. Let $T^m = T \times \cdots \times T$ be the Cartesian product of T in \mathbb{R}^{2m} . Then, a k -dimensional face of $P = T^m$ is the Cartesian product of k edges and $m-k$ vertices of T . Since $m-k > 0$ and a vertex of T lies outside $-T$, the face cannot intersect $-P = (-T)^m$. If $d = 2m + 1 \geq 3$ is odd, we consider $P = I \times T^m$, where $I = [-1, 2]$ is an interval. The remaining argument is analogous. \square

As mentioned in the introduction, repeated application of Theorem 3 immediately gives us the following:

Proposition 3. *Let k be a positive integer and let $d \leq 2^k$. Then, on the 1-skeleton of any d -dimensional convex polyhedron, there are 2^k points whose barycenter is at the origin.*

Proof. We use induction on k . The statement is true for $k = 1$ (Theorem 0). It follows from Theorem 3 that there are antipodal points $x \in F$ and $-x \in F'$, where F and F' are faces

from $S_{\lfloor d/2 \rfloor}(P)$ and $S_{\lceil d/2 \rceil}(P)$, respectively. By the induction hypothesis applied to F and F' (translated by $-x$ and x), we have 2^{k-1} points on the skeleton of F with barycenter x , and 2^{k-1} points on the skeleton of F' with barycenter $-x$. These 2^k points together satisfy our requirement. \square

We note that our method is constructive, and such a location of points can be computed in polynomial time for any fixed dimension.

Algorithmic aspects of the generalization of Proposition 3 by Dobbins [4] appear to be unexplored.

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